Probabilistic Approach to Large Time Behaviour of Mild Solutions of HJB Equations in Infinite Dimension by a Probabilistic Approach

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Objective

- Study the large time behaviour of solutions of the Cauchy problem in an infinite dimensional real Hilbert space H:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f(x, \nabla u(t,x)G), & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u(0,x) = g(x), & \forall x \in H, \end{cases} \tag{1}$$

- $$(\mathscr{L}h)(x) = \frac{1}{2}\operatorname{Tr}(GG^*\nabla^2 h(x)) + < Ax + F(x), \nabla h(x) > .$$

  is the formal generator of the Kolmogorov semigroup $\mathscr{P}_t$ of an H-valued random process solution of the following Ornstein-Uhlenbeck stochastic differential equation:

$$\begin{cases} \mathrm{d}X_t = (AX_t + F(X_t^x))\mathrm{d}t + G\,\mathrm{d}W_t, & t \in \mathbb{R}_+, \\ X_0 = x, & x \in H, \end{cases}$$

- W is a Wiener process with values in another real Hilbert space Ξ, assumed to be separable.

Method

First, let (v, λ) be the solution of the ergodic PDE:

$$\mathscr{L}v + f(x, \nabla v(x)G) - \lambda = 0, \quad \forall x \in H.$$

Then we have the following probabilistic representation. Let $(Y^{T,x}, Z^{T,x})$ be solution of the BSDE:

$$\begin{cases} \mathrm{d}Y_s^{T,x} = -f(X_s^x, Z_s^{T,x})\mathrm{d}s + Z_s^{T,x}\mathrm{d}W_s \\ Y_T^{T,x} = g(X_T^x), \end{cases}$$

and (Y, Z, λ) be solution of the EBSDE:

$$\mathrm{d}Y_s = -(f(X_s^x, Z_s^x) - \lambda)\mathrm{d}s + Z_s^x \mathrm{d}W_s.$$

Then

$$\begin{cases} Y_s^{T,x} = u(T - s, X_s^x), \\ Y_s^x = v(X_s^x). \end{cases}$$

Results

	Deterministic
First behaviour	$\frac{u(T,x)}{T} - \lambda \underset{T \to +\infty}{\longrightarrow} 0$
Second behaviour	$u(T,x) - \lambda T - v(x) \underset{T \to +\infty}{\longrightarrow} L$
Third behaviour	$\lvert u(T,x) - \lambda T - v(x) - L \rvert \leq C(1 + \lvert x \rvert^{2+\mu}) e^{-\hat{\eta} T}$

	Probabilistic
First behaviour	$\frac{Y_0^{T,x}}{T} - \lambda \underset{T \to +\infty}{\longrightarrow} 0$
Second behaviour	$Y_0^{T,x} - \lambda T - Y_0^x \underset{T \to +\infty}{\longrightarrow} L$
Third behaviour	$\lvert Y_0^{T,x} - \lambda T - Y_0^x - L \rvert \leq C(1 + \lvert x \rvert^{2+\mu}) e^{-\hat{\eta} T}$

Some references

- 1997 : Namah, Roquejoffre, (periodic, finite dimension, with speed of convergence)
- 2001 : Barles and Souganidis, (periodic, finite dimension)
- 2006 : Fujita, Ishii, Loreti, (finite dimension, $f(x, z) = H_1(x) + H_2(z)$, H_2 Lipschitz and H_1 locally Hölder)
- 2013 : Ichihara, Sheu (finite dimension, quadratic and convex with respect to z)

Preliminaries : some results about a perturbed forward SDE

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}F(s,X_s)\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}W_s, \qquad (2)$$

Hypothesis

1. A is an unbounded operator $A : D(A) \subset H \to H$, with $D(A)$ dense in H. We assume that A is dissipative and generates a stable C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. By this we mean that there exist constants $\eta > 0$ and $M > 0$ such that

$$\langle Ax, x\rangle \leq -\eta|x|^2, \quad \forall x \in D(A); \quad |e^{tA}|_{L(H,H)} \leq Me^{-\eta t}, \quad \forall t \geq 0.$$

2. For all $s > 0$, e^{sA} is a Hilbert-Schmidt operator. Moreover $|e^{sA}|_{L_2(H,H)} \leq Ms^{-\gamma}$ and $\gamma \in [0, 1/2)$.
3. $F : \mathbb{R}_+ \times H \to H$ is bounded and measurable.
4. G is a bounded linear operator in $L(\Xi, H)$.
5. G is invertible. We denote by G^{-1} its bounded inverse.

Some results about a perturbed forward SDE

Lemma

Assume that Hypothesis 1 (only points (1.)-(4.)) hold and that F is bounded and Lipschitz in x. Then for every $p \in [2, \infty)$, for every $T > 0$ there exists a unique process $X^x \in L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,T]; H))$ solution of (2). Moreover,

$$\sup_{0 \le t < +\infty} \mathbb{E}|X^x_t|^p \le C(1+|x|)^p, \tag{3}$$

for some constant C depending only on p, γ, M and $\sup_{t\ge 0}\sup_{x\in H}|F(t,x)|$. If F is only bounded and measurable, then the solution to equation 2 still exists but in the martingale sense. By this we mean that there exists a new $\mathscr{F}$-Wiener process $(\widehat{W}^x)_{t\ge 0}$ with respect to a new probability measure $\widehat{\mathbb{P}}$ (absolutely continuous with respect to $\mathbb{P}$), and an $\mathscr{F}$-adapted process $\widehat{X}^x$ with continuous trajectories for which (2) holds with W replaced by $\widehat{W}$. Moreover (3) still holds (with respect to the new probability). Finally such a martingale solution is unique in law.

Lemma (Basic Coupling Estimates)

Assume that Hypothesis above holds true and that F is a bounded and Lipschitz function. Then there exist $\hat{c} > 0$ and $\hat{\eta} > 0$ such that for all $\phi : H \to \mathbb{R}$ measurable with polynomial growth (i.e. $\exists C, \mu > 0$ such that $\forall x \in H$, $|\phi(x)| \leq C(1 + |x|^{\mu})$), $\forall x, y \in H$,

$$|\mathscr{P}_t[\phi](x) - \mathscr{P}_t[\phi](y)| \leq \hat{c}(1 + |x|^{1+\mu} + |y|^{1+\mu})e^{-\hat{\eta}t}. \qquad (4)$$

We stress the fact that $\hat{c}$ and $\hat{\eta}$ depend on F only through $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$.

Corollary

Relation (4) can be extended to the case in which F is only bounded measurable and for all $t \geq 0$, there exists a uniformly bounded sequence of Lipschitz functions in x $(F_n(t, \cdot))_{n \geq 1}$ (i.e. $\forall t \geq 0, \forall n \in \mathbb{N}$, $F_n(t, \cdot)$ is Lipschitz and $\sup_n \sup_t \sup_x |F_n(t, x)| < +\infty$) such that

$$\lim_n F_n(t, x) = F(t, x), \qquad \forall t \geq 0, \forall x \in H.$$

Clearly in this case in the definition of $\mathscr{P}_t[\phi]$ the mean value is taken with respect to the new probability $\widehat{\mathbb{P}}$.

Some results about a perturbed forward SDE

Lemma

Let $\zeta, \zeta' : \mathbb{R}_+ \times H \to \Xi^*$ *such that for all* $s \geq 0$, $\zeta(s,\cdot)$ *and* $\zeta'(s,\cdot)$ *are weakly* continuous with polynomial growth. We define*

$$\Upsilon(s,x) = \begin{cases} \frac{\psi(x,\zeta(s,x)) - \psi(x,\zeta'(s,x))}{|\zeta(s,x)-\zeta'(s,x)|^2}(\zeta(s,x) - \zeta'(s,x))^*, & \text{if } \zeta(s,x) \neq \zeta'(s,x), \\ 0, & \text{if } \zeta(s,x) = \zeta'(s,x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_n(s,\cdot))_{n\geq 1}$ *(i.e.* $\forall n$, $\Upsilon_n(s,\cdot)$ *is Lipschitz and* $\sup_n \sup_s \sup_x |\Upsilon_n(s,x)| < \infty$*) such that*

$$\lim_n \Upsilon_n(s,x) = \Upsilon(s,x), \quad \forall s \geq 0, \forall x \in H.$$

The BSDE and the EBSDE

BSDE

$$Y_t^{T,x} = g(X_T^x) + \int_t^T f(X_s^x, Z_s^{T,x})\mathrm{d}s - \int_t^T Z_s^{T,x}\mathrm{d}W_s, \qquad t \in [0, T]$$

EBSDE

$$Y_t = Y_T + \int_t^T f(X_s^x, Z_s^x) - \lambda\mathrm{d}s - \int_t^T Z_s^x\mathrm{d}W_s, \qquad \forall T > 0, \forall t \in [0, T]$$

We will assume the following assumptions.

Hypothesis

There exist $l > 0$, $\mu \geq 0$ such that the function $f : H \times \Xi^* \to \mathbb{R}$ and ξ^T satisfy :

1. $F : H \to H$ is a Lipschitz, bounded and belongs to the class $\mathscr{G}^1$,
2. $g(\cdot)$ is continuous and have polynomial growth : for all $x \in H$, $|g(x)| \leq C(1 + |x|^\mu)$,
3. $\forall x \in H$, $\forall z, z' \in \Xi^*$, $|f(x,z) - f(x,z')| \leq l|z - z'|$,
4. $f(\cdot, z)$ is continuous and $\forall x \in H$, $|f(x,0)| \leq C(1 + |x|^\mu)$.

First behaviour

Theorem

Assume that our hypothesis hold true. Then, $\forall T > 0$:

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \frac{C(1 + |x|^{1+\mu})}{T}. \tag{5}$$

In particular,

$$\frac{Y_0^{T,x}}{T} \underset{T \to +\infty}{\longrightarrow} \lambda,$$

uniformly in any bounded set of H.

Sketch of the proof

$$\left|\frac{Y_0^{T,x}}{T}-\lambda\right| \leq \left|\frac{Y_0^{T,x}-Y_0^x-\lambda T}{T}\right| + \left|\frac{Y_0^x}{T}\right|.$$

We have :

$$\begin{aligned}Y_0^{T,x}-Y_0^x-\lambda T &= g(X_T^x)-v(X_T^x)+\int_0^T (f(X_s^x,Z_s^{T,x})-f(X_s^x,Z_s^x))\mathrm{d}s \\ &\quad -\int_0^T (Z_s^{T,x}-Z_s^x)\mathrm{d}W_s \\ &= g(X_T^x)-v(X_T^x)+\int_0^T (Z_s^{T,x}-Z_s)\beta_s^T\mathrm{d}s-\int_0^T (Z_s^{T,x}-Z_s)\mathrm{d}W_s,\end{aligned}$$

where

$$\beta_s^T = \begin{cases} \frac{(f(X_s^x,Z_s^{T,x})-f(X_s^x,Z_s^x))(Z_s^{T,x}-Z_s^x)^*}{|Z_s^{T,x}-Z_s^x|^2}, & \text{if } Z_s^{T,x}-Z_s^x \neq 0 \\ 0, & \text{otherwise.}\end{cases}$$

Sketch of the proof

Taking the expectation with respect to $\mathbb{Q}^T$ we get

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T}(g(X_T^x) - v(X_T^x)). \tag{6}$$

So we have

$$\left|\frac{Y_0^{T,x} - Y_0^x - \lambda T}{T}\right| \leq C\frac{1 + \mathbb{E}^{\mathbb{Q}^T}\left(|X_T^x|^{1+\mu}\right)}{T}.$$

The process $(X_t^x)_{t\geq 0}$ is the mild solution of

$$\begin{cases} \mathrm{d}X_t^x = AX_t^x\mathrm{d}t + F(X_t^x)\mathrm{d}t + G\beta_t^T\mathbb{1}_{t<T}\mathrm{d}t + G\mathrm{d}\widetilde{W}_t^T, \quad t \in [0,T], \\ X_0^x = x. \end{cases}$$

Second and Third behaviour

Hypothesis

$$F \equiv 0.$$

Note that setting $F \equiv 0$ is not restrictive. Indeed we study

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f(x, \nabla u(t,x)G), & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u(0,x) = g(x), & \forall x \in H. \end{cases}$$

Now remark that

$$\langle Ax + F(x), \nabla u(t,x)\rangle + f(x, \nabla u(t,x)G) = \langle Ax, \nabla u(t,x)\rangle + \widetilde{f}(x, \nabla u(t,x)G),$$

where $\widetilde{f}(x,z) = f(x,z) + \langle F(x), zG^{-1}\rangle$ is a continuous function in x with polynomial growth in x and Lipschitz in z.

Second and Third behaviour

Theorem

Assume that our hypothesis hold true. Then there exists $L \in \mathbb{R}$ such that,

$$\forall x \in H, \quad Y_0^{T,x} - \lambda T - Y_0^x \underset{T \to +\infty}{\longrightarrow} L.$$

Furthermore the following speed of convergence holds

$$|Y_0^{T,x} - \lambda T - Y_0^x - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}.$$

Let us fix $T > 0$ and let us consider the following BSDE in finite horizon for an unknown process $(Y_s^{T,t,x}, Z_s^{T,t,x})_{s\in[t,T]}$ with values in $\mathbb{R}\times\Xi^*$:

$$Y_s^{T,t,x} = g(X_T^{t,x}) + \int_s^T f(X_r^{t,x}, Z_r^{T,t,x})\mathrm{d}r - \int_s^T Z_r^{T,t,x}\mathrm{d}W_r, \quad \forall s \in [t,T], \tag{7}$$

where $(X_s^{t,x})_{s\geq 0}$ is the mild solution of

$$\mathrm{d}X_s = [AX_s + F(X_s)]\mathrm{d}s + G\mathrm{d}W_t, \qquad X_t = x$$

If $t = 0$, we use the following standard notations $X_s^x = X_s^{0,x}$, $Y_s^{T,x} := Y_s^{T,0,x}$ and $Z_s^{T,x} := Z_s^{T,0,x}$.

Sketch of the proof

We define

$$u_T(t,x) := Y_t^{T,t,x}$$
$$w_T(t,x) := u_T(t,x) - \lambda(T-t) - v(x).$$

Key property

$u_T(0,x) = u_{T+S}(S,x)$

$$\Longrightarrow w_T(0,x) = w_{T+S}(S,x)$$

Lemma

Under the hypothesis of Theorem 2, there exist constant $C > 0$ and $C_{T'}$ such that $\forall x, y \in H$, $\forall T > 0$,

$$|w_T(0,x)| \leq C(1 + |x|^{1+\mu}),$$
$$|\nabla_x w_T(0,x)| \leq \frac{C_{T'}}{\sqrt{T'}}(1 + |x|^{1+\mu}), \quad \forall 0 < T' \leq T,$$
$$|w_T(0,x) - w_T(0,y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}.$$

First estimate of Lemma

$$
\begin{aligned}
|w_T(0,x)| &= |u_T(0,x) - \lambda T - v(x)| \\
&= |\,Y_0^{T,x} - Y_0^x - \lambda T\,| \\
&\leq C(1 + |x|^{1+\mu}).
\end{aligned} \tag{8}
$$

Second estimate of Lemma

$$
\begin{aligned}
w_T(s, X_s^{t,x}) = w_T(T, X_T^{t,x}) + \int_s^T (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))\mathrm{d}r \\
- \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r.
\end{aligned}
$$

$$
\begin{aligned}
w_T(s, X_s^{t,x}) &= w_T(T', X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))\mathrm{d}r \\
&\quad - \int_s^{T'} (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r \\
&= w_{T-T'}(0, X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x} - Z_r^{t,x} + Z_r^{t,x}) - f(X_r^{t,x}, Z_r^{t,x}))\mathrm{d}r \\
&\quad - \int_t^{T'} (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r,
\end{aligned}
$$

Second estimate of Lemma

$$Z_s^{T,t,x} = \nabla_x u_T(s, X_s^{t,x})G, \quad \text{and} \quad Z_s^x = \nabla_x v(X_s^{t,x})G.$$

Then we easily obtain that

$$Z_r^{T,t,x} - Z_r^{t,x} = \nabla_x w_T(r, X_r^{t,x})G.$$

Thus, applying the Bismut-Elworthy formula, we get $\forall x, h \in H$, $\forall t < T$,

$$\begin{aligned}\nabla_x w_T(t,x)h = \mathbb{E}\int_t^{T'} [f\left(X_s^{t,x}, \nabla_x w_T(r, X_r^{t,x})G + Z_s^{t,x}\right) - f\left(X_s^{t,x}, Z_s^{t,x}\right)]U^h(s,t,x)\mathrm{d}s \\ + \mathbb{E}\left[[w_{(T-T')}(0, X_{T'}^{t,x})]U^h(T',t,x)\right],\end{aligned}$$

Second estimate of Lemma

where, $\forall 0 \leq s \leq T$, $\forall x \in H$,

$$U^h(s,t,x) = \frac{1}{s-t}\int_t^s \langle G^{-1}\nabla_x X_u^{t,x} h, \mathrm{d}W_u\rangle.$$

Let us recall that

$$\nabla_x X_s^{t,x} h = e^{(s-t)A}h,$$

then,

$$\mathbb{E}|U^h(s,t,x)|^2 = \frac{1}{|s-t|^2}\int_t^s |G^{-1}\nabla_x X(u,t,x)h|^2 \mathrm{d}u \leq \frac{C|h|^2}{s-t},$$

where C is independent on t, s and x.

Third estimate of Lemma

We have

$$\begin{aligned} w_T(0,x) &= \mathbb{E}^{\mathbb{Q}^T}(g(X_T^x) - v(X_T^x)) \\ &= \mathbb{E}(g(U_T^x) - v(U_T^x)), \end{aligned}$$

where U^x is the mild solution of the following equation defined $\forall t \in \mathbb{R}$:

$$\mathrm{d}U_t^x = [AU_t^x + G\beta^T(t, U_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \quad U_0^x = x,$$

and where $\beta^T(t,x) =$

$$\begin{cases} \frac{(f(x,\nabla u_T(t,x)G) - f(x,\nabla v(x)G))(\nabla u_T(t,x)G - \nabla v(x)G)^*}{|(\nabla u_T(t,x) - \nabla v(x))G|^2} \mathbb{1}_{t<T}, & \text{if } \nabla u_T(t,x) - \nabla v(x) \neq 0 \\ 0 \ , & \text{otherwise.} \end{cases}$$

Therefore, $\forall x \in H$ $\forall T > 0$ we can write

$$|w_T(0,x) - w_T(0,y)| = |\mathbb{E}(g(U_T^x) + v(U_T^x)) - \mathbb{E}(g(U_T^y) + v(U_T^y))|.$$

$$|w_T(0,x) - w_T(0,y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}, \tag{9}$$

Proof of Theorem 2

By the three estimates of Lemma : $\exists (T_i)_i$ and $L_1 \in \mathbb{R}$ such that

$$\lim_i w_{T_i}(0,x) = L_1.$$

For any compact subset K of H, $\left\{ w_T(0,\cdot)_{|K} ; T > 1 \right\}$ is a relatively compact subspace of the space of continuous functions $K \to \mathbb{R}$ for the uniform distance (denoted by $(\mathscr{C}(K,\mathbb{R}), ||\cdot||_{K,\infty})$.

We show that the accumulation point is unique. We assume that there exists $(T_i')_i$ such that $w_{T_i'}(0,x) \to L_{2,K}$ uniformly.

Proof of Second behaviour

Let us write, $\forall x \in H$, $\forall T, S > 0$:

$$\begin{aligned}
w_{T+S}(0,x) &= Y_0^{T+S,x} - \lambda(T+S) - Y_0^x \\
&= Y_S^{T+S,x} - \lambda T - Y_S^x + \int_0^S \left(f(X_r^x, Z_r^{T+S,x}) - f(X_r^x, Z_r^x)\right)\mathrm{d}r \\
&\qquad - \int_0^S (Z_r^{T+S,x} - Z_r^x)\mathrm{d}W_r \\
&= Y_S^{T+S,x} - \lambda T - Y_S^x + \int_0^S (Z_r^{T+S,x} - Z_r^x)\mathrm{d}\widetilde{W}_r^{T,S},
\end{aligned}$$

with

$$\widetilde{W}_t^{T,S} = -\int_0^t \beta^{T,S}(s, X_s^x)\mathrm{d}s + W_t,$$

and where $\beta^{T,S}(t,x) =$

$$\begin{cases} \frac{(f(x,\nabla u_{T+S}(t,x)G) - f(x,\nabla v(x)G))((\nabla u_{T+S}(t,x) - \nabla v(x))G)^*}{|(\nabla u_{T+S}(t,x) - \nabla v(x))G|^2}\mathbb{1}_{t<S}, & \text{if } \nabla u_{T+S}(t,x) - \nabla v(x) \neq \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Second behaviour

$$
\begin{aligned}
w_{T+S}(0,x) &= \mathbb{E}^{\mathbb{Q}^{\boldsymbol{T},\boldsymbol{S}}}(Y_S^{T+S,x} - \lambda T - Y_S^x) \\
&= \mathbb{E}^{\mathbb{Q}^{\boldsymbol{T},\boldsymbol{S}}}(w_{T+S}(S, X_S^x)) \\
&= \mathbb{E}^{\mathbb{Q}^{\boldsymbol{T},\boldsymbol{S}}}(w_T(0, X_S^x)) \\
&= \mathbb{E}(w_T(0, U_S^x))
\end{aligned} \tag{10}
$$

where U^x is the mild solution of the following equation defined $\forall t \in \mathbb{R}_+$:

$$
\mathrm{d}U_t^x = [AU_t^x + G\beta^{T,S}(t, U_t^x)]\mathrm{d}t + G\,\mathrm{d}W_t, \quad U_0^x = x.
$$

$T \longleftarrow T_i'$ and $S \longleftarrow (T_i - T_i')$, for all $x \in H$,

$$
w_{T_i}(0,x) = \mathbb{E}(w_{T_i'}(0, U^x_{T_i - T_i'})).
$$

Finally we prove that this convergence holds with an explicit speed of convergence. Let us write, $\forall x \in H, \forall T > 0$,

$$\begin{aligned} |w_T(0,x) - L| &= \lim_{V \to +\infty} |w_T(0,x) - w_V(0,x)| \\ &= \lim_{V \to +\infty} |w_T(0,x) - \mathbb{E}(w_T(0, U^x_{V-T}))| \end{aligned}$$

thanks to equality (10), where U^x is the mild solution of the following equation defined $\forall t \in \mathbb{R}_+$:

$$\mathrm{d}U^x_t = [AU^x_t + \beta^V(t, U^x_t)]\mathrm{d}t + G\,\mathrm{d}W_t, \quad U^x_0 = x.$$

Now, thanks to the third estimate in Lemma 4, one have,

$$\begin{aligned} |w_T(0,x) - L| &\leq \lim_{V \to +\infty} C\mathbb{E}\left(1 + |x|^{2+\mu} + |U^x_{V-T}|^{2+\mu}\right) e^{-\hat{\eta} T} \\ &\leq C(1 + |x|^{2+\mu}) e^{-\hat{\eta} T}. \end{aligned}$$

Thank you for your attention

Some additional references

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