Some questions on minimal log discrepancies

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Setting

Let X be our ambient variety (assumed smooth or with mild singularities). We work over $k = \overline{k}$, with $\mathrm{char}(k) = 0$. Assume $\dim(X) \geq 2$.

Let $Z \subset X$ be defined by some $I_Z \subset \mathcal{O}_X$. We want to study the singularities of the pair (X, qZ), where $q \in \mathbf{Q}_{\geq 0}$ (or $q \in \mathbf{R}_{\geq 0}$).

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The invariant that we will discuss is obtained by considering all divisorial valuations of $k(X)$. Such a valuation corresponds to a prime divisor $E \subset Y$, where Y normal, with a birational morphism $Y \xrightarrow{f} X$. Get valuation ord_E on $k(X) = k(Y)$. Its center is $c_X(E) := \overline{f(E)}$.

Example. If $x \in X$ smooth point and E is the exceptional divisor on $\mathrm{Bl}_x(X)$, then $\mathrm{ord}_E = \mathrm{ord}_x$, where $\mathrm{ord}_x(f) = \max\{j \mid f \in I_x^j\}$.

The log discrepancy

Given a pair (X, qZ) as above and ord_E, one measures the singularities of the pair with respect to this valuation by considering

$$q \cdot \mathrm{ord}_E(Z) := q \cdot \min\{\mathrm{ord}_E(h) \mid h \in I_Z\}.$$

These numbers have to be normalized. This is done using the log discrepancy function.

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Suppose that X is smooth and E is a prime divisor on Y, with $f\colon Y \to X$ birational. May assume Y is smooth (replace Y by Y_{sm}). We have a morphism of vector bundles

$$f^*(\Omega_X) \to \Omega_Y$$

of the same rank, which drops rank along a divisor (defined by the determinant of the map), denoted by $K_{Y/X}$.
The log discrepancy of ord_E (with respect to X) is

$$A_X(\mathrm{ord}_E) := \mathrm{ord}_E(K_{Y/X}) + 1.$$

The log discrepancy, cont'd

The log discrepancy measures "how far the divisor lies over X".

Example. If $W \hookrightarrow X$ smooth subvariety and E is the exceptional divisor on $\mathrm{Bl}_W(X)$, then $A_X(\mathrm{ord}_E) = \mathrm{codim}_X(W)$.

Remark. If $Y' \to Y \to X$ are birational morphisms of smooth varieties and E is a prime divisor on Y', then

$$A_X(\mathrm{ord}_E) = A_Y(\mathrm{ord}_E) + \mathrm{ord}_E(K_{Y/X}).$$

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$$A_X(\mathrm{ord}_E) = A_Y(\mathrm{ord}_E) + \mathrm{ord}_E(K_{Y/X}).$$

Since X is smooth, every divisor E can be obtained by a sequence of smooth blow-ups. By the example and remark, $A_X(\mathrm{ord}_E)$ is determined by the codimensions of the centers and by which proper transforms of previous exceptional divisors contain the center at each step.

Remarks on the singular case

When X is singular, there are several possible definitions. In birational geometry the most useful is the following:

Assume that X is normal, hence K_X makes sense as a Weil divisor. Assume also that X is $\mathbf{Q}$-Gorenstein, that is, some mK_X is Cartier, $m \geq 1$.

One can still define $K_{Y/X}$ when $f\colon Y \to X$ is birational, with Y smooth, as the unique divisor supported on $\mathrm{Exc}(f)$, linearly equivalent to $K_Y - f^*(K_X)$ (note: it might not be effective). Then define $A_X(\mathrm{ord}_E)$ as before.

However: meaning is somewhat more subtle.

Minimal log discrepancy

Fix a pair (X, qZ) and a closed subset $W \subseteq X$ (most of the time will take W to be a point). The minimal log discrepancy of (X, qZ) with respect to W is

$$\mathrm{mld}_W(X, qZ) := \inf_{c_X(E) \subseteq W} \{A_X(\mathrm{ord}_E) - q \cdot \mathrm{ord}_E(Z)\}.$$

Note: "good singularities" of $(X, qZ) \leftrightarrow$ small $\mathrm{ord}_E(Z) \leftrightarrow$ large mld.

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Basic facts:

• If $\mathrm{mld}_W(X, qZ) < 0$, then $\mathrm{mld}_W(X, qZ) = -\infty$. Otherwise, one says that (X, qZ) is log canonical (in a neighborhood of W).

• If (X, qZ) log canonical and $f\colon Y \to X$ is a monomialization of I_Z such that $f^{-1}(W)$ is a divisor, then the infimum is a minimum over the divisors on Y.

Say: a divisor E computes $\mathrm{mld}_W(X, qZ)$ if $c_X(E) \subseteq W$ and

$$\mathrm{mld}_W(X, qZ) = A_X(\mathrm{ord}_E) - q \cdot \mathrm{ord}_E(Z).$$

Comparison with the log canonical threshold

It is instructive to compare $\mathrm{mld}_W(X, qZ)$ to another invariant of the pair (X, Z), the log canonical threshold $\mathrm{lct}(X, Z)$. This is defined as

$$\mathrm{lct}(X, Z) := \max\{t \geq 0 \mid (X, tZ) \text{ is log canonical}\}$$
$$= \min_E \frac{A_X(\mathrm{ord}_E)}{\mathrm{ord}_E(Z)}.$$

In spite of the similarity in the definition, the minimal log discrepancy turns out to be a much more subtle invariant than the log canonical threshold. In particular, the analogues of the questions we will see later are now well-understood for log canonical thresholds.

This difference is analogous to that between linear programming and integer programming.

Example 1: the monomial case

Let $Z \hookrightarrow X = \mathbf{A}^n$ defined by monomial ideal I. Let P_I be the Newton polyhedron

$$P_I = \mathrm{conv}\{u \in \mathbf{Z}^n_{\geq 0} \mid x^u \in I\}.$$

If E_u is the toric divisor corresponding to $u \in \mathbf{Z}^n_{\geq 0}$ (primitive), then $A_X(\mathrm{ord}_{E_u}) = u_1 + \ldots + u_n$.

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It follows that $(\mathbf{A}^n, qZ)$ is log canonical if and only if $(1, \ldots, 1) \in q \cdot P_I$. For such q, we have

$$\mathrm{mld}_0(\mathbf{A}^n, qZ) = \min\{u_1 + \ldots + u_n - q \cdot \min_{v \in P_I} \langle u, v \rangle\} \mid u \in \mathbf{Z}^n_{>0}\}.$$

Example 2: a cone with isolated singularities

Let $Z = V(f) \hookrightarrow \mathbf{A}^n$, with f homogeneous, such that Z has an isolated singularity at 0.

$\mathrm{Bl}_0\mathbf{A}^n \to \mathbf{A}^n$ is a monomialization of f, hence

$$(\mathbf{A}^n, qZ) \text{ is log canonical iff } q \leq \min\{1, n/d\}.$$

If this is the case, then

$$\mathrm{mld}_0(\mathbf{A}^n, qZ) = n - qd.$$

Two open questions

Semicontinuity conjecture (Ambro). Given (X, qZ), the function $X \ni x \to \mathrm{mld}_x(X, qZ)$ is lower semicontinuous, that is, all sets $\{x \in X \mid \mathrm{mld}_x(X, qZ) \geq t\}$ are open.

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ACC conjecture (Shokurov). Fix a DCC subset Γ of $\mathbf{R}_{\geq 0}$. If n is fixed, then the set

$$\{\mathrm{mld}_x(X, qZ) \mid x \in X, \dim(X) \leq n, q \in \Gamma\}$$

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The interest in these comes from the following
Theorem (Shokurov). If the above conjectures hold, then we have Termination of Flips in the Minimal Model Program.

A valuation-theoretic version of semicontinuity

Using Ambro's work, one can reduce the Semicontinuity Conjecture to the following purely valuation-theoretic conjecture. Here a divisorial valuation is a valuation of the form $q \cdot \mathrm{ord}_E$, with $q \in \mathbf{Z}_{>0}$.

Semicontinuity conjecture, strong version. Given X affine and two closed irreducible subsets $T_1 \subseteq T_2 \subsetneq X$, if v_2 is a divisorial valuation of $k(X)$ with $c_X(v_2) = T_2$, then there is a divisorial valuation v_1 of $k(X)$ such that:

i) $c_X(v_1) = T_1$.

ii) $v_1(f) \geq v_2(f)$ for every $f \in \mathcal{O}(X)$.

iii) $A_X(v_1) \leq A_X(v_2) + \mathrm{codim}_{T_2}(T_1)$.

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iii) $A_X(v_1) \leq A_X(v_2) + \mathrm{codim}_{T_2}(T_1)$.

This is known when

- X is toric (Ambro)
- X is smooth (Ein, M-, Yasuda)
- $\dim(X) = 2$ (in general, can reduce to X having terminal singularities)

Outline of proof in the smooth case

The proof uses the description of divisorial valuations on smooth varieties via arcs. For the sake of concreteness, say $X = \mathbf{A}^n$. Consider the space of arcs of X:

$$X_\infty = \mathrm{Hom}(\mathrm{Spec}\, k[[t]], X) = k[[t]]^n.$$

A cylinder C in X_∞ is the inverse image of a locally closed subset S of $(k[[t]]/(t^{m+1}))^n$ via the obvious projection map. We put

$$\mathrm{codim}_{X_\infty} C := \mathrm{codim}_{X_m} S.$$

For every $C \subseteq X_\infty$ and $f \in \mathcal{O}(X)$, we put

$$\mathrm{ord}_C(f) := \min_{\gamma \in C} \mathrm{ord}_t \gamma^*(f) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}.$$

The following result gives an approach to divisorial valuations via cylinders in X_∞.

Outline of proof in the smooth case, cont'd

Theorem (Ein, M-, Lazarsfeld). Let X be a smooth variety.

1) If C is an irreducible, closed cylinder in X_∞, then ord_C extends to a divisorial valuation of $k(X)$, whose center is the closure of the image of C in X.

2) Given any divisorial valuation v of $k(X)$, there is a unique maximal irreducible closed cylinder $C(v)$ such that $v = \mathrm{ord}_{C(v)}$.

3) We have $\mathrm{codim}_{X_\infty} C(v) = A_X(v)$.

What is $C(v)$: say $v = q \cdot \mathrm{ord}_E$, where E smooth divisor on smooth Y, with $f\colon Y \to X$ birational. We have

$$C(v) = \overline{f_\infty(\mathrm{Cont}^{\geq m}(E))}.$$

Outline of proof in the smooth case, cont'd

Goal: explain how the previous result implies the valuation-theoretic version of semicontinuity when X is smooth.
Recall that v_2 is a divisorial valuation with center T_2 and T_1 is an irreducible closed subset of T_2. Let $\pi\colon X_\infty \to X$ be the canonical projection and consider an irreducible component

$$C \subseteq C(v_2) \cap \pi^{-1}(T_1)$$

that dominates T_1. Then $v_1 = \mathrm{ord}_C$ satisfies our requirements:

- C dominates T_1 implies $c_X(v_1) = T_1$.
- $C \subseteq C(v_2)$ implies $v_1 = \mathrm{ord}_C \geq \mathrm{ord}_{C(v_2)} = v_2$.
- $A_X(v_1) = \mathrm{codim}_{X_\infty}(C(v_1)) \leq \mathrm{codim}_{X_\infty}(C) \leq$ $\mathrm{codim}_{X_\infty}(C(v_2)) + \mathrm{codim}_{T_2}(T_1) = A_X(v_2) + \mathrm{codim}_{T_2}(T_1)$.

A boundedness question on divisors computing mld's

We now turn to the ACC conjecture. We will be interested in a special case, when the ambient variety is fixed (this is sufficient, for example, if we are only interested in the case of ambient smooth varieties). What follows is joint work with Yusuke Nakamura.

The following question is motivated by the above problem:
Question (Nakamura). Let X and $x \in X$ be fixed, and let also $q \in \mathbf{R}_{>0}$ be fixed. Is there a fixed $M > 0$ such that for every subscheme $Z \hookrightarrow X$ with (X, qZ) log canonical, there is a divisor E over X such that

a) E computes $\mathrm{mld}_x(X, qZ)$, and
b) $A_X(\mathrm{ord}_E) \leq M$?

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a) E computes $\mathrm{mld}_x(X, qZ)$, and
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Remark. It is easy to see that if this has a positive answer, then for every DCC subset $\Gamma \subseteq \mathbf{R}_{\geq 0}$, the set

$$\{\mathrm{mld}_x(X, qZ) \mid q \in \Gamma, Z \hookrightarrow X\}$$

satisfies ACC.

A partial result

Theorem (M-, Nakamura). If X, $x \in X$, and $q \in \mathbf{R}_{>0}$ are fixed, then there a fixed $M > 0$ such that for every subscheme $Z \hookrightarrow X$ with (X, qZ) log canonical, there is a divisor E over X such that

a) E computes $\mathrm{mld}_x(X, qZ)$.

b) $\mathrm{ord}_E(I_x) \leq M$.

A partial result

Theorem (M-, Nakamura). If X, $x \in X$, and $q \in \mathbf{R}_{>0}$ are fixed, then there a fixed $M > 0$ such that for every subscheme $Z \hookrightarrow X$ with (X, qZ) log canonical, there is a divisor E over X such that

a) E computes $\mathrm{mld}_x(X, qZ)$.

b) $\mathrm{ord}_E(I_x) \leq M$.

Remark 1: Suppose that X is smooth and we describe E via a sequence of smooth blow-ups. To give a positive answer to the question: need to bound the length of this sequence. The theorem says that all but a bounded number of the blow-ups in the sequence are "free blow-ups" (the center is contained in only one proper transform of the previous exc. divisors).

Remark 2: $A_X(\mathrm{ord}_{E_m}) \geq \mathrm{ord}_{E_m}(I_x) \cdot \mathrm{lct}(X, \{x\})$, hence the assertion in the theorem would follow from a positive answer to the question.

Sketch of proof

The proof uses the generic limit construction (de Fernex, M-; Kollár).

Say we have a sequence of subschemes $Z_m \hookrightarrow X$ such that each (X, qZ_m) is log canonical and no matter how we choose divisors E_m computing $\mathrm{mld}_x(X, qZ_m)$, we have $\lim_{m\to\infty} \mathrm{ord}_{E_m}(I_x) = \infty$.
For simplicity: say $X = \mathbf{A}^n_k$, $x = 0$, and Z_m is defined by $f_m \in k[x_1, \ldots, x_n]$.

By "putting together the coefficients of the f_m", we get a formal power series $f \in K[[x_1, \ldots, x_n]]$, where K/k is a suitable field extension (obtained by a non-standard extension). This has the property that after passing to a subsequence, if $X' = \mathrm{Spec}\, K[[x_1, \ldots, x_n]]$ and Z' is defined by f, then we may assume

$$\mathrm{lct}_0(X', qZ') = \lim_{m\to\infty} \mathrm{lct}_0(X, qZ_m).$$

By assumption $q \leq \mathrm{lct}_0(X, Z_m)$ for every $m \geq 1$, hence $q \leq \mathrm{lct}_0(X', Z')$. There are two cases to consider.

Sketch of proof, cont'd

Case 1: $\mathrm{mld}_0(X', qZ') = 0$. If E' is a divisor over $0 \in X'$ computing this mld, after passing to a subsequence, this comes from a sequence of divisors E_m over $0 \in X$ such that

$$A_X(\mathrm{ord}_{E_m}) = A_{X'}(\mathrm{ord}_{E'})$$

$$\mathrm{ord}_{E_m}(I_0) = \mathrm{ord}_{E'}(I_0) \ \text{ and } \ \mathrm{ord}_{E_m}(f_m) = \mathrm{ord}_{E'}(f').$$

This is non-trivial: it uses the adic semicontinuity property of log canonical thresholds. One sees that in this case each E_m computes $\mathrm{mld}_0(X, qZ_m) = 0$, while $\{\mathrm{ord}_{E_m}(f_m)\}$ is bounded, a contradiction.

Sketch of proof, cont'd

Case 2: $\mathrm{mld}_0(X', qZ') > 0$. In this case there is $\delta > 0$ such that $\mathrm{lct}_0(X', qZ' + \delta\{0\}) = 1$.

Key point: after passing to a subsequence, we may assume that

$$\mathrm{lct}_0(X, qZ_m + \delta\{0\}) \geq 1 \text{ for all } m.$$

Indeed, we may assume that

$$\lim_{m\to\infty} \mathrm{lct}_0(X, qZ_m + \delta\{0\}) = 1$$

and the assertion follows from ACC for log canonical thresholds.

Sketch of proof, cont'd

Let's choose now for each m a divisor E_m that computes $\mathrm{mld}_0(X, qZ_m)$. By the key point, we have

$$A_X(\mathrm{ord}_{E_m}) \geq q \cdot \mathrm{ord}_{E_m}(Z_m) + \delta \cdot \mathrm{ord}_{E_m}(I_0).$$

By choice of E_m, we have

$$A_X(\mathrm{ord}_{E_m}) - q \cdot \mathrm{ord}_{E_m}(Z_m) = \mathrm{mld}_0(X, qZ_m).$$

Combining these, we get

$$\mathrm{ord}_{E_m}(I_0) \leq \frac{\mathrm{mld}_0(X, qZ_m)}{\delta} \leq \frac{\mathrm{mld}_0(X)}{\delta}.$$

The right-hand side is independent of m and this completes the proof of the theorem.

The boundedness question: the monomial case

Nakamura's question has a positive answer for monomial subschemes of $X = \mathbf{A}^n$:
Suppose that $Z_m = V(I_m)$, with $m \geq 1$, are such that each $I_m \subseteq k[x_1, \ldots, x_n]$ is a monomial ideal with $(\mathbf{A}^n, qZ_m)$ log canonical such that no matter how we choose E_m that computes $\mathrm{mld}_0(\mathbf{A}^n, qZ_m)$, we have $\lim_{m\to\infty} A_X(\mathrm{ord}_{E_m}) = \infty$. For simplicity, assume $q \in \mathbf{Q}$. One can easily reduce to the case when all I_m are $(x_1, \ldots, x_n)$-primary

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By a result of Maclagan, after restricting to a subset and reordering, we may assume that

$$I_1 \supseteq I_2 \supseteq \ldots, \quad \text{hence}$$

$$\mathrm{mld}_0(\mathbf{A}^n, qZ_1) \geq \mathrm{mld}_0(\mathbf{A}^n, qZ_2) \geq \ldots$$

All these mld's lie in $\frac{1}{r}\mathbf{Z}_{\geq 0}$ for some integer $r \geq 1$, hence they stabilize for $m \geq m_0$.
We now see that if E computes $\mathrm{mld}_0(\mathbf{A}^n, qZ_{m_0})$, it also computes $\mathrm{mld}_0(\mathbf{A}^n, qZ_m)$ for $m \geq m_0$, a contradiction.

The boundedness question: the surface case

Nakamura's question has a positive answer when X is a smooth surface:

Suppose that E is a divisor computing $\mathrm{mld}_x(X, qZ)$ with $A_X(\mathrm{ord}_E)$ minimal. Say $Z = Z(f)$. Consider the corresponding sequence of point blow-ups:

$$X_N \xrightarrow{\pi_N} X_{N-1} \ldots \longrightarrow X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X,$$

with $X_{i+1} = \mathrm{Bl}_{x_i}(X_i)$, with exceptional divisor E_i, where $x_i = c_{X_i}(E)$. We have $E = E_N$ and need to bound N in terms of q.

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For every i with $1 \leq i \leq N-1$, we put $p_i = q \cdot \mathrm{ord}_{x_i}(\widetilde{Z})$, where $\widetilde{Z}$ stands for the strict transform of Z. Note that we have

$$2 \geq p_0 \geq p_1 \geq \ldots \geq p_{N-1} > 0,$$

where the first inequality comes from the fact that (X, qZ) is log canonical. The p_i lie in a discrete set only depending on q.

The boundedness question: the surface case ,cont'd

Let $\tau_i := A_X(\mathrm{ord}_{E_i}) - q \cdot \mathrm{ord}_{E_i}(Z)$. For each blow-up $X_{i+1} \to X_i$ we have two cases:

Case 1: x_i only lies on E_i ("free blow-up"). In this case

$$\tau_{i+1} = \tau_i + 1 - p_i.$$

Case 2. x_i lies on E_i and on the strict transform of E_j, with $j < i$. In this case

$$\tau_{i+1} = \tau_i + \tau_j - p_i.$$

By the theorem, the number of blow-ups in Case 2 is bounded. Therefore we only need to show that also the number of blow-ups in Case 1 is bounded.

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Important point: if $p_i < 1$ and we are in Case 1, then $\tau_{i+1} > \tau_i$. In fact, there is $\epsilon > 0$ (only depending on q), such that if this is the case, then $\tau_{i+1} > \tau_i + \epsilon$.

The boundedness question: the surface case ,cont'd

(1) One can show that if $p_i = p_{i+1} = 1$, then $\mathrm{mld}_x(X, qZ)$ is computed by E_{i+1}, hence $i = N - 1$.

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(1) One can show that if $p_i = p_{i+1} = 1$, then $\mathrm{mld}_x(X, qZ)$ is computed by E_{i+1}, hence $i = N - 1$.

(2) Using the fact that (X, qZ) is log canonical, we can bound the number of $p_i > 1$, hence the number of those ≥ 1. Note that if $X_{i+1} \to X_i$ is in Case 1, then we still have $\tau_{i+1} - \tau_i \geq -1$.

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(1) One can show that if $p_i = p_{i+1} = 1$, then $\mathrm{mld}_x(X, qZ)$ is computed by E_{i+1}, hence $i = N - 1$.

(2) Using the fact that (X, qZ) is log canonical, we can bound the number of $p_i > 1$, hence the number of those ≥ 1. Note that if $X_{i+1} \to X_i$ is in Case 1, then we still have $\tau_{i+1} - \tau_i \geq -1$.

(3) Note also that in Case 2, we always have $\tau_{i+1} - \tau_i \geq -p_i \geq -2$. Since we have only finitely many steps in Case 2 or with $p_i \geq 1$ and otherwise $\tau_{i+1} > \tau_i + \epsilon$, and since

$$\tau_N = \mathrm{mld}_x(X, qZ) \leq \mathrm{mld}_x(X) = 2,$$

we conclude that N is bounded.