Optimal Control by Franchise and Deductible Amounts in the Classical Risk Model

Olena Ragulina

Taras Shevchenko National University of Kyiv, Ukraine

7 - 9 July, 2014

Overview

1 Preliminaries

2 Optimal Control by the Franchise Amount

3 Optimal Control by the Deductible Amount

Classical Risk Model

- the insurance company has an initial surplus $x > 0$
- $X_t(x)$ is a surplus at time $t \geq 0$ provided that the initial surplus equals x

Premium arrivals

- the insurance company receives premiums with constant intensity $c > 0$

Claim arrivals

- **the claim sizes** form a sequence $(Y_i)_{i \geq 1}$ of nonnegative i.i.d. random variables with c.d.f. $F(y) = \mathbb{P}[Y_i \leq y]$ and finite expectations μ; τ_i is the time when the ith claim arrives
- **the number of claims** on the time interval $[0, t]$ is a Poisson process $(N_t)_{t \geq 0}$ with constant intensity $\lambda > 0$; the random variables Y_i, $i \geq 1$, and the process $(N_t)_{t \geq 0}$ are independent
- **the total claims** on $[0, t]$ equal $\sum_{i=1}^{N_t} Y_i$; we set $\sum_{i=1}^{0} Y_i = 0$ if $N_t = 0$

Classical Risk Model

The surplus of the insurance company at time t equals

$$X_t(x) = x + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{1}$$

We assume that **the net profit condition** holds, i.e.

$$c > \lambda\mu.$$

The insurance company uses **the expected value principle** for premium calculation, i.e.

$$c = \lambda\mu(1+\theta),$$

where $\theta > 0$ is **a safety loading.**

Optimal Control Problems

- **optimal control by investments:** C. Hipp, M. Plum (2000); C. Hipp, M. Plum (2003); C. S. Liu, H. Yang (2004); P. Azcue, N. Muler (2009)
- **optimal control by reinsurance:** H. Schmidli (2001); C. Hipp, M. Vogt (2003)
- **optimal control by investments and reinsurance:** H. Schmidli (2002); M. I. Taksar, C. Markussen (2003); S. D. Promislow, V. R. Young (2005); C. Hipp, M. Taksar (2010)

Franchise and Deductible

Definitions

- **A franchise** is a provision in the insurance policy whereby the insurer does not pay unless damage exceeds the franchise amount.
- **A deductible** is a provision in the insurance policy whereby the insurer pays any amounts of damage that exceed the deductible amount.

Example 1

The franchise/deductible amount is 10.

- **Case 1:** the claim size is 5
  - If the franchise is used, then the insurance company pays nothing.
  - If the deductible is used, then the insurance company pays nothing.
- **Case 2:** the claim size is 100
  - If the franchise is used, then the insurance company pays 100.
  - If the deductible is used, then the insurance company pays $100 - 10 = 90$.

Motivation

- a franchise and a deductible are applied when the insured's losses are relatively small to deter a large number of trivial claims
- a deductible encourages the insured to take more care of the insured property

Additional assumptions

- the insurance company adjusts the franchise amount d_t at every time $t \geq 0$ on the basis of the information available before time t, i.e. every admissible strategy $(d_t)_{t\geq 0}$ ((d_t) for brevity) of the franchise amount choice is a predictable process w.r.t. the natural filtration generated by $(N_t)_{t\geq 0}$ and $(Y_i)_{i\geq 1}$
- $0 \leq d_t \leq d_{\max}$, where $d_{\max}$ is the maximum allowed franchise amount such that $0 < F(d_{\max}) < 1$; in particular, if $d_t = 0$, then the franchise is not used at time t
- the safety loading $\theta > 0$ is constant

The premium intensity at time t depends on the franchise amount at this time and it is given by

$$c(d_t) = \lambda(1+\theta) \int_{d_t}^{+\infty} y \, \mathrm{d}F(y).$$

Optimal Control by the Franchise Amount

Let $X_t^{(d_t)}(x)$ be the surplus of the insurance company at time t provided its initial surplus is x and the strategy (d_t) is used. Then

$$X_t^{(d_t)}(x) = x + \int_0^t c(d_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} Y_i \,\mathbb{I}_{\{Y_i > d_{\tau_i}\}}, \quad t \geq 0. \tag{2}$$

The ruin time under the admissible strategy (d_t) is defined as

$$\tau^{(d_t)}(x) = \inf\big\{t \geq 0\colon X_t^{(d_t)}(x) < 0\big\}.$$

The corresponding **infinite-horizon survival probability** is given by

$$\varphi^{(d_t)}(x) = \mathbb{P}\big[\tau^{(d_t)}(x) = \infty\big].$$

Optimal Control by the Franchise Amount

Our **aim** is to maximize the survival probability over all admissible strategies (d_t), i.e. to find

$$\varphi^*(x) = \sup_{(d_t)} \varphi^{(d_t)}(x),$$

and show that there exists an optimal strategy (d_t^*) such that $\varphi^*(x) = \varphi^{(d_t^*)}(x)$ for all $x \geq 0$.

The optimal strategy will be a function of the initial surplus only.

Hamilton-Jacobi-Bellman Equation

Proposition 1

Let the surplus process $\left(X_t^{(d_t)}(x)\right)_{t\geq 0}$ follow (2) under the above assumptions. If $\varphi^*(x)$ is differentiable on $\mathbb{R}_+$, then it satisfies the Hamilton-Jacobi-Bellman equation

$$\sup_{d\in[0,\, d_{\max}]}\Bigg((1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)\,\left(\varphi^*(x)\right)' + \left(F(d)-1\right)\varphi^*(x) + \int_d^{d\vee x}\varphi^*(x-y)\,\mathrm{d}F(y)\Bigg) = 0, \tag{3}$$

which is equivalent to

$$\left(\varphi^*(x)\right)' = \inf_{d\in[0,\, d_{\max}]}\left(\frac{\left(1-F(d)\right)\varphi^*(x) - \int_d^{d\vee x}\varphi^*(x-y)\,\mathrm{d}F(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)}\right). \tag{4}$$

Hamilton-Jacobi-Bellman Equation

Remark 1

Note that if there exists one solution to (3) or (4), then there exist infinitely many solutions to these equations which differ with a multiplicative constant.

Existence Theorem

Theorem 1

If the random variables Y_i, $i \geq 1$, have a p.d.f. $f(y)$, then there exists the solution $G(x)$ to (4) with $G(0) = \theta/(1+\theta)$, which is nondecreasing and continuously differentiable on $\mathbb{R}_+$, and $\theta/(1+\theta) \leq \lim_{x\to+\infty} G(x) \leq 1$.

The solution $G(x)$ to (4) that satisfies the conditions of Theorem 1 can be found as the limit of the sequence of functions $\big(G_n(x)\big)_{n\geq 0}$ on $\mathbb{R}_+$, where $G_0(x) = \varphi^{(0)}(x)$ is the survival probability provided that $d_t = 0$ for all $t \geq 0$, and

$$G_n'(x) = \inf_{d\in[0,\, d_{\max}]} \left(\frac{\big(1 - F(d)\big)\, G_{n-1}(x) - \int_d^{d\vee x} G_{n-1}(x-y)\,\mathrm{d}F(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)} \right),$$
$$G_n(0) = \theta/(1+\theta), \quad n \geq 1. \tag{5}$$

Verification Theorem

Theorem 2

Let the surplus process $\left(X_t^{(d_t)}(x)\right)_{t\geq 0}$ follow (2) and $G(x)$ be the solution to (4) that satisfies the conditions of Theorem 1. Then for any $x \geq 0$ and arbitrary admissible strategy (d_t), we have

$$\varphi^{(d_t)}(x) \leq \frac{G(x)}{\lim_{x\to+\infty} G(x)}, \tag{6}$$

and equality in (6) is attained under the strategy $(d_t^*) = \left(d_t^*\left(X_{t-}^{(d_t^*)}(x)\right)\right)$, where $\left(d_t^*(x)\right)$ minimizes the right-hand side of (4), i.e.

$$\varphi^*(x) = \varphi^{(d_t^*)}(x) = \frac{G(x)}{\lim_{x\to+\infty} G(x)}.$$

Verification Theorem

Remark 2

In Theorem 2 we used any solution to (4) that satisfies the conditions of Theorem 1. However, Theorem 2 also implies uniqueness of such a solution. The corresponding strategy (d_t^*) may not be unique in the general case.

Exponentially Distributed Claim Sizes

Theorem 3

Let the surplus process $\left(X_t^{(d_t)}(x)\right)_{t\geq 0}$ follow (2), the claim sizes be exponentially distributed with mean μ, and $d_{\max} = \mu$. Then the strategy (d_t) with $d_t = 0$ for all $t \geq 0$ is not optimal.

Remark 3

Theorem 3 implies that we can always increase the survival probability adjusting the franchise amount if the claim sizes are exponentially distributed.

Exponentially Distributed Claim Sizes

Example 2

If the claim sizes are exponentially distributed with mean $\mu = 10$, $d_{\max} = \mu$, and $\theta = 0.1$, then

$$\varphi^{(0)}(x) \approx 1 - 0.9090909\,\mathrm{e}^{-x/110}\,, \quad x \geq 0\,,$$

$$\varphi^{*}(x) \approx \begin{cases} 0.111048767\,\mathrm{e}^{x/22} & \text{if} \quad x \leq 8.93258, \\ 1 - 0.90382792\,\mathrm{e}^{x/110} & \text{if} \quad x > 8.93258, \end{cases}$$

$$d_t^{*}(x) = \begin{cases} 10 & \text{if} \quad x \leq 8.93258, \\ 0 & \text{if} \quad x > 8.93258. \end{cases}$$

Constant Franchise Amount

Theorem 4

Let the surplus process $\big(X_t^{(d_t)}(x)\big)_{t\ge 0}$ follow (2) with $d_t \equiv d$ where $d > 0$ is constant, and the claim sizes be exponentially distributed with mean μ. Then $\varphi^{(d)}(x) = \varphi_{n+1}^{(d)}(x)$ for all $x \in [nd, (n+1)d)$, $n \ge 0$, where

$$\varphi_1^{(d)}(x) = C_{1,1}\,\mathrm{e}^{x/\gamma}, \qquad \varphi_2^{(d)}(x) = \big(C_{2,1} + A_{2,0}\,x\big)\,\mathrm{e}^{x/\gamma} + C_{2,2}\,\mathrm{e}^{-x/\mu},$$

$$\varphi_{n+1}^{(d)}(x) = \left(C_{n+1,1} + \sum_{i=0}^{n-1} A_{n+1,i}\,x^{i+1}\right)\mathrm{e}^{x/\gamma} + \left(C_{n+1,2} + \sum_{i=0}^{n-2} B_{n+1,i}\,x^{i+1}\right)\mathrm{e}^{-x/\mu}, \quad n \ge 2.$$

Constant Franchise Amount

Theorem 4

Here

$$\gamma = (1+\theta)(\mu+d),$$
$$C_{1,1} = \theta/(1+\theta),$$
$$A_{2,0} = -\frac{\theta}{(1+\theta)(\gamma+\mu)}\,\mathrm{e}^{-d/\gamma},$$
$$C_{2,1} = \frac{\theta}{1+\theta}\left(1+\frac{\gamma\mu+d(\gamma+\mu)}{(\gamma+\mu)^2}\,\mathrm{e}^{-d/\gamma}\right),$$
$$C_{2,2} = -\frac{\theta\gamma\mu}{(1+\theta)(\gamma+\mu)^2}\,\mathrm{e}^{d/\mu},$$

and other coefficients can be found by recurrent formulas.

Constant Franchise Amount

Theorem 5

Let conditions of Theorem 4 hold.

- If $d > 0$ and $x \in \left[0,\ \min\left\{\frac{\mu(1+\theta)}{\theta}\ln\left(1+\frac{\theta d}{\mu(1+\theta)}\right),\ d\right\}\right]$, then the correspoding survival probability is less than the classical one.
- If $d \in \left(0,\ \frac{\mu(1+\theta)\ln(1+\theta)}{\theta}\right)$ and x is large enough, then the correspoding survival probability is greater than the classical one.

Additional assumptions

- the insurance company adjusts the deductible amount $\bar{d}_t$ at every time $t \geq 0$ on the basis of the information available before time t, i.e. every admissible strategy $(\bar{d}_t)_{t\geq 0}$ ($(\bar{d}_t)$ for brevity) of the deductible amount choice is a predictable process w.r.t. the natural filtration generated by $(N_t)_{t\geq 0}$ and $(Y_i)_{i\geq 1}$
- $0 \leq \bar{d}_t \leq \bar{d}_{\max}$, where $\bar{d}_{\max}$ is the maximum allowed deductible amount such that $0 < F(\bar{d}_{\max}) < 1$; in particular, if $\bar{d}_t = 0$, then the deductible is not used at time t
- the safety loading $\theta > 0$ is constant

The premium intensity at time t depends on the deductible amount at this time and it is given by

$$c(\bar{d}_t) = \lambda(1+\theta)\int_{\bar{d}_t}^{+\infty}(y - \bar{d}_t)\,\mathrm{d}F(y).$$

Optimal Control by the Deductible Amount

Let $X_t^{(\bar{d}_t)}(x)$ be the surplus of the insurance company at time t provided its initial surplus is x and the strategy $(\bar{d}_t)$ is used. Then

$$X_t^{(\bar{d}_t)}(x) = x + \int_0^t c(\bar{d}_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} (Y_i - \bar{d}_{\tau_i})^+. \tag{7}$$

The ruin time under the admissible strategy $(\bar{d}_t)$ is defined as

$$\tau^{(\bar{d}_t)}(x) = \inf\big\{t \geq 0\colon X_t^{(\bar{d}_t)}(x) < 0\big\}.$$

The corresponding **infinite-horizon survival probability** is given by

$$\varphi^{(\bar{d}_t)}(x) = \mathbb{P}\big[\tau^{(\bar{d}_t)}(x) = \infty\big].$$

Optimal Control by the Deductible Amount

Our **aim** is to maximize the survival probability over all admissible strategies $(\bar{d}_t)$, i.e. to find

$$\varphi^*(x) = \sup_{(\bar{d}_t)} \varphi^{(\bar{d}_t)}(x),$$

and show that there exists an optimal strategy $(\bar{d}_t^*)$ such that $\varphi^*(x) = \varphi^{(\bar{d}_t^*)}(x)$ for all $x \geq 0$.

Hamilton-Jacobi-Bellman Equation

Proposition 2

Let the surplus process $\left(X_t^{(\bar{d}_t)}(x)\right)_{t\geq 0}$ follow (7) under the above assumptions. If $\varphi^*(x)$ is differentiable on $\mathbb{R}_+$, then it satisfies the Hamilton-Jacobi-Bellman equation

$$\sup_{\bar{d}\in[0,\,\bar{d}_{\max}]}\left((1+\theta)\int_{\bar{d}}^{+\infty}(y-\bar{d})\,\mathrm{d}F(y)\,\left(\varphi^*(x)\right)' + \left(F(\bar{d})-1\right)\varphi^*(x) + \int_{\bar{d}}^{x+\bar{d}}\varphi^*(x+\bar{d}-y)\,\mathrm{d}F(y)\right) = 0, \tag{8}$$

which is equivalent to

$$\left(\varphi^*(x)\right)' = \inf_{\bar{d}\in[0,\,\bar{d}_{\max}]}\left(\frac{\left(1-F(\bar{d})\right)\varphi^*(x) - \int_{\bar{d}}^{x+\bar{d}}\varphi^*(x+\bar{d}-y)\,\mathrm{d}F(y)}{(1+\theta)\int_{\bar{d}}^{+\infty}(y-\bar{d})\,\mathrm{d}F(y)}\right). \tag{9}$$

Existence Theorem

Theorem 6

If the random variables Y_i, $i \geq 1$, have a p.d.f. $f(y)$, then there exists the solution $G(x)$ to (9) with $G(0) = \theta/(1+\theta)$, which is nondecreasing and continuously differentiable on $\mathbb{R}_+$, and $\theta/(1+\theta) \leq \lim_{x \to +\infty} G(x) \leq 1$.

The solution $G(x)$ to (9) that satisfies the conditions of Theorem 6 can be found as the limit of the sequence of functions $\big(G_n(x)\big)_{n \geq 0}$ on $\mathbb{R}_+$, where $G_0(x) = \varphi^{(0)}(x)$ is the survival probability provided that $d_t = 0$ for all $t \geq 0$, and

$$G_n'(x) = \inf_{\bar{d} \in [0,\, \bar{d}_{\max}]} \left(\frac{(1 - F(\bar{d}))\, G_{n-1}(x) - \int_{\bar{d}}^{x+\bar{d}} G_{n-1}(x + \bar{d} - y)\, \mathrm{d}F(y)}{(1+\theta) \int_{\bar{d}}^{+\infty} (y - \bar{d})\, \mathrm{d}F(y)} \right),$$
$$G_n(0) = \theta/(1+\theta), \quad n \geq 1. \tag{10}$$

Verification Theorem

Theorem 7

Let the surplus process $\left(X_t^{(\bar{d}_t)}(x)\right)_{t\geq 0}$ follow (7) and $G(x)$ be the solution to (9) that satisfies the conditions of Theorem 6. Then for any $x \geq 0$ and arbitrary admissible strategy $(\bar{d}_t)$, we have

$$\varphi^{(\bar{d}_t)}(x) \leq \frac{G(x)}{\lim_{x\to+\infty} G(x)}, \tag{11}$$

and equality in (11) is attained under the strategy $(\bar{d}_t^*) = \left(\bar{d}_t^*(X_{t_-}^{(\bar{d}_t^*)}(x))\right)$, where $\left(\bar{d}_t^*(x)\right)$ minimizes the right-hand side of (9), i.e.

$$\varphi^*(x) = \varphi^{(\bar{d}_t^*)}(x) = \frac{G(x)}{\lim_{x\to+\infty} G(x)}.$$

Exponentially Distributed Claim Sizes

Theorem 8

Let the surplus process $\left(X_t^{(\bar{d}_t)}(x)\right)_{t\geq 0}$ follow (7) and the claim sizes be exponentially distributed. Then $\varphi^*(x) = \varphi^{(\bar{d}_t)}(x)$ for every admissible strategy $(\bar{d}_t)$, i.e. every admissible strategy is optimal.

Remark 4

Theorem 8 implies that we cannot increase the survival probability adjusting the deductible amount for the exponentially distributed claim sizes.

References

P. Azcue, N. Muler (2009)
Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constrains
Insurance: Mathematics and Economics 44(1), 26–34.

C. Hipp, M. Plum (2000)
Optimal investment for insurers
Insurance: Mathematics and Economics 27(2), 215–228.

C. Hipp, M. Plum (2003)
Optimal investment for investors with state dependent income, and for insurers
Finance and Stochastics 7(3), 299–321.

C. Hipp, M. Taksar (2010)
Optimal non-proportional reinsurance control
Insurance: Mathematics and Economics 47(2), 246–254.

References

C. Hipp, M. Vogt (2003)
Optimal dynamic XL reinsurance
ASTIN Bulletin 33(2), 193–207.

C. S. Liu, H. Yang (2004)
Optimal investment for an insurer to minimize its probability of ruin
North American Actuarial Journal 8(2), 11–31.

S. D. Promislow, V. R. Young (2005)
Minimizing the probability of ruin when claims follow Brownian motion with drift
North American Actuarial Journal 9(3), 109–128.

O. Ragulina (2014)
Maximization of the survival probability by franchise and deductible amounts in the classical risk model
Springer Optimization and Its Applications 90, 287–300.

References

H. Schmidli (2001)
Optimal proportional reinsurance polisies in a dynamic setting
Scandinavian Actuarial Journal 2001(1), 55–68.

H. Schmidli (2002)
On minimizing the ruin probability by investment and reinsurance
The Annals of Applied Probability 12(3), 890–907.

M. I. Taksar, C. Markussen (2003)
Optimal dynamic reinsurance policies for large insurance portfolios
Finance and Stochastics 7(1), 97–121.

Thank you very much for your attention!