Complete vector fields on affine surfaces

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In 2004 Brunella published the remarkable classification of complete algebraic vector fields (CAVFs) on $\mathbb{C}^2$. The proof for this classification is roughly divided into two parts. Using the classification of foliations on complete algebraic surfaces developed by himself, McQuillan and others, Brunella showed that for every CAVF there is an algebraic fibration (with $\mathbb{C}$- or $\mathbb{C}^*$-fibers) that is preserved by the flow of the vector field. Brunella's second step was to use the description of such fibrations by Suzuki in order to give the explicit form of the CAVFs.

In this talk we present a generalization of Brunella's first step to any normal affine algebraic surface, namely we prove that for every CAVF there is a fibration that is preserved by the flow of the vector field. As an application of this result we are able to give a classification of affine algebraic surfaces where the flows of CAVF act quasi-transitively. This is generalizing the famous work of Gizatullin for purely algebraic automorphisms. It is worth mentioning that among these surfaces there are many surfaces with only few algebraic automorphisms, showing that there can be a huge difference between algebraic and holomorphic transitivity of the action by the respective automorphism group.

Moreover given a specific affine surface an explicit classification of $\mathbb{C}$- and $\mathbb{C}^*$-fibrations is needed in order to fulfill the second part in Brunella's proof and get the explicit list of the CAVFs. This can and has been done for several families of surfaces. For instance the list of CAVFs on affine toric surfaces has been done and the result shows that there is an integer n depending only on the surface such that all CAVFs have a zero of order at least n in the fixed point of the torus action. More families are in progress.

This is a joint work with many authors, prominently with Kaliman and Kutzschebauch.