A cubature based algorithm to solve forward and forward-backward stochastic differential equation of McKean-Vlasov type

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8 juillet 2014

FBSDE of Mckean-Vlasov type (weakly coupled)

Let $T > 0$, we consider, on $[0, T]$, the system

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- System of M players

$$\begin{aligned} \mathrm{d}\tilde{X}_t^1 &= \tilde{b}(t, \tilde{X}_t^1, \mu_t^M)\mathrm{d}t + \tilde{\sigma}(t, \tilde{X}_t^1, \mu_t^M)\mathrm{d}B_t^1 \\ &\vdots \\ \mathrm{d}\tilde{X}_t^M &= \tilde{b}(t, \tilde{X}_t^M, \mu_t^M)\mathrm{d}t + \tilde{\sigma}(t, \tilde{X}_t^M, \mu_t^M)\mathrm{d}B_t^M \end{aligned}$$

where $\mu_t^M = \frac{1}{M}\sum_{j=1}^{M} \delta_{\tilde{X}_t^j}$

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- System of M players + one *marked player*

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$$\mathrm{d}\bar{X}_t^\alpha = b'(t, \bar{X}_t^\alpha, \mu_t^M, \alpha_t)\mathrm{d}t + \sigma'(t, \bar{X}_t^\alpha, \mu_t^M)\mathrm{d}\bar{B}_t, \quad \bar{X}_0^\alpha = \bar{x}$$

where $\mu_t^M = \frac{1}{M}\sum_{j=1}^M \delta_{\tilde{X}_t^j}$, α a control

- Cost functional : $\inf_{\alpha\in\mathcal{A}} J(t, \bar{x}, \alpha) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\phi(\bar{X}_T^{\alpha;t,\bar{x}}, \mu_T^M) + \int_0^T h(s, \bar{X}_s^{\alpha;t,\bar{x}}, \mu_t^M)ds\right]$

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- **Problem could be hard to solve for large M**
  ↳ Asymptotic approximation ? (Lasry, Lions)

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where $\mu_t^M = \frac{1}{M}\sum_{j=1}^{M} \delta_{\tilde{X}_t^j}$, α a control

- $M \to +\infty \quad \xrightarrow{\text{law of large number}} \mu^M \to \mu, \qquad X = (\tilde{X}^1, \bar{X})^*$
- Cost functional : $\inf_{\alpha \in \mathcal{A}} J(t, \bar{x}, \alpha) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\phi(\bar{X}_T^{\alpha;t,\bar{x}}, \mu_T) + \int_0^T h(s, \bar{X}_s^{\alpha;t,\bar{x}}, \mu_t)ds\right]$
- Solution of a HJB :
$(\partial_t + \mathcal{L})u(t,x) = H(t, x, u(t,x), \sigma(x, \mu_t)D_x u(t,x), \mu_t), \quad H = \inf_\alpha\{b' + h\}$

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- Control problem of a marked player in a mean field environment
- We have $Y_t = u(t, X_t)$ where u is the solution of the PDE :

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- Objective : approximate μ_T et u

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↳ Approximation based on cubature methods

Cubature on Wiener space

↳ *Lyons, Victoir (2002)*

- $T > 0$, $(\mathcal{C}([0, T], \mathbb{R}), \mathbb{P})$, Wiener space.
- Approximate the Wiener measure $\mathbb{P}$ with a discrete measure $\mathbb{Q}$ with finite support on the set of continuous functions with bounded variation such that :

$$(\mathbb{E}_{\mathbb{P}} - \mathbb{E}_{\mathbb{Q}}) \left[\int_{0=t_0<t_1<\ldots<t_l=T} \circ \mathrm{d}B_{t_1} \cdots \circ \mathrm{d}B_{t_l} \right] = 0,$$

  for all $l \leqslant m$, $m \in \mathbb{N}$ given.

- Existence : "Tchakaloff theorem".
- For all smooth functional F :

$$(\mathbb{E}_{\mathbb{P}} - \mathbb{E}_{\mathbb{Q}})[F(B_T)] \leqslant CT^{(m+1)/2} \sup_{j \leqslant m+2} ||\nabla_x^j F||_\infty .$$

- The SDE is replaced by a weighted system of ODE.

Framework

Let $T > 0$, we consider on $[0, T]$ the following one-dimensional system :

$$\begin{cases} dX_t^x = V(X_t^x, \mu_t) \circ \mathrm{d}B_t \\ dY_t^x = -f(t, X_t^x, Y_t^x, Z_t^x, \mu_t^{X,Y})\mathrm{d}t + Z_t^x \mathrm{d}B_t \\ X_0^x = x, \quad Y_T^x = \phi(X_T^x), \end{cases}$$

- Approximation based on cubature method :
  The coefficients V and f are smooth in space, and at least Lipschitz continuous w.r.t. the measure

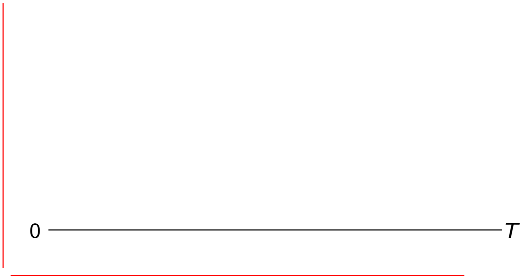
- Two main steps :
  1. build a cubature tree $\mathcal{T}(m)$ (m is the cubature order) approaching the law of the forward component
  2. go backward the tree to compute the values of the process (Y, Z)

Algorithm, first step : built the cubature tree $\mathcal{T}(m)$

Initialisation :

- Interval $[0, T]$,

example : order 3 cubature, $N = 3$.



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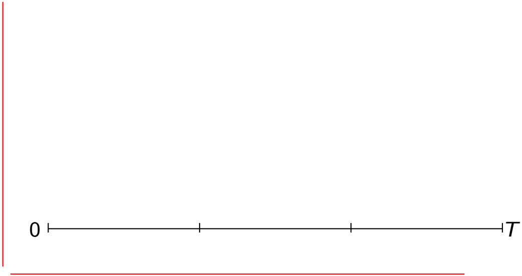

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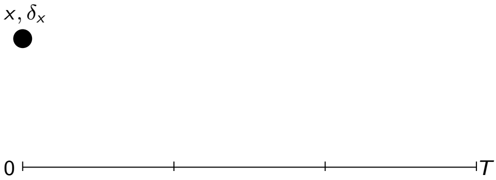

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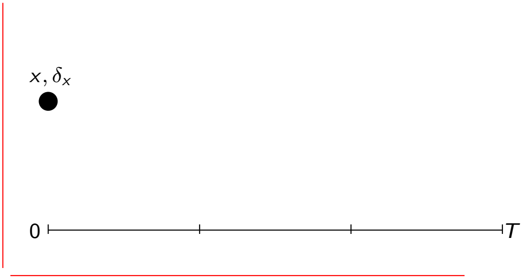

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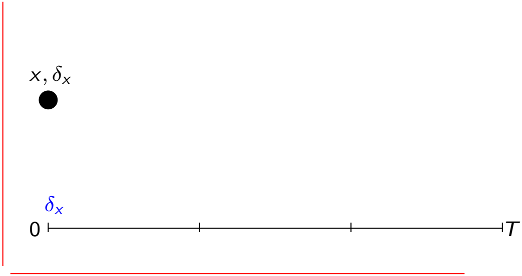

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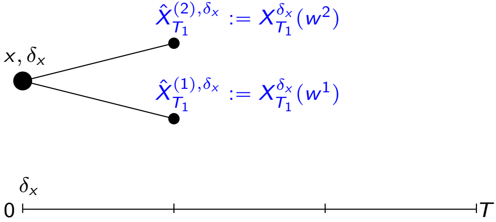

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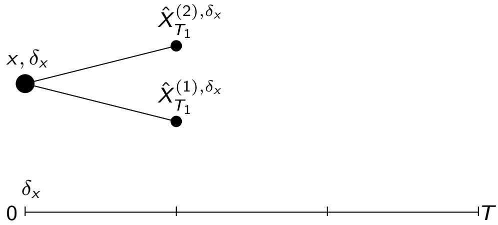

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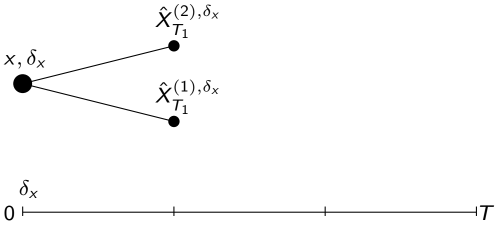

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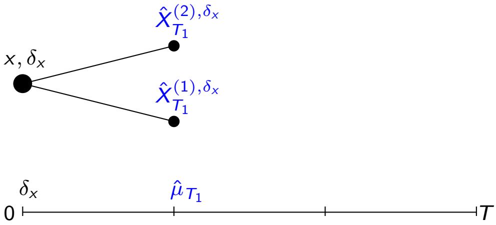

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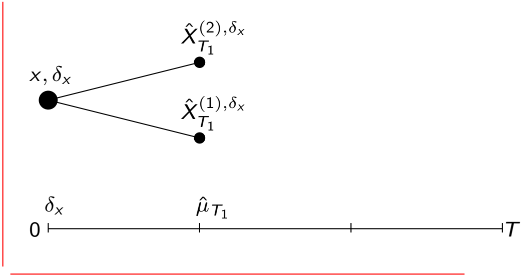

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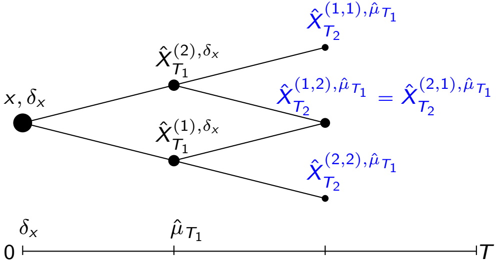

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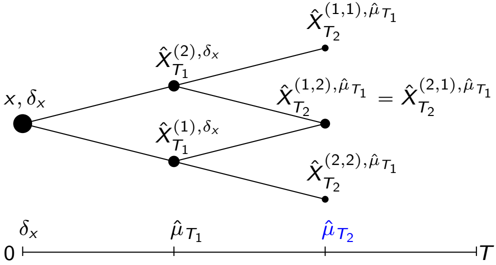

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- $\hat{\mu}_{T_2} = \sum_{i,j=1}^{n} \lambda_i \lambda_j \delta_{\hat{X}_{T_2}^{(i,j),\hat{\mu}}}$

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step $[T_k, T_{k+1}[$

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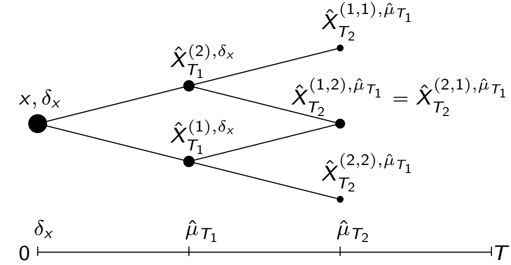

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- $\hat{\mu}_{T_k} = \displaystyle\sum_{\pi \in \{1,\cdots,n\}^k}^{n} \left(\prod_{i_j \in \pi} \lambda_{i_j} \right) \delta_{\hat{X}_{T_k}^{\pi,\hat{\mu}}}$

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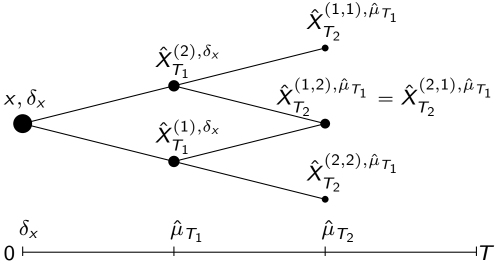
step $[T_k, T_{k+1}[$

example : order 3 cubature, $N = 3$.



- Interval $[0, T]$, $N > 0$
- $T_0 = 0 < \ldots < T_N = T$
- Cubature of order m :

$$\{w_t^1, \cdots, w_t^n\}, \ \{\lambda_1, \ldots, \lambda_n\}$$

- True dynamic : $dX_t = V(X_t, \mu_t) \circ \mathrm{d}B_t$
- Approximation : $d\hat{X}_t^{(\pi,j)} = V(\hat{X}_t^{(\pi,j)}, \hat{\mu}_{T_k})\mathrm{d}w_t^j, \quad \hat{X}_{T_1}^{(\pi,j)} = \hat{X}_{T_1}^{\pi,\delta_x} \qquad j = 1, \ldots, n, \quad \pi \in \{1, \cdots, n\}^k$
- $$\hat{\mu}_{T_k} = \sum_{\pi\in\{1,\cdots,n\}^k}^{n} \left(\prod_{i_j\in\pi} \lambda_{i_j}\right) \delta_{\hat{X}_{T_k}^{\pi,\hat{\mu}}}$$

Error (weak) $\mathcal{T}$

- Objective : control $|\langle \mu_T - \hat{\mu}_T, \phi \rangle|$
  - Markov property : **global error = sum of local errors**
- Local error $[T_{k-1}, T_k]$: $E_{T_k} = |\langle \mu_{T_k} - \hat{\mu}_{T_k}, \psi \rangle|$, ψ smooth function.

$$\begin{array}{lll} \text{True dynamic :} & dX_t = V(X_t, \mu_t) \circ \mathrm{d}B_t & X_{T_{k-1}} = y \\ \text{Approximation :} & d\hat{X}_t^{(j),\hat{\mu}} = V(\hat{X}_t^{(j),\hat{\mu}}, \hat{\mu}_{T_k}) \mathrm{d}w_t^j, \quad j = 1, \ldots, n, & \hat{X}_{T_{k-1}}^{(j),\hat{\mu}} = y \end{array}$$

$E_{T_k} \leqslant$?

Error (weak) $\mathcal{T}$

- Objective : control $|\langle \mu_T - \hat{\mu}_T, \phi \rangle|$
  - Markov property : **global error = sum of local errors**
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$$E_{T_k} = \text{Euler error} \approx \Delta_{T_k}^{3/2} + \text{propagation error} \approx \Delta_{T_k} E_{T_{k-1}} + \text{cubature error} \approx \Delta_{T_k}^{(m+1)/2}$$

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Approximation : $d\hat{X}_t^{(j),\hat{\mu}} = V(\hat{X}_t^{(j),\hat{\mu},}, \hat{\mu}_{T_k})\mathrm{d}w_t^j, \quad j = 1, \ldots, n, \quad \hat{X}_{T_{k-1}}^{(j),\hat{\mu}} = y$

E_{T_k}= Euler error $\approx \Delta_{T_k}^{3/2}$ + propagation error $\approx \Delta_{T_k} E_{T_{k-1}}$ + cubature error $\approx \Delta_{T_k}^{(m+1)/2}$

Convergence rate : Let $\mathcal{T}(m)$ a tree and $T_k = T(1-(1-\frac{k}{N}))$, $k = 1, \cdots, N$, a subdivision :

- if ϕ smooth then

$$|\langle \mu_T - \hat{\mu}_T, \phi \rangle| \leqslant CN^{-([(m-1)/2] \wedge 1/2)} \sup\{||\nabla_x^j \phi||_\infty ;\ j \leqslant m+2\},$$

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$$\begin{array}{lll} \text{True dynamic :} & dX_t = V(X_t, \langle \mu_t, \varphi\rangle) \circ \mathrm{d}B_t & X_{T_{k-1}} = y \\ \text{Approximation :} & d\hat{X}_t^{(j),\hat{\mu}} = V(\hat{X}_t^{(j),\hat{\mu}}, \langle \hat{\mu}_{T_k}, \varphi\rangle)\mathrm{d}w_t^j, \quad j = 1, \dots, n, & \hat{X}_{T_{k-1}}^{(j),\hat{\mu}} = y \end{array}$$

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Convergence rate : Let $\mathcal{T}(m)$ a tree and $T_k = T(1 - (1 - \frac{k}{N}))$, $k = 1, \cdots, N$, a subdivision :

- if ϕ and φ smooth then

$$|\langle \mu_T - \hat{\mu}_T, \phi\rangle| \leqslant CN^{-([(m-1)/2] \wedge 1/2)} \sup\{||\nabla_x^j \phi||_\infty;\ j \leqslant m+2\},$$

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Approximation : $d\hat{X}_t^{(j),\hat{\mu}} = V(\hat{X}_t^{(j),\hat{\mu}}, \sum_{p=0}^{q-1} \frac{1}{p!}(t - T_k)\langle \hat{\mu}_{T_k}, (\mathcal{L}^{\hat{\mu}})^p \varphi \rangle)\mathrm{d}w_t^j$,

E_{T_k} = Euler error $\approx \Delta_{T_k}^{q+1}$ + propagation error $\approx \Delta_{T_k} E_{T_{k-1}}$ + cubature error $\approx \Delta_{T_k}^{(m+1)/2}$

Convergence rate : Let $\mathcal{T}(m, q)$ a tree and $T_k = T(1-(1-\frac{k}{N}))$, $k = 1, \cdots, N$, a subdivision :

- if ϕ and φ smooth then

$$|\langle \mu_T - \hat{\mu}_T, \phi \rangle| \leqslant CN^{-([(m-1)/2] \wedge q)} \sup\{||\nabla_x^j \phi||_\infty;\ j \leqslant m+2\},$$

Error (weak) $\mathcal{T}$

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- if ϕ and φ smooth then

$$|\langle \mu_T - \hat{\mu}_T, \phi\rangle| \leqslant CN^{-([(m-1)/2]\wedge q)} \sup\{||\nabla_x^j \phi||_\infty ;\ j \leqslant m+2\},$$

if ϕ Lipschitz ?

Error (weak) $\mathcal{T}$

- Objective : control $|\langle \mu_T - \hat{\mu}_T, \phi \rangle|$
  - Markov property : **global error = sum of local errors**
- Local error $[T_{k-1}, T_k]$: $E_{T_k} = |\langle \mu_{T_k} - \hat{\mu}_{T_k}, \psi \rangle|$, ψ smooth function φ smooth.

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E_{T_k} = Euler error $\approx \Delta_{T_k}^{q+1}$ + propagation error $\approx \Delta_{T_k} E_{T_{k-1}}$ + cubature error $\approx \Delta_{T_k}^{(m+1)/2}$

Convergence rate : Let $\mathcal{T}(m, q, \gamma)$ a tree and $T_k = T(1 - (1 - \frac{k}{N})^\gamma)$, $k = 1, \cdots, N$, a subdivision :

- if $\gamma = 1$, if ϕ and φ smooth then

$$|\langle \mu_T - \hat{\mu}_T, \phi \rangle| \leqslant CN^{-([(m-1)/2] \wedge q)} \sup\{||\nabla_x^j \phi||_\infty;\ j \leqslant m+2\},$$

- if $\gamma > m - 1$, if ϕ Lipschitz , V uniformly elliptic and φ smooth, then

$$|\langle \mu_T - \hat{\mu}_T, \phi \rangle| \leqslant CN^{-([(m-1)/2] \wedge q \wedge \gamma/2)} ||\phi||_{\mathrm{Lip}}.$$

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

↳ *inspired by Zhang (2004) and Crisan & Manolarakis (2010)*

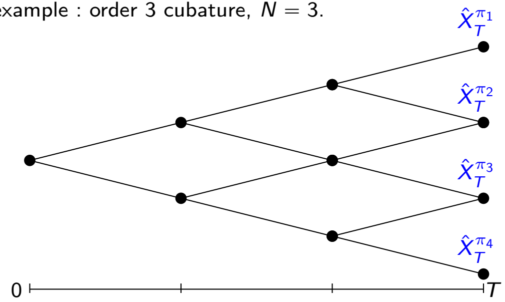
Data :

- interval $[0, T]$, $N > 0$
- $T_0 = 0 < \ldots < T_N = T$
- order m cubature :

$$\{w_t^1, \cdots, w_t^n\}, \ \{\lambda_1, \ldots, \lambda_n\}$$

- tree $\mathcal{T}(m)$, ϕ
- $\pi_j \in \{1, \cdots, n\}^N$ (genealogy)

example : order 3 cubature, $N = 3$.



Algorithm, second step : going backward the tree $\mathcal{T}(m)$

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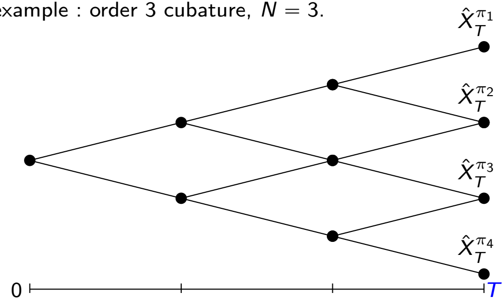
example : order 3 cubature, $N = 3$.



Initialisation :

- interval $[0, T]$, $N > 0$
- $T_0 = 0 < \ldots < T_N = T$
- order m cubature :

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- tree $\mathcal{T}(m)$, ϕ
- $\pi_j \in \{1, \cdots, n\}^N$ (genealogy)

- True value : $Y_T^{\pi_j} = u(T, \hat{X}_T^{\pi}) = \phi(X_T^{\pi_j})$, $\quad Z_T^{\pi_j} =: v(T, X_T^{\pi_j}) = V(X_T^{\pi_j}, \mu_T) D_x u(T, X_T^{\pi_j})$

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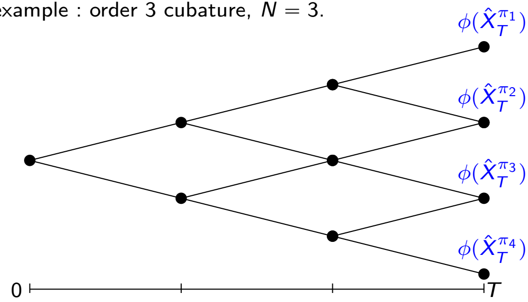

Initialisation :

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- $\pi_j \in \{1, \cdots, n\}^N$ (genealogy)

- True value : $Y_T^{\pi_j} = u(T, \hat{X}_T^{\pi}) = \phi(X_T^{\pi_j}), \quad Z_T^{\pi_j} =: v(T, X_T^{\pi_j}) = V(X_T^{\pi_j}, \mu_T) D_x u(T, X_T^{\pi_j})$
- Approximation : $\hat{Y}_T^{\pi} = \hat{u}(T, \hat{X}_T^{\pi}) = \phi(\hat{X}_T^{\pi}), \quad \hat{Z}_T^{\pi} = \hat{v}(T, \hat{X}_T^{\pi}) = 0$

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

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step $[T_{N-1}, T_N[$

- interval $[0, T]$, $N > 0$
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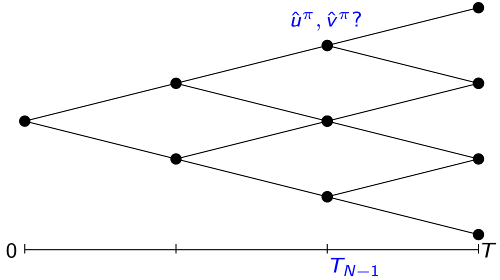
- $\pi \in \{1, \cdots, n\}^{N-1}$ (genealogy)

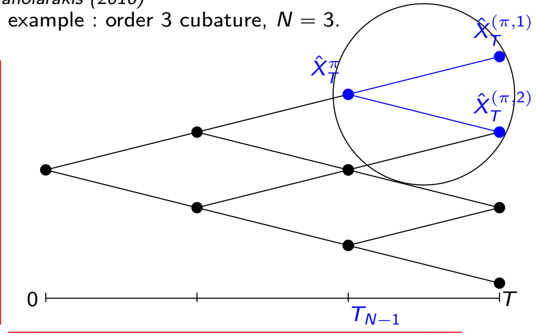


- True value : $Y^\pi_{T_{N-1}} = u(T_{N-1}, X^\pi_{T_{N-1}}) = \mathbb{E}\left[u(T_N, X^\pi_{T_N}) + \int_{T_{N-1}}^{T_N} f(\cdots)\mathrm{d}s \;\middle|\; \mathcal{F}_{T_{N-1}} \right]$
- Approximation : $\hat{Y}^\pi_{T_{N-1}} = \hat{u}^\pi, \quad \hat{Z}^\pi_{T_{N-1}} = \hat{v}^\pi$

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

↳ *inspired by Zhang (2004) and Crisan & Manolarakis (2010)*

example : order 3 cubature, $N = 3$.



step $[T_{N-1}, T_N[$

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- $T_0 = 0 < \ldots < T_N = T$
- order m cubature :

$$\{w_t^1, \cdots, w_t^n\},\ \{\lambda_1, \ldots, \lambda_n\}$$

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  ↳ The local cubature tree gives an approximation of the conditional law

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

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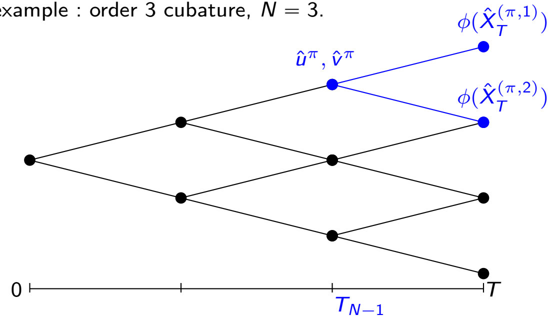

- True value : $Y^\pi_{T_{N-1}} = u(T_{N-1}, X^\pi_{T_{N-1}}) = \mathbb{E}\left[u(T_N, X^\pi_{T_N}) + \int_{T_{N-1}}^{T_N} f(\cdots)\mathrm{d}s \;\middle|\; \mathcal{F}_{T_{N-1}} \right]$

  ↳ The local cubature tree gives an approximation of the conditional law

- $v(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) \leftarrow \mathbb{E}\left[u\left(T_N, X^{T_{N-1}, \hat{X}^\pi_{T_N}}\right) \Delta W_{T_N}\right] \leftarrow \sum_{j=1}^n \lambda_j \phi(\hat{X}^{(\pi,j)}_{T_N}) \Delta w^{(j)}_{T_N}$

$$u(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) \leftarrow \mathbb{E}\left[u(T_{k+1}, X^{T_k,y}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\cdot, \hat{\mu}_{T_{N-1}}) \right] \leftarrow \sum_{j=1}^n \lambda_j \phi(\hat{X}^{(\pi,j)}_{T_N}) + \Delta_{T_N} f(\cdot, \hat{\mu}_{T_{N-1}})$$

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

↳ *inspired by Zhang (2004) and Crisan & Manolarakis (2010)*

example : order 3 cubature, $N = 3$.

step $[T_{N-1}, T_N[$

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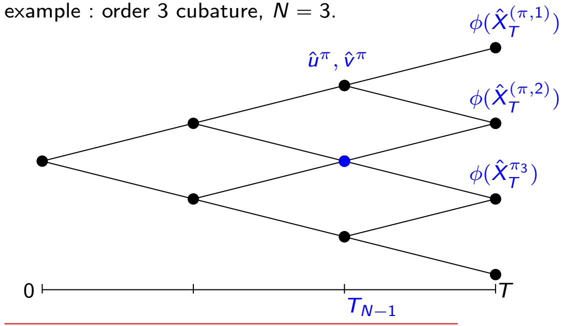

- True value : $Y^\pi_{T_{N-1}} = u(T_{N-1}, X^\pi_{T_{N-1}}) = \mathbb{E}\left[u(T_N, X^\pi_{T_N}) + \int_{T_{N-1}}^{T_N} f(\cdots)\mathrm{d}s \;\middle|\; \mathcal{F}_{T_{N-1}} \right]$

  ↳ The local cubature tree gives an approximation of the conditional law

- $v(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) \leftarrow \mathbb{E}\left[u\left(T_N, X^{T_{N-1}, \hat{X}^\pi_{T_N}} \right) \Delta W_{T_N} \right] \leftarrow \sum_{j=1}^{n} \lambda_j \phi(\hat{X}^{(\pi,j)}_{T_N}) \Delta w^{(j)}_{T_N}$

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Algorithm, second step : going backward the tree $\mathcal{T}(m)$

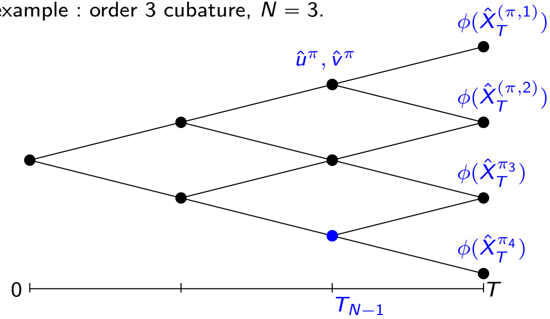
↳ *inspired by Zhang (2004) and Crisan & Manolarakis (2010)*

step $[T_{N-1}, T_N[$

- interval $[0, T]$, $N > 0$
- $T_0 = 0 < \ldots < T_N = T$
- order m cubature :

  $$\{w_t^1, \cdots, w_t^n\}, \ \{\lambda_1, \ldots, \lambda_n\}$$
- tree $\mathcal{T}(m)$, ϕ
- $\pi \in \{1, \cdots, n\}^{N-1}$ (genealogy)

example : order 3 cubature, $N = 3$.



- True value : $Y^\pi_{T_{N-1}} = u(T_{N-1}, X^\pi_{T_{N-1}}) = \mathbb{E}\left[u(T_N, X^\pi_{T_N}) + \int_{T_{N-1}}^{T_N} f(\cdots)\mathrm{d}s \;\middle|\; \mathcal{F}_{T_{N-1}} \right]$

  ↳ The local cubature tree gives an approximation of the conditional law

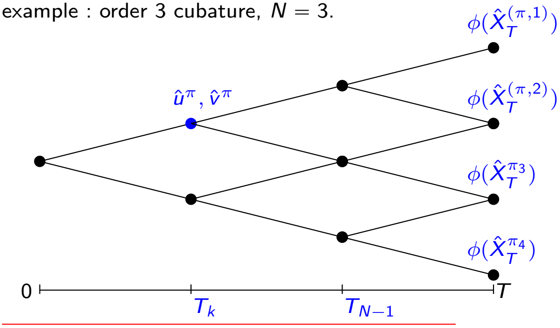
- $v(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) \leftarrow \mathbb{E}\left[u\left(T_N, X^{T_{N-1}, \hat{X}^\pi_{T_N}}\right) \Delta W_{T_N} \right] \leftarrow \sum_{j=1}^{n} \lambda_j \phi(\hat{X}^{(\pi,j)}_{T_N}) \Delta w^{(j)}_{T_N}$

  $u(T_{N-1}, \hat{X}^\pi_{T_{N-1}}) \leftarrow \mathbb{E}\left[u(T_{k+1}, X^{T_k, y}_{T_{k+1}}) + \Delta_{T_{k+1}} f(\cdot, \hat{\mu}_{T_{N-1}}) \right] \leftarrow \sum_{j=1}^{n} \lambda_j \phi(\hat{X}^{(\pi,j)}_{T_N}) + \Delta_{T_N} f(\cdot, \hat{\mu}_{T_{N-1}})$

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

↳ *inspired by Zhang (2004) and Crisan & Manolarakis (2010)*

example : order 3 cubature, $N = 3$.



step $[T_k, T_{k+1}[$

- interval $[0, T]$, $N > 0$
- $T_0 = 0 < \ldots < T_N = T$
- order m cubature :

$$\{w_t^1, \cdots, w_t^n\}, \; \{\lambda_1, \ldots, \lambda_n\}$$

- tree $\mathcal{T}(m)$, ϕ
- $\pi \in \{1, \cdots, n\}^k$ (genealogy)

- True value : $Y_{T_k}^\pi = u(T_k, \hat{X}_{T_k}^\pi)$, $\quad Z_{T_k}^\pi =: v(T_k, X_{T_k}^{\pi_j})$
  ↳ The local cubature tree gives an approximation of the conditional law
- Approximation :
  - $$\hat{v}(T_k, \hat{X}_{T_k}^\pi) = \sum_{j=1}^n \lambda_j u(T_{k+1}, X_{T_{k+1}}^{T_k,\pi,j}) \Delta w_{T_{k+1}}^{(j)}$$
  - $$\hat{u}(T_k, \hat{X}_{T_k}^\pi) = \sum_{j=1}^n \lambda_j \left(u(T_{k+1}, X_{T_{k+1}}^{T_k,\pi,j}) + \Delta_{T_{k+1}} f(\cdot, \hat{\mu}_{T_k}) \right)$$

Algorithm, second step : going backward the tree $\mathcal{T}(m)$

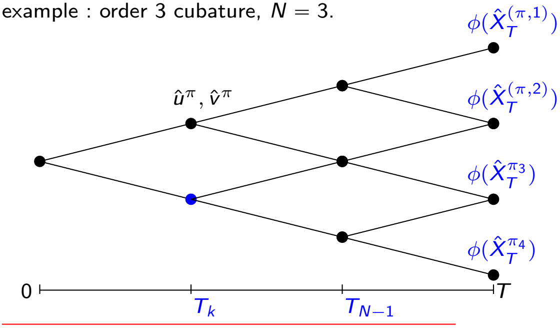
↳ *inspired by Zhang (2004) and Crisan & Manolarakis (2010)*

step $[T_k, T_{k+1}[$

- interval $[0, T]$, $N > 0$
- $T_0 = 0 < \ldots < T_N = T$
- order m cubature :

$$\{w_t^1, \cdots, w_t^n\}, \ \{\lambda_1, \ldots, \lambda_n\}$$

- tree $\mathcal{T}(m)$, ϕ
- $\pi \in \{1, \cdots, n\}^k$ (genealogy)



- True value : $Y_{T_k}^\pi = u(T_k, \hat{X}_{T_k}^\pi)$, $\quad Z_{T_k}^\pi =: v(T_k, X_{T_k}^{\pi_j})$
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  - $\hat{u}(T_k, \hat{X}_{T_k}^\pi) = \sum_{j=1}^n \lambda_j \left(u(T_{k+1}, X_{T_{k+1}}^{T_k,\pi,j}) + \Delta_{T_{k+1}} f(\cdot, \hat{\mu}_{T_k}) \right)$

Error estimate

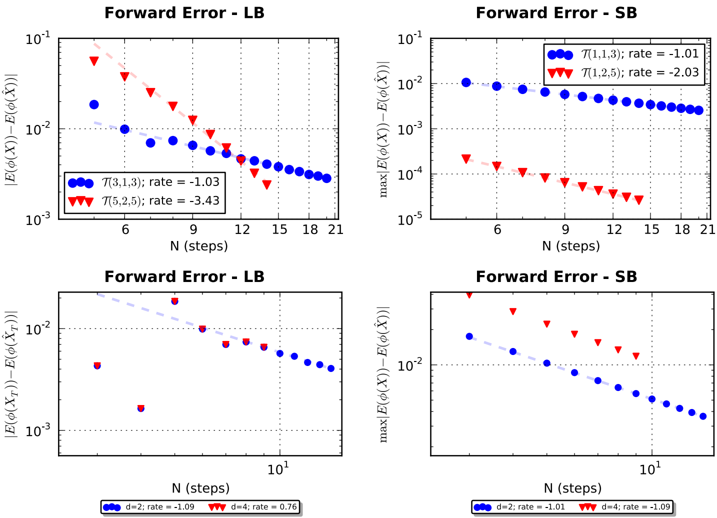
Let N an integer, $T_k = T(1-(1-\frac{k}{N})^\gamma)$, $k = 1, \cdots, N$, a subdivision and $\mathcal{T}(m,q,\gamma)$ a tree. Under the appropriate assumptions, one can show that there exists C, independent of N, such that

$$\max_{k\in\{0,\cdots,N-1\}} \max_{\pi\in\{1,\cdots,n\}^k} |\hat{Y}^\pi_{T_k} - Y^\pi_{T_k}| + \Delta^{1/2}_{T_{k+1}} |\hat{Z}^\pi_{T_k} - Z^\pi_{T_k}| \leqslant C\frac{1}{N}.$$

Moreover, one can use a predictor-corrector scheme to obtain that :

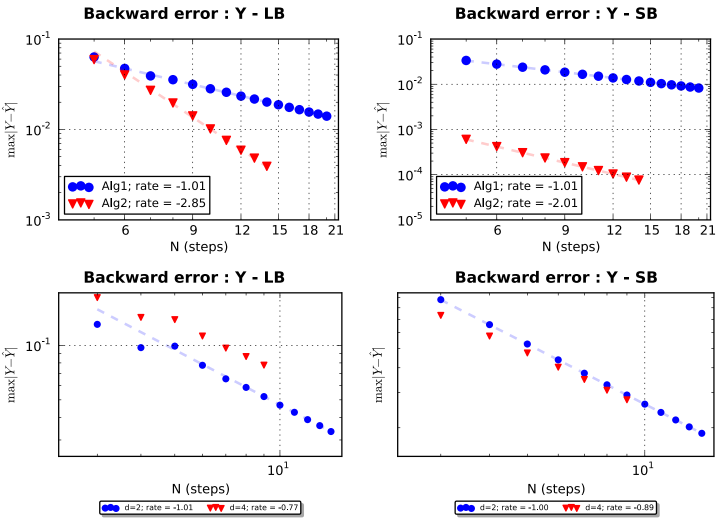
$$\max_{k\in\{0,\cdots,N-1\}} \max_{\pi\in\{1,\cdots,n\}^k} |\hat{Y}^\pi_{T_k} - Y^\pi_{T_k}| + \Delta^{1/2}_{T_{k+1}} |\hat{Z}^\pi_{T_k} - Z^\pi_{T_k}| \leqslant C\left(\frac{1}{N}\right)^2.$$

Illustration on toy model



$$dX_t = \mathbb{E}[\sin(X_t)]dt + \mathrm{d}B_t; \implies X = B$$

Illustration on toy model



$$\mathrm{d}X_t = \mathbb{E}[\sin(X_t)]\mathrm{d}t + \mathrm{d}B_t; \; -\mathrm{d}Y_t = \left(\frac{\mathbf{1} \cdot \cos(X_t)}{2} + \mathbb{E}\left[(\mathbf{1} \cdot \sin(X_t)) \exp(-Y_t^2) \right] \right) dt - Z_t \cdot \mathrm{d}B_t,$$

Thank you !