Robust Duality without Refence Measure

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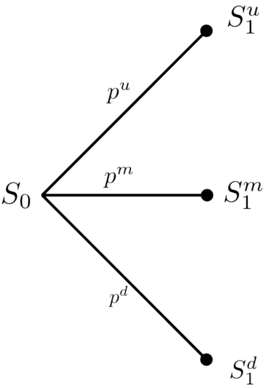
Arbitrage and superhedging on a probability space

Financial Market

- Probability space $(\Omega, \mathcal{F}, P)$
- Stock Price $S = (S_0, S_1)$
- Investment strategies ϑ
- Gains of trade:
  $G(\vartheta) = \vartheta \Delta S = \vartheta(S_1 - S_0)$
- Contingent claim ξ

Single period



minimal superhedging price

$$\inf\{x : x + \vartheta\Delta S \geq \xi \quad \text{for some strategy } \vartheta\}$$

$$=$$

$$\sup\{E_Q[\xi] : Q \sim P \text{ martingale measure}\}$$

largest non arbitrage price

Theorem (Drapeau et. al)

g convex, lsc and positive: for all $\xi \in L^\infty(P)$

$$\operatorname*{ess\,inf}\left\{ Y_t : Y_t - \int_t^T g(Y_u, Z_u)\,du + \int_t^T Z_u\,dW_u \geq \xi \right\}$$

$$=$$

$$\operatorname*{ess\,sup}_{Q \sim P, \beta} E_Q\left[e^{-\int_t^T \beta_u\,du}\xi - \int_t^T e^{-\int_t^u \beta_s\,ds} g^*(q_u, \beta_u)\,du \mid \mathcal{F}_t \right]$$

"Das Signal an die Praxis des Risikomanagements ist jedenfalls klar: sich nicht binden an ein einziges Modell, flexibel bleiben, die Modelle je nach Fragestellung variieren, immer mit Blick auf den 'worst case' "

Hans Föllmer, "Alles richtig und trotzdem falsch?", MDMV 2009

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$P \rightsquigarrow \mathcal{P}$: whole family of models

Goal

In the robust setting $(\Omega, \mathcal{F}, \mathcal{P})$

- Arbitrage free market $\overset{?}{\Longleftrightarrow}$ equivalent martingale measures
- Superhedging duality?

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Existing litterature

- Path-wise consideration: $\mathcal{P} = \{\delta_\omega : \omega \text{ possible path}\}$
  - Acciaio, et. al: finite discrete time, possibility to trade options.
  - Obłój: existence of a maximal set of paths
- Quasi-sure consideration: $\mathcal{P}$ general
  - Bouchard & Nutz: discrete time, possibility to trade finitely many options.
  - Bion-Nadal & Kervarec: Robust representation with capacity

Ansatz

$$\phi(\xi) = \inf \left\{ x : x + \int_0^T \vartheta \, dS \geq \xi \quad \text{for some strategy } \vartheta \right\}$$

$$=$$

$$\sup_{Q \text{ martingale measures}} E_Q[\xi]$$

$\rightsquigarrow$ coherent risk measures.

Risk Measures: $(\Omega, \mathcal{F})$

Given an ordered space $\mathcal{X}$, we consider a function

$$\phi : \mathcal{X} \to \mathbb{R}$$

such that:

(M) $\xi \geq \eta$ implies $\phi(\xi) \geq \phi(\eta)$

(T) $\phi(\xi + m) = \phi(\xi) + m$

(C) $\phi(\lambda\xi + (1-\lambda)\eta) \leq \lambda\phi(\xi) + (1-\lambda)\phi(\eta)$, $\lambda \in [0, 1]$.

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Applications:

- pricing: (El Karoui, Soner, Touzi)
- risk management: (Artzner et. al, Föllmer et al, Frittelli et al)
- utility: (Delbaen, Maccheroni et. al.)
- ...

Representation

$$\phi(\xi) = \sup_{Q \in \mathcal{M}_1(P)} \{E_Q[\xi] - \phi^*(Q)\}$$

- $(\Omega, \mathcal{F}, P)$ probability space
- $\mathcal{X} = L^\infty(P)$
- ϕ satisfies the Fatou property $\leadsto \sigma(L^\infty(P), L^1(P))$-lsc
  $\leadsto$ Delbaen, Föllmer and Schied

Representation

$$\phi(\xi) = \sup_{Q \in \mathcal{M}_{1,f}} \{E_Q[\xi] - \phi^*(Q)\}$$

- Ω polish space
- $\mathcal{X} = B(\Omega)$ bounded measurable functions
- Proof: ϕ norm-continuous, $B(\Omega)^* = ba(\Omega)$ and Fenchel-Moreau

Representation

$$\phi(\xi) = \sup_{Q \in \mathcal{M}_1} \{E_Q[\xi] - \phi^*(Q)\}$$

- Ω polish space and compact
- $\mathcal{X} = C(\Omega)$ continuous functions
- Proof: ϕ norm-continuous, $C(\Omega)^* = ca(\Omega)$ and Fenchel-Moreau

Representation

$$\phi(\xi) = \sup_{Q \in \mathcal{M}_1} \{E_Q[\xi] - \phi^*(Q)\}$$

- Ω polish space
- $\mathcal{X} = C_b(\Omega)$ continuous bounded functions
- ϕ tight: for an increasing sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compacts,

$$\phi(\lambda 1_{K_n^c}) \to \phi(0) \quad \text{for all } \lambda \geq 1$$

$\rightsquigarrow$ Föllmer and Schied

- Ω polish space and $\Omega = \cup K_n$, K_n compact
- $\mathcal{X} = C_b(\Omega)$ continuous bounded functions

Theorem

Suppose that ϕ satisfies

- $\phi(\xi^n) \uparrow \phi(\xi)$ *for all* $\xi^n \uparrow \xi$ *and* $\xi^n = \xi$ *on* K_n

Then

$$\phi(\xi) = \sup_{Q \in \mathcal{M}_1} \{E_Q[\xi] - \phi^*(Q)\}$$

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Then

$$\phi(\xi) \leq \sup_{Q \in \mathcal{M}_1} \{E_Q[\xi] - \phi^*(Q)\}$$

for all $\xi \in usc_b(\Omega)$ *upper semi-continuous bounded function.*

Fundamental theorem of asset pricing

- $\Omega = C([0, T])$
- Stock price: S, canonical process
- Information: canonical filtration
- Strategies: simple processes

$$\vartheta = \sum_{i=1}^{N-1} \alpha_i 1_{(\tau_i, \tau_{i+1}]} \quad , \alpha_i \in C_b(\mathcal{F}_{\tau_i})$$

- Gains from trading:

$$G(\vartheta) := \sum_{i=1}^{N-1} \alpha_i (S_{\tau_{i+1}} - S_{\tau_i}) = \int_0^T \vartheta \, dS$$

- Set of reference measures: $\mathcal{P} \subseteq \mathcal{M}$
- $\Omega = \cup_n K_n \ \mathcal{P}$ -q.s.

$$\xi^1 = \xi^2 \, \mathcal{P}\text{-q.s.} \quad \text{if and only if} \quad \xi^1 = \xi^2 \; P\text{-a.s. for all } P \in \mathcal{P}$$

- Contingent claims:

$$\|\xi\|_\infty := \inf\{m : \sup_{P \in \mathcal{P}} P(|\xi| \geq m) = 0\} < \infty$$

$\rightsquigarrow L^\infty(\mathcal{P})$

- $\mathcal{M}(S, \mathcal{P})$ = local-martingale measures Q s.t. $Q \ll \mathcal{P}$

1. Ω is compact

2. $\mathcal{P}$ is tight

3. Further Examples

   a) $\Omega = \mathbb{R}^d, \quad K_n = [-n, n]^d$

   b) $\Omega = C([0, T]; \mathbb{R})$

   $$K_n = \left\{ w : [0, T] \to \mathbb{R} : \|w\|_\infty \le n, \sup_{s \neq t} \frac{|w(s) - w(t)|}{|s-t|^{1/n}} \le 1/n \right\}$$

   $\Omega = \bigcup_{n \ge 1} K_n$ $\mathcal{P}$-q.s., where the probabilistic models in $\mathcal{P}$ are supported on Hölder continuous paths.

The market admits a *free lunch with disappearing risk (FLDR)* if there exists $\xi \in L^\infty(\mathcal{P})_+$ with $P(\xi > 0) > 0$ for some $P \in \mathcal{P}$ such that for every counting measure $(q_j) \in l^1_+$, there exists a sequence of strategies (ϑ^n) with

$$\sum_j q_j \left\| \left(\int_0^T \vartheta^n \, dS - \xi \right)^- 1_{K_{j+1} \setminus K_j} \right\|_\infty \longrightarrow 0.$$

Theorem

The following are equivalent:

(i) *The market does not admit FLDR*

(ii) $\mathcal{M}(S,\mathcal{P})$ *is non-empty and* $\mathcal{M}(S,\mathcal{P}) \sim \mathcal{P}$

Superhedging under model uncertainty

S continuous process.

- $\phi(\xi) = \inf\left\{x : x + \int_0^T \vartheta \, dS \geq \xi\right\} \geq \sup_{Q \in \mathcal{M}(S,\mathcal{P})} E_Q[\xi]$
- if $m > \sup_{Q \in \mathcal{M}(S,\mathcal{P})} E_Q[\xi]$,

FTAP on $(m - \xi, S) \Longrightarrow$ there exists a strategy ϑ such that

$$m + \int_0^T \vartheta \, dS \geq \xi.$$

$\Longrightarrow \phi(\xi) \leq \sup_{Q \in \mathcal{M}(S,\mathcal{P})} E_Q[\xi]$

- For $\Omega = D([0, T])$, S càdlàg,

Further assumption: we need $\mathcal{M}(S, \mathcal{P})$ locally of compact support.

$$\rightsquigarrow \phi(\xi) = \sup_{Q \in \mathcal{M}(S,\mathcal{P})} E_Q[\xi], \quad \xi \in L^\infty(\mathcal{P})$$

Summary

- We discussed robust representation results for convex risk measures.
- Based on these duality results we derive a FTAP when there is no reference probability model.
- Robust superhedging duality

Thank You!