# Regularity of Constrained BSDEs

## 2nd Young Researchers Meeting on BSDEs, Numerics and Finance


Ludovic Moreau

ETH, Zürich – ludovic.moreau@math.ethz.ch


Joint work with Bruno Bouchard and Romuald Elie

July, 07th 2014

# Table of content

# Table of content

# Introduction

Given a forward process

$$X^{t,x} = x + \int_t^{\cdot} b_s\left(X_s^{t,x}\right) ds + \int_t^{\cdot} \sigma_s\left(X_s^{t,x}\right) dW_s.$$

We are interested in the minimal solution of the constrained BSDE

$$\mathcal{Y}^{t,x} \geq g\left(X_T^{t,x}\right) + \int_{\cdot}^{T} f_s\left(X_s^{t,x}, \mathcal{Y}_s^{t,x}, \mathcal{Z}_s^{t,x}\right) ds - \int_{\cdot}^{T} \mathcal{Z}_s^{t,x} dW_s$$

with

$$\mathcal{Z}^{t,x}\sigma^{-1}\left(X^{t,x}\right) \in K \quad dt \otimes d\mathbb{P}\text{-a.e.}$$

with $K$ convex set of support function $\delta(u) := \sup\{k^\top u\ ;\ k \in K\}$.

# Introduction

Given a forward process

$$X^{t,x} = x + \int_t^{\cdot} b_s\left(X_s^{t,x}\right) ds + \int_t^{\cdot} \sigma_s\left(X_s^{t,x}\right) dW_s.$$

We are interested in the minimal solution of the constrained BSDE

$$\mathcal{Y}^{t,x} \geq g\left(X_T^{t,x}\right) + \int_{\cdot}^{T} f_s\left(X_s^{t,x}, \mathcal{Y}_s^{t,x}, \mathcal{Z}_s^{t,x}\right) ds - \int_{\cdot}^{T} \mathcal{Z}_s^{t,x} dW_s$$

with

$$\mathcal{Z}^{t,x}\sigma^{-1}\left(X^{t,x}\right) \in K \quad dt \otimes d\mathbb{P}\text{-a.e.}$$

with $K$ convex set of support function $\delta(u) := \sup\{k^\top u \ ; \ k \in K\}$.

More precisely, we are interested in the regularity of

$$(t,x) \in [0,T] \times \mathbb{R}^d \longmapsto \mathcal{Y}_t^{t,x}.$$

# Table of content

# Motivation

- Super-Replication in Differential Games (Bouchard and Nutz) ;

# Motivation

- Super-Replication in Differential Games (Bouchard and Nutz) ;
- Numerical Schemes ;

# Motivation

- Super-Replication in Differential Games (Bouchard and Nutz) ;
- Numerical Schemes ;
- A priori regularity for PPDEs.

# Motivation

## Super–Replication in Differential Games (1)

- We want to extend

$$v(t,x,p) := \inf\left\{y:\ \exists\ \nu \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}$$

to

$$\inf\left\{y:\ \exists\ ``\nu \in K" \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}.$$

# Motivation

## Super–Replication in Differential Games (1)

- We want to extend

$$v(t,x,p) := \inf\left\{y:\ \exists\ \nu \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}$$

  to

$$\inf\left\{y:\ \exists\ \text{"}\nu \in K\text{"} \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}.$$

- Difficulty to establish DPP ;

# Motivation

## Super–Replication in Differential Games (1)

- We want to extend

$$v(t,x,p) := \inf\left\{y:\ \exists\ \nu \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}$$

  to

$$\inf\left\{y:\ \exists\ \text{"}\nu \in K\text{"} \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}.$$

- Difficulty to establish DPP ;
- In Bouchard, M. & Nutz, done by covering $\oplus$ continuity ;

# Motivation

## Super–Replication in Differential Games (1)

- We want to extend

$$v(t,x,p) := \inf\left\{y :\ \exists\ \nu \text{ s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}$$

  to

$$\inf\left\{y :\ \exists\ \text{"}\nu \in K\text{" s.t. } \mathbb{E}\left[\Psi\left(X_T^{t,x,\nu,\alpha}, Y_T^{t,x,y,\nu,\alpha}\right)\right] \geq p\ \forall\ \alpha\right\}.$$

- Difficulty to establish DPP ;
- In Bouchard, M. & Nutz, done by covering $\oplus$ continuity ;
- This does not allow to consider the Super-Replication :

$$v(t,x) := \inf\left\{y :\ \exists\ \text{"}\nu \in K\text{" s.t. } Y_T^{t,x,y,\nu,\alpha} \geq g\left(X_T^{t,x,\nu,\alpha}\right)\ \forall\ \alpha\right\}$$

  where $\Psi(x,y) := \mathbf{1}_{\{y \geq g(x)\}}$ and $p = 1$.

# Motivation

## Super–Replication in Differential Games (2)

In Bouchard & Nutz, they establish the DPP in the SR case by

1. Constructing a smooth super-solution $w$ "not so far" from $v$ ;

# Motivation

## Super–Replication in Differential Games (2)

In Bouchard & Nutz, they establish the DPP in the SR case by

1. Constructing a smooth super-solution $w$ "not so far" from $v$ ;
2. Construct an almost optimal control based on $w$ and verification type arguments ;

# Motivation

## Super–Replication in Differential Games (2)

In Bouchard & Nutz, they establish the DPP in the SR case by

1. Constructing a smooth super-solution $w$ "not so far" from $v$ ;
2. Construct an almost optimal control based on $w$ and verification type arguments ;
3. Deduce the DPP for $v$ using this Markovian Control.

# Motivation

## Super–Replication in Differential Games (2)

In Bouchard & Nutz, they establish the DPP in the SR case by

1. Constructing a smooth super-solution $w$ "not so far" from $v$ ;
2. Construct an almost optimal control based on $w$ and verification type arguments ;
3. Deduce the DPP for $v$ using this Markovian Control.

The point $(1)$ need to have continuity for an object of this type

$$(t,x) \longmapsto \sup_{\alpha} \mathcal{Y}_t^{t,x,\alpha}.$$

# Motivation

## Regularity for Solution of PPDE

- Singular Control Problem :

$$v^{\varepsilon}(t,\omega,x) := \sup_{L} \mathbb{E}\left[U\left(X_T^{t,\omega,x,L,\varepsilon}\right)\right]$$

with

$$X^{t,\omega,x,L,\varepsilon} = x + \int_t^{\cdot} b_s^{t,\omega}\left(X_s\right)ds + \int_t^{\cdot} \sigma_s^{t,\omega}\left(X_s\right)dW_s + \int_t^{\cdot} f(\varepsilon)dL_s;$$

# Motivation

## Regularity for Solution of PPDE

- Singular Control Problem :

$$v^{\varepsilon}(t,\omega,x) := \sup_{L} \mathbb{E}\left[U\left(X_T^{t,\omega,x,L,\varepsilon}\right)\right]$$

with

$$X^{t,\omega,x,L,\varepsilon} = x + \int_t^{\cdot} b_s^{t,\omega}\left(X_s\right) ds + \int_t^{\cdot} \sigma_s^{t,\omega}\left(X_s\right) dW_s + \int_t^{\cdot} f(\varepsilon) dL_s;$$

- Family of non-dominated measure indexed by $L$ which is not-bounded ;

# Motivation

## Regularity for Solution of PPDE

- Singular Control Problem :

$$v^{\varepsilon}(t,\omega,x) := \sup_{L} \mathbb{E}\left[U\left(X_T^{t,\omega,x,L,\varepsilon}\right)\right]$$

with

$$X^{t,\omega,x,L,\varepsilon} = x + \int_t^{\cdot} b_s^{t,\omega}(X_s)\,ds + \int_t^{\cdot} \sigma_s^{t,\omega}(X_s)\,dW_s + \int_t^{\cdot} f(\varepsilon) dL_s;$$

- Family of non-dominated measure indexed by $L$ which is not-bounded ;
- BUT "PPDE of quasi-variational" type ;

# Motivation

## Regularity for Solution of PPDE

- Singular Control Problem :

$$v^{\varepsilon}(t,\omega,x) := \sup_{L} \mathbb{E}\left[U\left(X_T^{t,\omega,x,L,\varepsilon}\right)\right]$$

with

$$X^{t,\omega,x,L,\varepsilon} = x + \int_t^{\cdot} b_s^{t,\omega}(X_s)\,ds + \int_t^{\cdot} \sigma_s^{t,\omega}(X_s)\,dW_s + \int_t^{\cdot} f(\varepsilon)dL_s;$$

- Family of non-dominated measure indexed by $L$ which is not-bounded ;
- BUT "PPDE of quasi-variational" type ;
- No-Trade Region : the optimal control $L \equiv 0$ ;

# Motivation

## Regularity for Solution of PPDE

- Singular Control Problem :

$$v^{\varepsilon}(t,\omega,x) := \sup_{L} \mathbb{E}\left[U\left(X_T^{t,\omega,x,L,\varepsilon}\right)\right]$$

with

$$X^{t,\omega,x,L,\varepsilon} = x + \int_t^{\cdot} b_s^{t,\omega}\left(X_s\right)ds + \int_t^{\cdot} \sigma_s^{t,\omega}\left(X_s\right)dW_s + \int_t^{\cdot} f(\varepsilon)dL_s;$$

- Family of non-dominated measure indexed by $L$ which is not-bounded ;
- BUT "PPDE of quasi-variational" type ;
- No-Trade Region : the optimal control $L \equiv 0$ ;
- Need for regularity of $(t,\omega,x) \longmapsto v^{\varepsilon}(t,\omega,x)$ ;

# Motivation

## Regularity for Solution of PPDE

- Singular Control Problem :

$$v^{\varepsilon}(t,\omega,x) := \sup_{L} \mathbb{E}\left[U\left(X_T^{t,\omega,x,L,\varepsilon}\right)\right]$$

with

$$X^{t,\omega,x,L,\varepsilon} = x + \int_t^{\cdot} b_s^{t,\omega}(X_s)\,ds + \int_t^{\cdot} \sigma_s^{t,\omega}(X_s)\,dW_s + \int_t^{\cdot} f(\varepsilon)dL_s;$$

- Family of non-dominated measure indexed by $L$ which is not-bounded ;
- BUT "PPDE of quasi-variational" type ;
- No-Trade Region : the optimal control $L \equiv 0$ ;
- Need for regularity of $(t,\omega,x) \longmapsto v^{\varepsilon}(t,\omega,x)$ ;
- Done by showing that $\mathcal{Y}^{t,\omega,x,\varepsilon} = v^{\varepsilon}(t,\omega,x)$.

# Table of content

# Dual Formulation ($\zeta := (t, x)$)

## Equivalent families of Measure

For each predictable bounded process $\nu$ we introduce

$$\frac{d\mathbb{P}^{t,x,\nu}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_\tau^T |\sigma^{-1}(X_s^{t,x})\nu_s|^2 ds + \int_\tau^T \sigma^{-1}(X_s^{t,x})\nu_s dW_s}$$

$$W^{t,x,\nu} := W - \int_\tau^{\tau\vee\cdot} \sigma^{-1}(X_s^{t,x})\nu_s ds$$

so that

$$X^{t,x} = x + \int_t^\cdot \left[b\left(X_s^{t,x}\right) + \nu_s\right] ds + \int_t^\cdot \sigma\left(X_s^{t,x}\right) dW_s^{t,x,\nu}.$$

# Dual Formulation

## The family of standard BSDEs

Let $(\bar{Y}^{t,x,\nu}, \bar{Z}^{t,x,\nu})$ be the unique $\mathbf{S}^2(\mathbb{P}^{t,x,\nu}) \times \mathbf{H}^2(\mathbb{P}^{t,x,\nu})$-solution of

$$\bar{Y}^{\zeta,\nu} = g\left(X_T^{\zeta}\right) + \int_{\cdot}^{T} \left[ f\left(X_s^{\zeta}, \bar{Y}_s^{\zeta,\nu}, \bar{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s) \right] ds - \int_{\cdot}^{T} \bar{Z}_s^{\zeta,\nu} dW_s^{\zeta,\nu}.$$

Then

$$\begin{aligned} \mathcal{Y}_\theta^{t,x} &:= \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \bar{Y}_\theta^{t,x,\nu} \\ &= \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathbb{E}_\theta^{\mathbb{P}^{t,x,\nu}} \left[ g\left(X_T^{t,x}\right) + \int_\theta^T \left( f_s\left(X_s^{t,x}, \bar{Y}_s^{t,x,\nu}, \bar{Z}_s^{t,x,\nu}\right) - \delta(\nu_s) \right) ds \right] \end{aligned}$$

is the minimal solution of the previous constrained BSDE.
(see Cvitanic, Karatzas and Soner for the case of a convex driver)

# Dual Formulation

$\mathcal{Y}$ is the minimal solution of the constrained BSDE

We have

$$\bar{Y}^{\zeta,\nu} = g\left(X_T^{\zeta}\right) + \int_{\cdot}^{T} \left[f\left(X_s^{\zeta}, \bar{Y}_s^{\zeta,\nu}, \bar{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s) + \bar{Z}_s^{\zeta,\nu}\sigma^{-1}(X_s^{t,x})\nu_s\right] ds$$
$$- \int_{\cdot}^{T} \bar{Z}_s^{\zeta,\nu} dW_s$$

and the following characterization of $K$ :

$$z \in K \quad \Longrightarrow \quad \min\{\delta(u) - z \cdot u, u \in \mathbb{R}^d, |u| = 1\} \geq 0. \tag{3.1}$$

1. The minimality comes from standard comparison and by $(3.1)$ :

$$f(X, y, z) + z\sigma^{-1}(X)\nu - \delta(\nu) \leq f(X, y, z) \quad \text{whenever} \quad z \in K.$$

2. $\mathcal{Y}$ solution of the constrained BSDE comes from non-linear Doob-Meyer decomposition, the fact that $\nu$ lives in a cone, $\delta$ positively-homogeneous and $(3.1)$.

# Table of content

# Main difficulties when considering $\left|\mathcal{Y}_t^{t,x} - \mathcal{Y}_{t'}^{t',x'}\right|$

$$\mathcal{Y}_t^{t,x} = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathbb{E}_t^{\mathbb{P}^{t,x,\nu}} \left[ g\left(X_T^{t,x}\right) + \int_t^T \left( f_s\left(X_s^{t,x}, \bar{Y}_s^{t,x,\nu}, \bar{Z}_s^{t,x,\nu}\right) - \delta(\nu_s)\right) ds \right].$$

- Different probabilities, conditional expectations (Strong Formulation) ;
- Each $\nu$ is bounded, but all $\nu$'s are not uniformly bounded (face-lift).

# Table of content

# Strong Formulation

## The deterministic case

We have

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \bar{Y}_{\theta}^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \hat{Y}_{\theta}^{\zeta,\nu} \tag{5.1}$$

with

$$\bar{Y}^{\zeta,\nu} = g\left(X_T^{\zeta}\right) + \int_{\cdot}^{T} \left[f\left(X_s^{\zeta}, \bar{Y}_s^{\zeta,\nu}, \bar{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds - \int_{\cdot}^{T} \bar{Z}_s^{\zeta,\nu} dW_s^{\zeta,\nu},$$

$$\hat{Y}^{\zeta,\nu} = g\left(\hat{X}_T^{\zeta,\nu}\right) + \int_{\cdot}^{T} \left[f_s\left(\hat{X}_s^{\zeta,\nu}, \hat{Y}_s^{\zeta,\nu}, \hat{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds - \int_{\cdot}^{T} \hat{Z}_s^{\zeta,\nu} dW_s.$$

and

$$X^{\zeta} = x + \int_t^{\cdot} \left[b\left(X_s^{\zeta}\right) + \nu_s\right] ds + \int_t^{\cdot} \sigma\left(X_s^{\zeta}\right) dW_s^{\zeta,\nu},$$

$$\hat{X}^{\zeta,\nu} = x + \int_t^{\cdot} \left[b\left(\hat{X}_s^{\zeta,\nu}\right) + \nu_s\right] ds + \int_t^{\cdot} \sigma\left(\hat{X}_s^{\zeta,\nu}\right) dW_s.$$

# Strong Formulation

## The deterministic case

We have

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \bar{Y}_\theta^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \hat{Y}_\theta^{\zeta,\nu} \tag{5.1}$$

This is done by :

1. The supremums in $(5.1)$ may be taken among piecewise constant $\nu$ ;

# Strong Formulation

## The deterministic case

We have

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \bar{Y}_\theta^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \hat{Y}_\theta^{\zeta,\nu} \tag{5.1}$$

This is done by :

1. The supremums in $(5.1)$ may be taken among piecewise constant $\nu$ ;
2. Then for each $\nu \in \mathcal{U}_{pc}$, find $\bar{\nu} \in \mathcal{U}_{pc}$ such that

$$\bar{Y}_\tau^{\zeta,\nu} = \hat{Y}_\tau^{\zeta,\bar{\nu}}.$$

# Strong Formulation

## With an Adverse Control

We have

$$\operatorname*{ess\,sup}_{(\nu,\alpha)\in\mathcal{U}\times\mathcal{A}} \bar{Y}_\theta^{\zeta,\nu,\alpha} = \operatorname*{ess\,sup}_{(\nu,\alpha)\in\mathcal{U}\times\mathcal{A}} \hat{Y}_\theta^{\zeta,\nu,\alpha} \tag{5.2}$$

# Strong Formulation

## With an Adverse Control

We have

$$\operatorname*{ess\,sup}_{(\nu,\alpha)\in\mathcal{U}\times\mathcal{A}} \bar{Y}_{\theta}^{\zeta,\nu,\alpha} = \operatorname*{ess\,sup}_{(\nu,\alpha)\in\mathcal{U}\times\mathcal{A}} \hat{Y}_{\theta}^{\zeta,\nu,\alpha} \tag{5.2}$$

This is done by :

1. The supremums in $(5.2)$ may be taken among piecewise constant $\nu$ ;

# Strong Formulation

## With an Adverse Control

We have

$$\operatorname*{ess\,sup}_{(\nu,\alpha)\in\mathcal{U}\times\mathcal{A}} \bar{Y}_\theta^{\zeta,\nu,\alpha} = \operatorname*{ess\,sup}_{(\nu,\alpha)\in\mathcal{U}\times\mathcal{A}} \hat{Y}_\theta^{\zeta,\nu,\alpha} \tag{5.2}$$

This is done by :

1. The supremums in $(5.2)$ may be taken among piecewise constant $\nu$ ;
2. Then for each $(\nu, \alpha) \in \mathcal{U}_{pc} \times \mathcal{A}$, find $(\bar{\nu}, \bar{\alpha}) \in \mathcal{U}_{pc} \times \mathcal{A}$ such that

$$\bar{Y}_\tau^{\zeta,\nu,\alpha} = \hat{Y}_\tau^{\zeta,\bar{\nu},\bar{\alpha}}.$$

# Strong Formulation

## When the coefficients depend on $\omega$

We have

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \bar{Y}_{\theta}^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \hat{Y}_{\theta}^{\zeta,\nu} \tag{5.3}$$

# Strong Formulation

## When the coefficients depend on $\omega$

We have

$$\operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \bar{Y}_{\theta}^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \hat{Y}_{\theta}^{\zeta,\nu} \tag{5.3}$$

In this case, even the definition of the strong formulation is not that clear

1. We added a delay on the dependence in $\omega$ ;

# Strong Formulation

## When the coefficients depend on $\omega$

We have

$$\operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \bar{Y}_\theta^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \hat{Y}_\theta^{\zeta,\nu} \tag{5.3}$$

In this case, even the definition of the strong formulation is not that clear

1. We added a delay on the dependence in $\omega$ ;
2. The supremums in $(5.1)$ may be taken among piecewise constant $\nu$,
   $\Rightarrow$ Need for uniform continuity ;

# Strong Formulation

## When the coefficients depend on $\omega$

We have

$$\operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \bar{Y}_\theta^{\zeta,\nu} = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \hat{Y}_\theta^{\zeta,\nu} \tag{5.3}$$

In this case, even the definition of the strong formulation is not that clear

1. We added a delay on the dependence in $\omega$ ;
2. The supremums in $(5.1)$ may be taken among piecewise constant $\nu$,
   $\Rightarrow$ Need for uniform continuity ;
3. Then for each $\nu \in \mathcal{U}_{pc}$, find $\bar{\nu} \in \mathcal{U}_{pc}$ such that

$$\bar{Y}_\tau^{\zeta,\nu} = \hat{Y}_\tau^{\zeta,\bar{\nu}}.$$

# Table of content

# The *Face-Lift*

## Definition of the *Face–Lift*

Let

$$\hat{g}(x) := \sup_y \{g(x+y) - \delta(y)\}.$$

# The *Face-Lift*

## Definition of the *Face–Lift*

Let

$$\hat{g}(x) := \sup_{y}\{g(x+y) - \delta(y)\}.$$

1. One wants to hedge a digicall of strike $K$ ;

# The *Face-Lift*

## Definition of the *Face–Lift*

Let

$$\hat{g}(x) := \sup_y \{g(x+y) - \delta(y)\}.$$

1. One wants to hedge a digicall of strike $K$ ;
2. Under the delta constraint that $\Delta \in [0, 1]$.

# The *Face-Lift*

## Definition of the *Face–Lift*

Let

$$\hat{g}(x) := \sup_{y} \{g(x+y) - \delta(y)\}.$$

1. One wants to hedge a digicall of strike $K$ ;
2. Under the delta constraint that $\Delta \in [0,1]$.

In this case :

$$\delta : u \in \mathbb{R} \longmapsto \begin{cases} u & \text{if} \quad u \geq 0 \ ; \\ 0 & \text{if} \quad u \leq 0 \ . \end{cases}$$

# The *Face-Lift*

## Definition of the *Face–Lift*

Let

$$\hat{g}(x) := \sup_{y}\{g(x+y) - \delta(y)\}.$$

1. One wants to hedge a digicall of strike $K$ ;
2. Under the delta constraint that $\Delta \in [0,1]$.

In this case :

$$\delta : u \in \mathbb{R} \longmapsto \begin{cases} u & \text{if} \quad u \geq 0 \ ; \\ 0 & \text{if} \quad u \leq 0 \ . \end{cases}$$

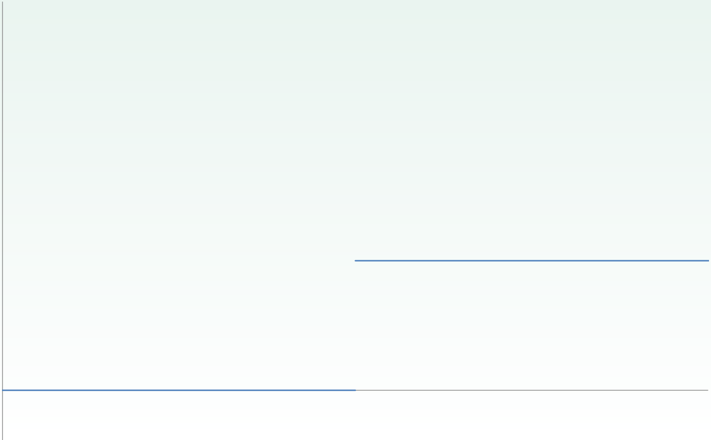
And then :

$$\hat{g} : x \in \mathbb{R}_+ \longmapsto \begin{cases} 0 & \text{if} \quad x \in [0, K-1] \ ; \\ x + (1-K) & \text{if} \quad x \in [K-1, K] \ ; \\ 1 & \text{if} \quad x \geq K \ . \end{cases}$$

# The *Face-Lift*

Super-Replication price of digiCall with $\Delta \in [0,1]$ :

$$\operatorname*{ess\,sup}_{\nu} \mathbb{E}^{\mathbb{P}^{\zeta,\nu}} \left[ g\left(X_T^{\zeta}\right) - \int_{\tau}^{T} \delta(\nu_s) ds \right].$$



# The *Face-Lift*

is the Super-Replication price of a callSpread with no constraints :

$$\operatorname*{ess\,sup}_{\nu} \mathbb{E}^{\mathbb{P}^{\zeta,\nu}} \left[ g\left(X_T^{\zeta}\right) - \int_{\tau}^{T} \delta(\nu_s) ds \right] = \mathbb{E}^{\mathbb{Q}} \left[ \hat{g}\left(X_T^{\zeta}\right) \right].$$

# The *Face-Lift*

## In the constrained BSDE case

We have

$$\mathcal{Y}_\theta^\zeta = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathbb{E}\left[\hat{g}\left(\hat{X}_T^{\zeta,\nu}\right) + \int_\cdot^T \left[f\left(\hat{X}_s^{\zeta,\nu}, \tilde{Y}_s^{\zeta,\nu}, \tilde{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds\right]$$

# The *Face-Lift*

## In the constrained BSDE case

We have

$$\mathcal{Y}_\theta^\zeta = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathbb{E}\left[\hat{g}\left(\hat{X}_T^{\zeta,\nu}\right) + \int_\cdot^T \left[f\left(\hat{X}_s^{\zeta,\nu}, \tilde{Y}_s^{\zeta,\nu}, \tilde{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds\right]$$

In fact, one has the face-lift

$$\mathcal{Y}_t^{t,x} \geq \operatorname*{ess\,sup}_{u \in L_\infty} \left\{\mathcal{Y}_t^{t,x+u} - \delta(u)\right\}.$$

# The *Face-Lift*

## In the constrained BSDE case

We have

$$\mathcal{Y}_\theta^\zeta = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathbb{E}\left[\hat{g}\left(\hat{X}_T^{\zeta,\nu}\right) + \int_\cdot^T \left[f\left(\hat{X}_s^{\zeta,\nu}, \tilde{Y}_s^{\zeta,\nu}, \tilde{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds\right]$$

Said differently

$$v(t,x) = \hat{v}(t,x) := \sup_y \{v(t,x+y) - \delta(y)\}.$$

# The *Face-Lift*

## In the constrained BSDE case

We have

$$\mathcal{Y}_\theta^\zeta = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathbb{E}\left[\hat{g}\left(\hat{X}_T^{\zeta,\nu}\right) + \int_\cdot^T \left[f\left(\hat{X}_s^{\zeta,\nu}, \tilde{Y}_s^{\zeta,\nu}, \tilde{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds\right]$$

Said differently

$$v(t,x) = \hat{v}(t,x) := \sup_y \{v(t, x+y) - \delta(y)\}.$$

The face-lift has the nice property that

$$\hat{\hat{g}}(x) := \sup_y \{\hat{g}(x+y) - \delta(y)\} = \hat{g}(x).$$

# The *Face-Lift*

## In the constrained BSDE case

We have

$$\mathcal{Y}_\theta^\zeta = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \mathbb{E}\left[\hat{g}\left(\hat{X}_T^{\zeta,\nu}\right) + \int_\cdot^T \left[f\left(\hat{X}_s^{\zeta,\nu}, \tilde{Y}_s^{\zeta,\nu}, \tilde{Z}_s^{\zeta,\nu}\right) - \delta(\nu_s)\right] ds\right]$$

Said differently

$$v(t,x) = \hat{v}(t,x) := \sup_y\{v(t,x+y) - \delta(y)\}.$$

The face-lift has the nice property that

$$\hat{\hat{g}}(x) := \sup_y\{\hat{g}(x+y) - \delta(y)\} = \hat{g}(x).$$

The $\delta(\nu)$ will kill the $\nu$ in $\hat{X}_T^{\zeta,\nu}$.

# Table of content

# The Stability Result

Finally we have

$$\left|\mathcal{Y}_t^{t,x} - \mathbb{E}_t\left[\mathcal{Y}_{t'}^{t',x'}\right]\right| \leq C\left(\left|t'-t\right|^{1/2} + \left|x-x'\right|\right)$$

and we split this stability into

1. Stability in space ;
2. Stability in time.

# Stability in Space

- Easy with standard estimates when $\hat{g}$ is Lipschitz–continuous :

$$\left|\mathcal{Y}_t^{t,x} - \mathcal{Y}_t^{t,x'}\right| \leq \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \left|\bar{Y}_t^{t,x,\nu} - \bar{Y}_t^{t,x',\nu}\right|.$$

# Stability in Time

- We split in 2 inequalities

$$\cdots \leq \mathcal{Y}_t^{t,x} - \mathbb{E}_t\left[\mathcal{Y}_{t'}^{t',x'}\right] \leq \cdots$$

and use the DPP

$$\mathcal{Y}_t^{t,x} = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}} \mathcal{E}_{t,\theta}^{t,x,\nu}\left[\mathcal{Y}_\theta^{t,x}\right].$$

# Stability in Time

- We split in 2 inequalities

$$-C\left(\left|t'-t\right|^{1/2}+\left|x-x'\right|\right)\leq \mathcal{Y}_t^{t,x}-\mathbb{E}_t\left[\mathcal{Y}_{t'}^{t',x'}\right]$$

  and use the DPP

$$\mathcal{Y}_t^{t,x}=\operatorname*{ess\,sup}_{\nu\in\mathcal{U}}\mathcal{E}_{t,\theta}^{t,x,\nu}\left[\mathcal{Y}_\theta^{t,x}\right].$$

- Taking $\nu=0$ gives one inequality ;

# Stability in Time

- We split in 2 inequalities

$$\mathcal{Y}_t^{t,x} - \mathbb{E}_t\left[\mathcal{Y}_{t'}^{t',x'}\right] \leq C\left(\left|t'-t\right|^{1/2} + \left|x-x'\right|\right)$$

  and use the DPP

$$\mathcal{Y}_t^{t,x} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \mathcal{E}_{t,\theta}^{t,x,\nu}\left[\mathcal{Y}_\theta^{t,x}\right].$$

- Taking $\nu = 0$ gives one inequality ;
- For the other one, we have to "kill" $\nu$ on $[t, t']$.

# Stability in Time

- We split in 2 inequalities

$$\mathcal{Y}_t^{t,x} - \mathbb{E}_t\left[\mathcal{Y}_{t'}^{t',x'}\right] \leq C\left(\left|t'-t\right|^{1/2} + \left|x-x'\right|\right)$$

and use the DPP

$$\mathcal{Y}_t^{t,x} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \mathcal{E}_{t,\theta}^{t,x,\nu}\left[\mathcal{Y}_\theta^{t,x}\right].$$

- Taking $\nu = 0$ gives one inequality ;
- For the other one, we have to "kill" $\nu$ on $[t, t']$.
- This is done by face-lifting :

$$\mathcal{Y}^{\theta,\bar{X}_\theta^{t,x}} - \int_t^\theta \delta\left(\nu_s ds\right) \leq \mathcal{Y}^{\theta,\bar{X}_\theta^{t,x}} - \delta\left(\int_t^\theta \nu_s ds\right) \leq \mathcal{Y}^{\theta,\bar{X}_\theta^{t,x}-\int_t^\theta \nu_s ds}$$

since $\delta(\int_t^\theta \nu_s ds) \leq \int_t^\theta \delta(\nu_s ds)$ and $\int_t^\theta \nu_s ds \in \mathcal{U}$.

# Stability in Time

- We split in 2 inequalities

$$\mathcal{Y}_t^{t,x} - \mathbb{E}_t\left[\mathcal{Y}_{t'}^{t',x'}\right] \leq C\left(\left|t'-t\right|^{1/2} + \left|x-x'\right|\right)$$

and use the DPP

$$\mathcal{Y}_t^{t,x} = \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \mathcal{E}_{t,\theta}^{t,x,\nu}\left[\mathcal{Y}_\theta^{t,x}\right].$$

- Taking $\nu = 0$ gives one inequality ;
- For the other one, we have to "kill" $\nu$ on $[t,t']$.
- This is done by face-lifting :

$$\mathcal{Y}^{\theta,\bar{X}_\theta^{t,x}} - \int_t^\theta \delta\left(\nu_s ds\right) \leq \mathcal{Y}^{\theta,\bar{X}_\theta^{t,x}} - \delta\left(\int_t^\theta \nu_s ds\right) \leq \mathcal{Y}^{\theta,\bar{X}_\theta^{t,x} - \int_t^\theta \nu_s ds}$$

since $\delta(\int_t^\theta \nu_s ds) \leq \int_t^\theta \delta(\nu_s ds)$ and $\int_t^\theta \nu_s ds \in \mathcal{U}$.

# Continuity at the Terminal Time

- The previous reasoning applies on $[0, T)$ ;

# Continuity at the Terminal Time

- The previous reasoning applies on $[0,T)$ ;
- Then $\mathcal{Y}$ is continuous on $[0,T)$ ;

# Continuity at the Terminal Time

- The previous reasoning applies on $[0,T)$ ;
- Then $\mathcal{Y}$ is continuous on $[0,T)$ ;
- Finally the stability in time implies

$$\mathcal{Y}^{\zeta}_{T-} = \hat{g}\left(X^{\zeta}_{T}\right).$$

# Continuity at the Terminal Time

- The previous reasoning applies on $[0, T)$ ;
- Then $\mathcal{Y}$ is continuous on $[0, T)$ ;
- Finally the stability in time implies

$$\mathcal{Y}^{\zeta}_{T-} = \hat{g}\left(X^{\zeta}_{T}\right).$$

- And then

$$\mathcal{Y}^{t,x}\mathbf{1}_{[0,T)} + \hat{g}(X^{t,x}_{T})\mathbf{1}_{\{T\}}.$$