Convexifying positive polynomials and a proximity algorithm

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In honor of Bernard Teissier's 70th birthday
June 22 – 26, 2015 Aussois

• We prove that if f is a positive C^2 function on a convex compact set $X \subset \mathbb{R}^n$ then

$$\varphi_N = f(x)(1 + |x|^2)^N$$

is strongly convex for N large enough.

• For f polynomial we give an explicit estimate for N, which depends on the size of the coefficients of f and on the lower bound of f on X.

• Application: an algorithm which for a given polynomial f on a convex compact semialgebraic set X produces a sequence (starting from an arbitrary point in X) which converges to a (lower) critical point of f on X. The convergence is based on the method of talweg which is a generalization of the Łojasiewicz gradient inequality

$$|\nabla f| \geq |f|^\rho,$$

with $\rho < 1$ for f analytic in a ngbh. of $0 \in R^n$, $f(0) = 0$.

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We denote by $\mathbb{R}[x]$ or $\mathbb{R}[x_1, \dots, x_n]$ the ring of polynomials in $x = (x_1, \dots, x_n)$ with coefficients in $\mathbb{R}$.

A set $X \subset \mathbb{R}^n$ is called *semialgebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_j(x) \geq 0, g_{j+1}(x) > 0, \dots, g_r(x) > 0\},$$

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Convexifying

The aim of the lecture is convexification of polynomials.

Let X be a convex closed semialgebraic subset of $\mathbb{R}^n$
and let f be a polynomial which is positive on X.

We give necessary and sufficient conditions for the existence of an exponent $N \in \mathbb{N}$ such that

$$(1+|x|^2)^N f(x) \quad \text{is a strongly convex function on } X.$$

A C^1 function $g : X \to \mathbb{R}$ is called μ-strongly convex if

$$g(y) \geq g(x) + \langle y - x, \nabla g(x) \rangle + \frac{\mu}{2}|y - x|^2 \quad \text{for } x, y \in X,$$

where $\mu > 0$ and $\langle \cdot\,, \cdot \rangle$ is the standard scalar product.

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The key point in further considerations is the following lemma.

Lemma (1)

Let $f \in \mathbb{R}[t]$, and let f be positive on a closed interval $I \subset \mathbb{R}$. Then there exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the polynomial

$$\varphi_N(t) := (1+t^2)^N f(t)$$

is strongly convex on I.

Remark (2)

Let $f(t) = \sum_{i=0}^{d} a_i t^{d-i}$, $a_0, \ldots, a_d \in \mathbb{R}$, $a_0 \neq 0$, $d = \deg f$. The number N_0 we may effectively estimate. Namely

$$N_0 = [\mathcal{N}(m, K, D)] + 1,$$

where $\mathcal{N}(m, K, D) := \max\left\{\frac{D}{m} + \frac{m}{16D}, \frac{(1+K^2)D}{Km} + 1, \frac{4D^2}{m^2} + 2, \frac{(1+K^2)D}{2m}\right\}$.

$$K = 1 + 2\max_{1\leq i\leq d} |a_i/a_0|^{1/i}, \quad m = \min\{f(t) : t \in I\},$$

$$|f'(t)| \leq D, \quad |f''(t)| \leq D \quad \text{for } |t| \leq K.$$

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Convexifying polynomials on compact sets

Theorem (3, KS 2014)

Let $f \in \mathbb{R}[x]$ be positive on a compact convex set $X \subset \mathbb{R}^n$. Then there exists $N_0 \in \mathbb{N}$ such that for any integer $N \geq N_0$ the polynomial

$$\varphi_N(x) = (1 + x_1^2 + \cdots + x_n^2)^N f(x)$$

is strongly convex in X.

Sketch of the proof. Let $R = \max\{|x| : x \in X\}$, and let

$$\mathcal{A} = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n : \langle \alpha, \beta \rangle = 0,\ |\alpha| \leq R,\ |\beta| = 1\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. Let

$$\gamma_{\alpha,\beta}(t) := \sqrt{1 + |\alpha|^2}\beta t + \alpha.$$

Clearly the family of all $\gamma_{\alpha,\beta}$ with $(\alpha, \beta) \in \mathcal{A}$ parametrizes all affine lines in $\mathbb{R}^n$ which intersects X. Since

$$\varphi_N \circ \gamma_{\alpha,\beta}(t) = (1 + |\alpha|^2)^N (1 + t^2)^N f \circ \gamma_{\alpha,\beta}(t),$$

applying Lemma 1 we deduce the assertion. □

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Question. Whether convexity of a polynomial φ_{N_0} implies convexity of all polynomials φ_N for $N \geq N_0$.

It turns out to be false, namely we have

Example

Let $f(x) = 7x^2 - 22x + 19$. The polynomial f is strictly positive on $\mathbb{R}$. Moreover,

$$\varphi_N''(1) = 2^{N+1}[2N^2 - 8N + 7].$$

Then $\varphi_1''(1) = 4$, $\varphi_2''(1) = -8$, nd $\varphi_3''(1) = 16$. Hence, there exists a closed interval $I \subset \mathbb{R}$ centered at 1 such that

- $\varphi_1''(x) > 0$ for $x \in I$, so φ_1 is strongly convex in I,
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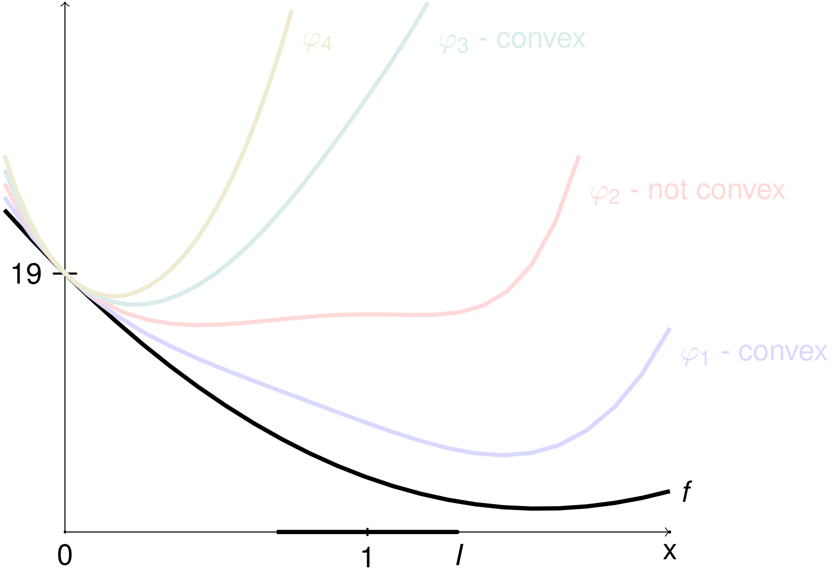
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f
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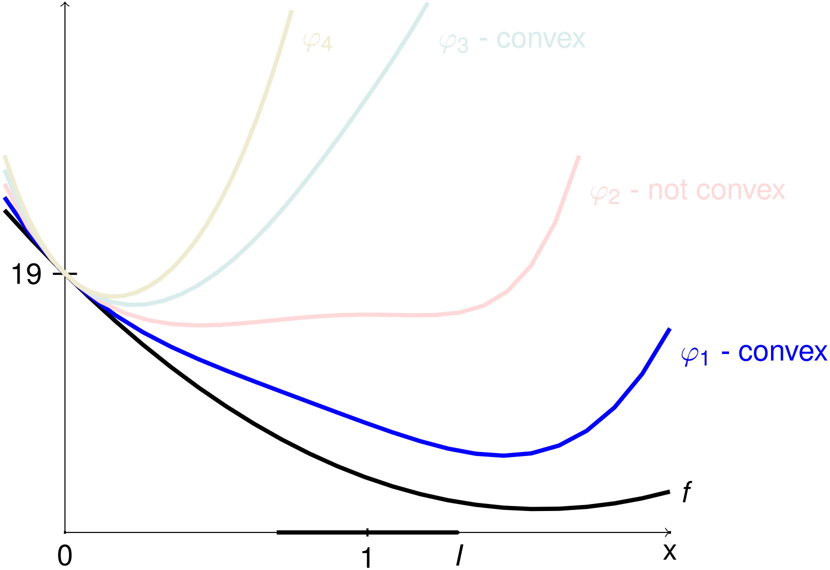

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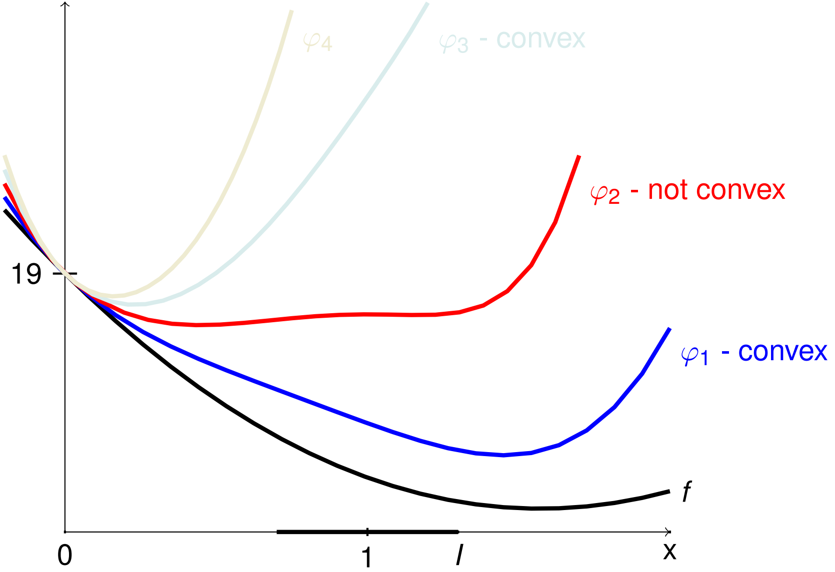

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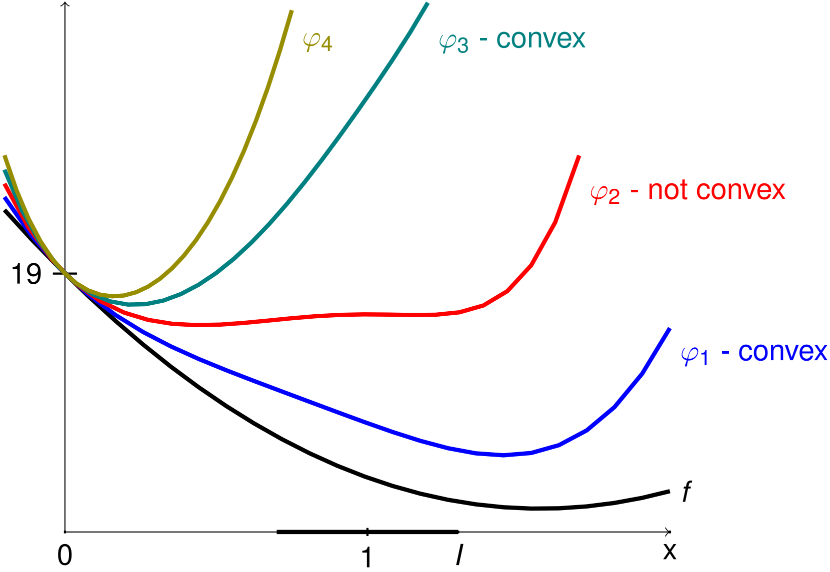

A proximity algorithm for a polynomial on a convex set

Let f be a C^1 function in a neighborhood U of a closed set $X \subset \mathbb{R}^n$.
Recall that $a \in X$ is a *lower critical point of f on X* if

$$\langle \nabla f(a), x - a \rangle \geq 0 \quad \text{for any } x \in X \text{ in a ngbh. of a.}$$

We denote by $\Sigma_X f$ the set of lower critical points of f on X, and by $\Sigma f := \{x \in U : \nabla f(x) = 0\}$ the set of ordinary critical points of f.

Proposition (4)

If $X \subset \mathbb{R}^n$ is closed convex and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, then:

1. $X \cap \Sigma f \subset \Sigma_X f$;
2. *if f restricted to X has a local minimum at a, then $a \in \Sigma_X f$;*
3. *if $M \subset X$ is a smooth manifold and $a \in M \cap \Sigma_X f$, then for any $z \in T_a M$,*
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Let $X \subset \mathbb{R}^n$ be a compact convex semialgebraic set.

Using a translation and a dilatation we may assume that X is contained in a ball of radius $R = 1/2$ centered at zero.

Replacing f by $f + c$, where c is a constant large enough we may assume that $m = \inf\{f(x) : x \in X\} = D > 0$, where D is a bound for the absolute value of the first and the second directional derivatives of f (along vectors of norm 1).

Then we have $\mathcal{N}(m, 2R, D) = 6$. So, for $N = 6$ and some $\mu > 0$ the function

$$\varphi_{N,\xi}(x) := (1 + |x - \xi|^2)^N f(x)$$

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Choose an arbitrary point $a_0 \in X$, and by induction set

$$a_\nu := \operatorname{argmin}_X \varphi_{N,a_{\nu-1}}. \tag{1.1}$$

The main corollary of the convexification method is

Theorem (A, KS 2015)

Let $X \subset \mathbb{R}^n$ be a compact convex semialgebraic set and $f : \mathbb{R}^n \to \mathbb{R}$ a polynomial positive on X. Let a_ν be the sequence defined by (1.1) with $a_0 \in X$. Then the limit

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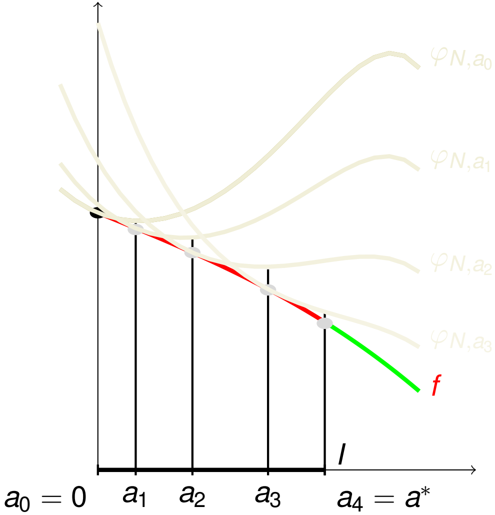

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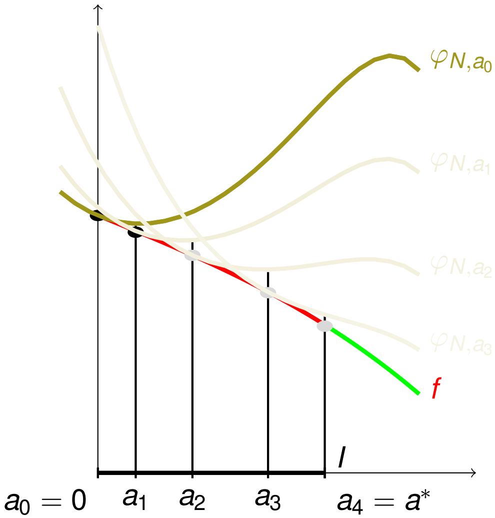

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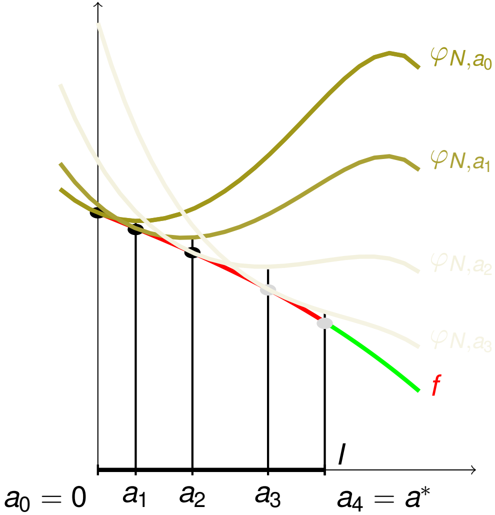

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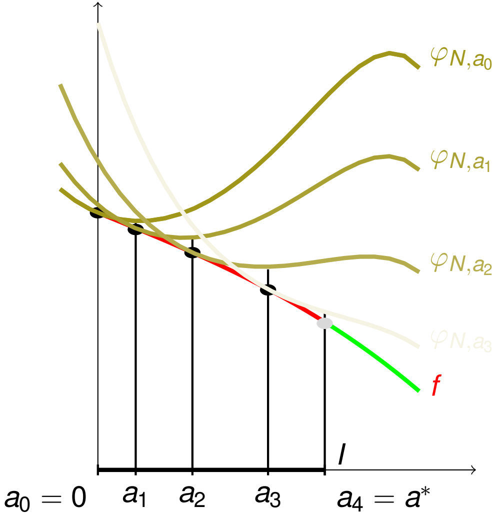

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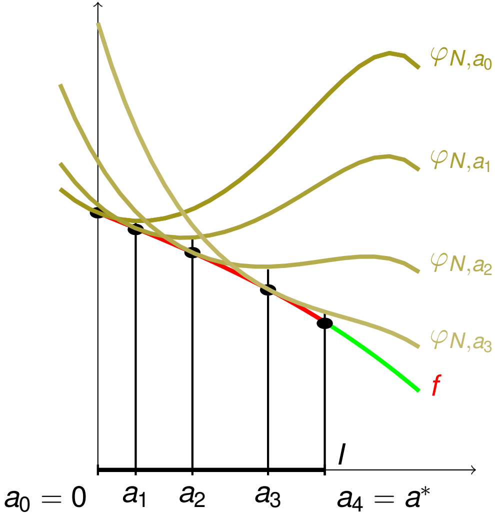

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Key points in the proof of Theorem A.

Step 1. Assuming that $a^* = \lim_{\nu\to\infty} a_\nu$ exists, we prove that $a^* \in \Sigma_X f$.

The main difficulty is to prove that $\lim_{\nu\to\infty} a_\nu$ exists.

Step 2. From the definition of $\varphi_{N,\xi}$ we obtain: for any $\nu \in \mathbb{N}$ we have

$$|a_{\nu+1} - a_\nu| = \operatorname{dist}(a_\nu, f^{-1}(f(a_{\nu+1})) \cap X).$$

Step 3. From Theorem 3 we obtain: for any $\nu \in \mathbb{N}$ we have

$$f(a_{\nu+1}) \leq \frac{f(a_\nu) - \frac{\mu}{2}|a_{\nu+1} - a_\nu|^2}{(1 + |a_{\nu+1} - a_\nu|^2)^N}.$$

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Step 4. A key point in the proof is the use of the Comparison Principle due to **D. D'Acunto** and **K. K** (2006).

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let $M \subset \mathbb{R}^n$ be a smooth bounded semialgebraic set. Let $\nabla f(x)$ denote the gradient of f with respect to the standard Euclidean scalar product, and

$\nabla_M f(x)$ - the projection of $\nabla f(x)$ on $T_x M$, the tangent space to M at x.

Let $\Gamma_M \subset \overline{M}$ be a semialgebraic curve meeting each level set of f and such that for every point $y \in \Gamma$ we have

$$|\nabla_M f(y)| \leq |\nabla_M f(x)| \quad \text{for all } x \in f^{-1}(f(y)).$$

By standard arguments (semialgebraic choice) such a curve always exists; it is called a *talweg* or *a ridge-valley line of f in X*.

Lemma (**Comparison Principle**)

For every pair of values $a < b$ taken by f, the length of any trajectory of ∇f lying in $f^{-1}((a, b)) \cap M$ is bounded by the length of $\Gamma_M \cap f^{-1}((a, b))$.

To prove that $\lim_{\nu\to\infty} a_\nu$ exists recall first that by Step 3. we have

$$f(a_\nu) \geq f(a_{\nu+1}) \geq \cdots \geq f_* := \lim_{\nu\to\infty} f(a_\nu).$$

By Proposition 5 the set $f(\Sigma_X f)$ is finite, so we may assume that either the sequence $f(a_\nu)$ is eventually constant, or

$$(f(a_\nu), f_*) \cap f(\Sigma_X f) = \emptyset \quad \text{for } \nu \text{ large enough.}$$

Clearly in the first case, by Step 3., also the sequence a_ν is eventually constant. So we assume from now on that the sequence $f(a_\nu)$ is strictly decreasing and

$$(f(a_0), f_*) \cap f(\Sigma_X f) = \emptyset.$$

The set X is semialgebraic, so there exists a stratification

$$X = \bigcup_{i \in I} M_i,$$

i.e., a finite disjoint union of connected smooth semialgebraic sets, called ***strata***.
Moreover $\overline{M}_i \setminus M_i$ is a union of some of the M_j's of dimension smaller than $\dim M_i$.

We can refine this stratification in such a way that f is of constant rank on each M_i, $i \in I$; then

our polynomial f restricted to M_i is either a constant or a submersion.

Let $I^* = \{i \in I : \operatorname{rank} f|_{M_i} = 1\}$.
Note that $C_X f = \bigcup_{i \in I \setminus I^*} f(M_i)$ is a finite set. Since the sequence $f(a_\nu)$ is strictly decreasing we may assume that

$$(f(a_0), f_*) \cap C_X f = \emptyset.$$

To each M_i, $i \in I^*$, we can associate a semialgebraic curve $\Gamma_i := \Gamma_{M_i}$ which is a talweg of f in M_i. Set

$$\Gamma := \bigcup_{i \in I^*} \Gamma_i.$$

Recall that, by Step 2.,

$a_{\nu+1}$ is the point closest to a_ν on the fiber $f^{-1}(f(a_{\nu+1})) \cap X$.

To estimate $|a_{\nu+1} - a_\nu|$ we will construct a continuous curve

$\gamma_\nu : [t_\nu, t_{\nu+1}] \to X$ such that $\gamma_\nu(t_\nu) = a_\nu$ and $f(\gamma_\nu(t_{\nu+1})) = f(a_{\nu+1})$.

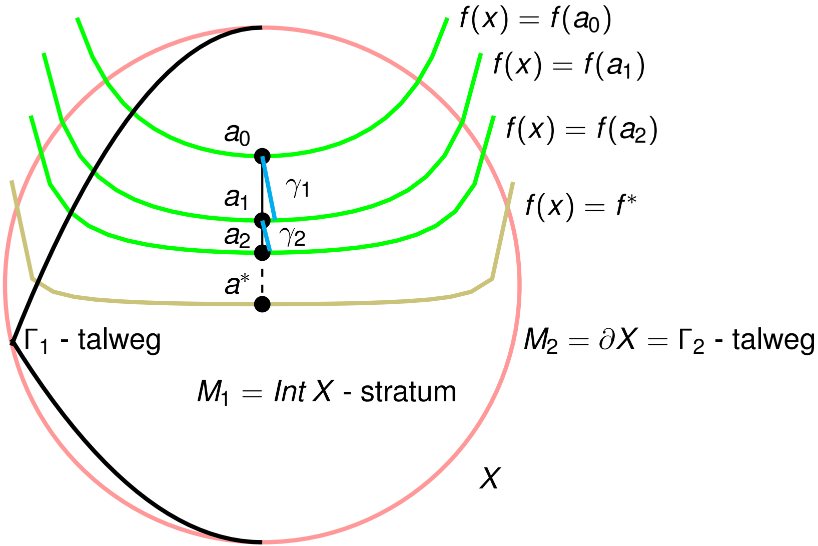
By Step 2., we will then have

$$|a_{\nu+1} - a_\nu| \leq \text{length}(\gamma_\nu).$$

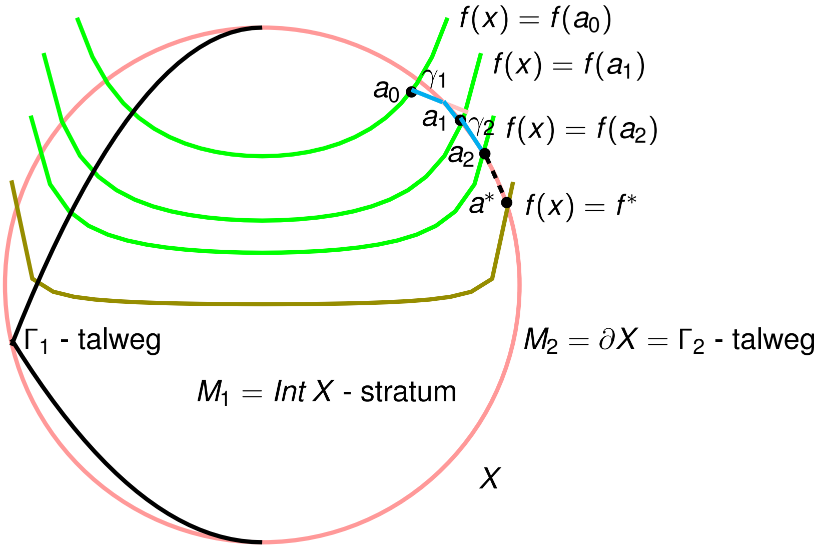
The curve γ_ν will be a piecewise trajectory of $-\nabla_{M_i} f$ (more precisely, of $-\nabla_{M_i} f / |\nabla_{M_i} f|$).

Hence, by the Comparison Principle,

$$|a_{\nu+1} - a_\nu| \leq \text{length}(\gamma_\nu) \leq \text{length}\{\Gamma \cap f^{-1}(f(a_{\nu+1}), f(a_\nu))\}.$$


$f(x) = f(a_0)$
$f(x) = f(a_1)$
$f(x) = f(a_2)$
$f(x) = f^*$
a_0
a_1
a_2
a^*
γ_1
γ_2
Γ_1 - talweg
$M_2 = \partial X = \Gamma_2$ - talweg
$M_1 = Int\, X$ - stratum
X

Simple case



More complicated case

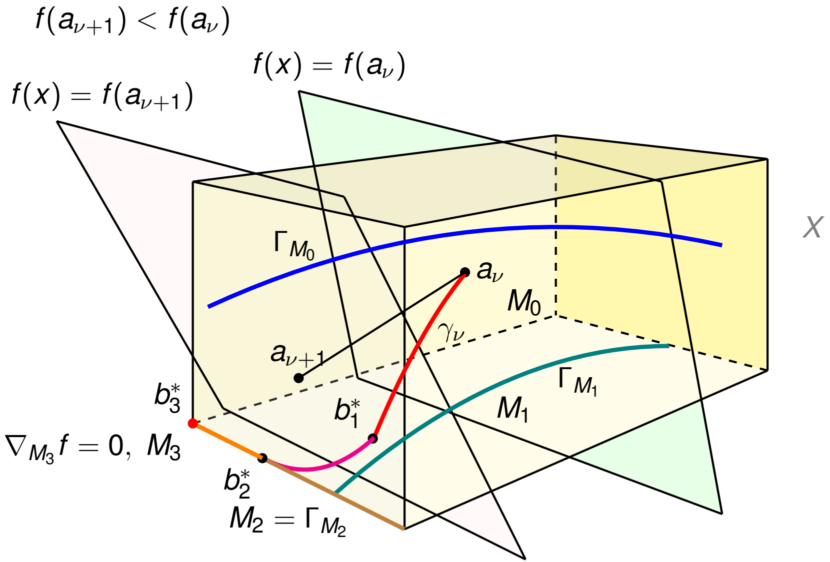
Recall thet Γ, being a bounded semialgebraic curve, has finite length; therefore

$$\sum_{\nu=0}^{\infty} |a_{\nu+1} - a_\nu| \leq \text{length}(\Gamma \cap f^{-1}(f_*, f(a_0))) < \infty.$$

So the series $\sum_{\nu=0}^{\infty} |a_{\nu+1} - a_\nu|$ is convergent, which implies that $a^* = \lim_{\nu\to\infty} a_\nu$ exists.

To complete the proof it is sufficient to construct the curves γ_ν.

We obtain this by using of Comparison Principle.



3-dimensional case

Let f be an analytic function in an nghb. of $\overline{U}$ where $U \subset \mathbb{R}^n$ is open and bounded. Let $\gamma(t)$ be a trajectory of ∇f starting at some point of U. By the Łojasiewicz gradient inequality either $\gamma(t)$ leaves U or it has a limit $\gamma^* = \lim_{t\to\infty} \gamma(t) \in U$. Clearly $\nabla(\gamma^*) = 0$.

Gradient Conjecture of R. Thom 70's

$$\lim_{\nu\to\infty} \frac{\gamma^* - \gamma(t)}{|\gamma^* - \gamma(t)|}$$

exists.

Answered affirmatively by KK, T. Mostowski, A. Parusiński in 2000.

Discrete Thom's Gradient Conjecture

Let $X \subset \mathbb{R}^n$ be a compact convex semialgebraic set and $f : \mathbb{R}^n \to \mathbb{R}$ a polynomial positive on X. Choose an arbitrary point $a_0 \in X$, and by induction set

$$a_\nu := \operatorname{argmin}_X \varphi_{N,a_{\nu-1}}.$$

We have proved that

$$a^* = \lim_{\nu\to\infty} a_\nu$$

exists and $a^* \in \Sigma_X f$.

Conjecture

$$\lim_{\nu\to\infty} \frac{a^* - a_\nu}{|a^* - a_\nu|}$$

exists.

There is some numerical evidence supporting this conjecture.

Happy Birthday Bernard

Sto Lat !!!

we wish you (at least) 100 years