Tamed Euler schemes for FBSDEs

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FBSDEs

Let $(W_t)_{t\geq 0}$ be m-dimensional Brownian motion.

$$\begin{cases} X_t &= x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t &= g(X_T) + \int_t^T f(Y_s, Z_s)ds - \int_t^T Z_s dW_s \end{cases}$$

Assumption (HX0):

$$\|g(x) - g(y)\|^2 \leq L\|x - y\|^2,$$

$$|\mu(x)| \leq L(1 + |y|^r) \quad |\sigma(x)| \leq L(1 + |x|^q), \quad r + 1 \geq 2q,$$

$$\langle x - y, \mu(x) - \mu(y)\rangle + \frac{p-1}{2}\|\sigma(x) - \sigma(y)\|^2 \leq L\|x - y\|^2.$$

Assumption (HY0):

$$|f(y, z)| \leq L + L_y|y|^m + L_z\|z\|, \quad m \geq 1,$$

$$\langle y' - y, f(t, x, y', z) - f(t, x, y, z)\rangle \leq L_y|y' - y|^2.$$

Motivation

Biology model for electrical distribution in the heart (here in 1-d)

$$\begin{cases} -\partial_t u - \frac{1}{2}\sigma^2 \Delta u & = & cu - u^3 \\ u(T,\cdot) & = & g(\cdot) \end{cases}$$

and $u \geq 0$ for any $t \in [0, T]$ - A FitzHugh-Nagumo type PDE.

Often reaction diffusion PDEs have the form

$$\begin{cases} -\partial_t u - \frac{1}{2}\sigma^2 \Delta u & = & cu - u^{2n}, \quad n \in \mathbb{N} \\ u(T,\cdot) & = & g(\cdot) \end{cases}$$

Hence monotone condition only holds on the positive domain.

FBSDEs - Approximations

Forward process: 1st idea - use explicit Euler scheme with $h = T/N$ and $\Delta w_{k+1} = w((k+1)\,h) - w(k\,h)$

$$X_{k+1} = X_k + \mu(X_k)h + \sigma(X_k)\Delta w_{k+1}, \quad X_0 = x$$

Backward process: Consider $Y_N := g(X_N)$, $Z_N := 0$, and for $i = N-1, \ldots, 0$,

$$Y_i := \mathbb{E}_i\Big[Y_{i+1} + f\big(Y_{i+1}, Z_{i+1}\big)h\Big],$$
$$Z_i := \mathbb{E}_i\Big[\frac{\Delta W_{t_{i+1}}}{h} Y_{i+1}\Big],$$

where $\mathbb{E}_i[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_i]$.

- In this talk we only deal with time-discretization error
- in practical simulations of backward part, even Lipschitz case causes trouble (big Lipschitz constant, big terminal condition)

A motivating example for backward part

Let $\xi \in \mathcal{F}_1$ and $\xi \in L^p$ for $p \geq 2$.

$$Y_t = \xi - \int_t^1 Y_s^3 ds - \int_t^1 Z_s dW_s, \quad t \in [0,1]$$

For the explicit scheme we have

$$Y_i = \mathbb{E}[Y_{i+1} - Y_{i+1}^3 h | \mathcal{F}_i] = \mathbb{E}[Y_{i+1}(1 - hY_{i+1}^2) | \mathcal{F}_i], \quad i = 0, \ldots, N-1,$$

$$|\xi| \geq 2\sqrt{N} \quad \text{then} \quad |Y_i| \geq 2^{2^{N-i}} \sqrt{N} \quad \text{for} \quad i = 0, \ldots, N.$$

Consequently

$$\lim_{N \to \infty} \mathbb{E}[\,|Y_{\frac{1}{2}}^{(N)}|\,] = +\infty,$$

Possible fix - Tamed schemes

Consider $Y_N := \frac{g(X_N)}{1+|g(X_N)|h^{\alpha_1}}$, $Z_N := 0$, and for $i = N-1, \dots, 0$,

$$Y_i := \mathbb{E}_i\Big[Y_{i+1} + \frac{f\big(Y_{i+1}, Z_{i+1}\big)}{1+|f\big(Y_{i+1}, Z_{i+1}\big)|h^{\alpha_2 t}}h\Big],$$

$$Z_i := \mathbb{E}_i\Big[\frac{\Delta W_{t_{i+1}}}{h}Y_{i+1}\Big],$$

- Tamed scheme converges with rate= Euler rate $\wedge \min\{\alpha_1, \alpha_2\}$.
- Simple and Robust
- No knowledge of a priori estimates is needed.

In the context of SDEs see work by Kloeden, Jentzen, Hutzenhaler.

But how to tame? Examples for SDEs

$$f^h(x) = T_{h,x}(f(x)) \quad \text{or} \quad f^h(x) = f(T_{h,x}(x)).$$

For example:

- state dependent taming: $T_{h,x}(x) = x/(1 + |x|h^{\alpha})$
  - simple black box implementation
  - not obvious access to large deviation estimates
- state independent taming : $T_{h,x}(x) = (-h^{\alpha} < x_1 < h^{\alpha}, \ldots, -h^{\alpha} < x_d < h^{\alpha})$
  - level at which we truncate is model dependent
  - easy access to large deviation estiamtes

We also could tame $T_{h,x}(f(x)h)$. Key observation being that

$$T_{h,x}(x) \approx x + R(x), \quad R(x) = O(h^{\alpha}) \text{ a.s. or in } L^p$$

Key Questions

- Convergence (What happens when $h \to 0$)
- Stability (Qualitative properties when $h > 0$)
- "Structure" preservation (positivity, domain preservation)

SDEs

Khasminskii Theorem

$$dx(t) = \mu(x(t))dt + \sigma(x(t))dw(t), \quad \mu : \mathbb{R}^d \to \mathbb{R}^d \quad \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$$

Let D be an open subset of $\mathbb{R}^d$. If there exists a C^2-function $V : D \to \mathbb{R}_+$ such that

$$L_{\mu,\sigma} V(x) := \sum_{i=1}^{d} \Big(\frac{\partial V}{\partial x_i}\Big)(x)\mu_i(x) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{s=1}^{m} \Big(\frac{\partial^2 V}{\partial x_i \partial x_j}\Big)(x)\sigma_{i,s}(x)\sigma_{j,s}(x)$$

$$= V^{'}(x)\mu(x) + \frac{1}{2} \sum_{s=1}^{m} V^{''}(x)(\sigma_s(x), \sigma_s(x)) \leq \rho V(x) \quad \text{for } \rho \in \mathbb{R}.$$

Then the solution is V-stable

$$\mathbb{E}[V(x(t))] \leq e^{\rho t}\mathbb{E}[V(x(0))] \quad \text{for } t \geq 0.$$

V-stability

Consequences of the V-stability property:

- Moments bound: If we further assume that there exists $l \geq 1$, such that $\|x\|^l \leq V(x)$ then inequality V-stability implies that $\sup_{t\in[0,T]} \mathbb{E}[\|x(t)\|^l] < \infty$ (provided $\mathbb{E}[V(x(0))] \leq \infty$).
- Stability: If we further assume that $\rho < 0$ then V-stability implies that $\mathbb{E}[V(x(t))] \leq e^{-\alpha t}\mathbb{E}[V(x(0))]$ for $0 < \alpha < \rho$ (to see it apply Itô's formula to the function $e^{\alpha t}V(x)$).

To clarify the idea consider scalar SDE

$$dx(t) = \mu(x(t))dt + \sigma(x(t)dw(t)$$

and assume that

$$2x\mu(x) + |\sigma(x)|^2 \leq \rho(1 + |x|^2) \quad \forall x \in \mathbb{R}.$$

Khasminskii theorem gives that for all $t > 0$

$$\mathbb{E}[V(x(t))] \leq e^{\rho t}\mathbb{E}[V(x(0))],$$

with Lyapunov function $V(x) = 1 + |x|^2$.

By squaring both sides of the explicit Euler scheme

$$X_{k+1} = X_k + \mu(X_k)h + \sigma(X_k)\Delta w_{k+1},$$

we get

$$\mathbb{E}_k[|X_{k+1}|^2] = |X_k|^2 + \big(2X_k\mu(X_k) + |\sigma(X_k)|^2\big)h + |\mu(X_k)|^2 h^2,$$

were $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_k]$.

- V-stability cannot be recovered due to $|\mu(X_k)|^2 h^2$ (in non-Lipschitz case).
- Consequently for Euler scheme with super-linearly growing coefficients, it can be shown that if $X_k \notin (-L_h, L_h)$, for sufficiently big $L_h > 0$, then $\{X_r\}_{r>k}$ growths double exponentially fast.
- Hence even if $\mathbb{P}(X_k \notin (-L_h, L_h))$ is exponentially small, explosion happens.

Tamed schemes

A general tamed Euler approximation is given by

$$X_{k+1} = X_k + \mu^h(X_k)h + \sigma^h(X_k)\Delta w_{k+1},$$

where $\mu^h \to \mu$ and $\sigma^h \to \sigma$, as $h \to 0$ [1].

[1] Precise definition of theses limits may vary by applications.

By squaring both sides of tamed Euler scheme

$$\mathbb{E}_k[|X_{k+1}|^2] = |X_k|^2 + \left(2X_k\mu^h(X_k) + |\sigma^h(X_k)|^2\right)h + |\mu^h(X_k)|^2 h^2,$$

were $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_k]$, Now we are seeking taming such that for $\tilde{\rho} > 0$

$$(*) \quad |\mu^h(x)|^2 h \leq \tilde{\rho} V(x) \quad \forall x \in \mathbb{R}.$$

and

$$(**) \quad 2x\mu^h(x) + |\sigma^h(x)|^2 \leq \rho V(x) \quad \forall x \in \mathbb{R}.$$

so that for any $k, N \in \mathbb{N}$, $k < N$,

$$\begin{aligned} \mathbb{E}_k[V(X_{k+1})] &= V(X_k) + (\rho + \tilde{\rho})V(X_k)h \\ &\implies \mathbb{E}[V(X_N)] \leq e^{(\rho+\tilde{\rho})Nh}\mathbb{E}[V(X_0)]. \end{aligned}$$

Main Stability Theorem

We denote by $\mathcal{C}_p^n(\mathbb{R}^n, [1,\infty))$ space of Lyapunov functions such that $\left\| V^{(i)}(x) \right\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \leq c_V |V(x)|^{1-i/p}$ for $i = 1, 2, \ldots p$.

Theorem

Let $\mu^h : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma^h : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, $\rho \in \mathbb{R}$ *and* $V \in \mathcal{C}_p^n(\mathbb{R}^d, [1,\infty))$, $n, p \geq 3$, *be a function such that*

$$(**) \quad L_{\mu^h,\sigma^h} V(x) \leq \rho V(x).$$

Moreover, we consider tamed Euler scheme and assume that there exist constants $c_\mu, c_\sigma > 0$ *such that*

$$(*) \quad \left\| \mu^h(x) \right\|_{L^2(\mathbb{R}^d)} h^{\beta_0} \leq c_\mu V(x)^{1/p} \quad \text{and} \quad \left\| \sigma^h(x) \right\|_{L^2(\mathbb{R}^m,\mathbb{R}^d)} h^{\beta_1} \leq c_\sigma V(x)^{1/p},$$

where $\beta_0 \leq 1/2$ *and* $\beta_1 \leq 1/2 - 1/\min\{n, 4\}$. *Then there exists a constant* $\tilde{\rho} := \tilde{\rho}(c_\mu, c_\sigma)$ *such that*

$$\mathbb{E}\Big[V(X(t_k))\Big] \leq e^{(\rho+\tilde{\rho})kh}\mathbb{E}[V(X_0)] \quad \forall k \geq 0.$$

Convergence of Tamed Euler scheme

Assumption (Strong monotone)

For any constant $p \geq 2$, there exists a constant $L > 0$ such that

$$\langle x - y, \mu(x) - \mu(y) \rangle + \frac{p-1}{2} \|\sigma(x) - \sigma(y)\|^2 \leq L \|x - y\|^2 .$$

Theorem

Under one-sided Lipschitz and strong monotone conditions, for arbitrary $T > 0$ and $p > 2$, there exists $c > 0$ such that we have

$$\mathbb{E}\left[\max_{0 \leq t_k \leq T} |X_{t_k} - x(t_k)|^p \right] \leq c\, h^{p/2}.$$

BSDEs

joint work with: G.d. Reis (Edinburgh), A. Lionnet (Oxford)

Assumption (HX0): Terminal condition is Lipschitz
Assumption (HY0):

$$|f(y,z)| \leq L + L_y|y|^m + L_z\|z\|, \quad m \geq 1,$$
$$\langle y' - y, f(t,x,y',z) - f(t,x,y,z)\rangle \leq L_y|y'-y|^2.$$

Assumptions $HY0_{loc}$:

$$|f(y,z) - f(y',z)| \leq L_y(1 + |y|^{m-1} + |y'|^{m-1})|y - y'|, \quad m \geq 1.$$

We analyse following error:

$$\mathrm{ERR}_\pi(Y,Z) := \Big(\max_{i=0,\ldots,N} \mathbb{E}\big[\,|Y_{t_i} - Y_i|^2\big] + \sum_{i=0}^{N-1} \mathbb{E}\big[|\bar{Z}_{t_i} - Z_i|^2\big]\, h\Big)^{\frac{1}{2}},$$

where

$$\bar{Z}_{t_i} = \frac{1}{t_{i+1} - t_i}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} Z_s ds \Big| \mathcal{F}_{t_i}\Big], \quad 0 \leq i \leq N-1, \quad \text{and} \quad \bar{Z}_{t_N} = Z_T.$$

Tamed Terminal Condition

Consider $Y_N := T_{L_h}\big(g(X_N)\big)$, $Z_N := 0$, and for $i = N-1, \dots, 0$,

$$Y_i := \mathbb{E}_i\Big[Y_{i+1} + f\big(Y_{i+1}, Z_{i+1}\big)h\Big],$$
$$Z_i := \mathbb{E}_i\Big[\frac{\Delta W_{t_{i+1}}}{h}\Big(Y_{i+1} + f\big(Y_{i+1}, Z_{i+1}\big)h\Big)\Big],$$

where the levels L_h satisfy $e^{c_1 T}\big(L_h^2 + c_2 T\big) \leq h^{-1/(m-1)}$, with

$$c_1 = 2\big(L_y + 12dL_z^2 + 2L_y^2\big) \quad \text{and} \quad c_2 = \max\Big\{\frac{L^2}{4dL_z^2}, \frac{L_x^2}{4dL_z^2}\Big\}.$$

For $h \leq h^*$, where h^* satisfies $e^{c_1 T} c_2 T \leq (h^*)^{-1/(m-1)}/3$ and $h^* \leq 1/(32dL_z^2)$ we can take

$$L_h = \frac{1}{\sqrt{3}} e^{-\frac{1}{2}c_1 T}\Big(\frac{1}{h}\Big)^{\frac{1}{2(m-1)}}.$$

Theorem

Let (HX0), (HY0$_{loc}$) hold and $h \leq h^*$. Assume that the order γ of the approximation $\{X_i\}_{i=0,\cdots,N}$ of X is at least $1/2$. Then for the tamed explicit scheme there exists a constant c such that

$$\mathrm{ERR}_\pi(Y,Z) := \Big(\max_{i=0,\dots,N} \mathbb{E}\big[\,|Y_{t_i} - Y_i|^2\big] + \sum_{i=0}^{N-1} \mathbb{E}\big[|\bar{Z}_{t_i} - Z_i|^2\big]\,h \Big)^{\frac{1}{2}} \leq c\; h^{1/2}.$$

Sketch of the proof

We decompose the local error into two parts. Given $i \in \{0, \cdots, N-1\}$ we write

$$Y_{t_i} - Y_i = \underbrace{\left(Y_{t_i} - Y_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})}\right)}_{\text{one-step disretization error}} + \underbrace{\left(Y_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})} - Y_{i,(Y_{i+1},Z_{i+1})}\right)}_{\text{stability of the scheme}},$$

and similarly for Z

$$\bar{Z}_{t_i} - Z_i = \underbrace{\left(\bar{Z}_{t_i} - Z_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})}\right)}_{\text{one-step discretization error}} + \underbrace{\left(Z_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})} - Z_{i,(Y_{i+1},Z_{i+1})}\right)}_{\text{stability of the scheme}}.$$

Sketch of the proof

Definition (Scheme stability)

We say that the numerical scheme $\{(Y_i, Z_i)\}_{i=0,\cdots,N}$ is *stable* if for some $\rho > 0$ there exists a constant $c > 0$ such that

$$\mathbb{E}[|Y_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})} - Y_{i,(Y_{i+1},Z_{i+1})}|^2] + \rho\mathbb{E}[|Z_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})} - Z_{i,(Y_{i+1},Z_{i+1})}|^2]h$$
$$\leq (1+ch)\left(\mathbb{E}[|Y_{t_{i+1}} - Y_{i+1}|^2] + \frac{\rho}{4}\mathbb{E}[|\bar{Z}_{t_{i+1}} - Z_{i+1}|^2]h\right) + \mathbb{E}[H_i],$$

where $H_i \in L^1(\mathcal{F}_i)$ and moreover $\{H_i\}_{i=0,\cdots,N-1}$ satisfies

$$\mathcal{R}^{\mathcal{S}}(H) := \max_{i=0,\ldots,N-1} \sum_{j=i}^{N-1} e^{c(j-i)h}\mathbb{E}[H_j] \longrightarrow 0, \quad \text{as} \quad h \to 0.$$

The quantity $\mathcal{R}^{\mathcal{S}}(H)$ is called the *stability remainder*.

Sketch of the proof

Lemma (Fundamental Lemma)

Assume that the numerical scheme $\{(Y_i, Z_i)\}_{i=0,\cdots,N}$ is stable. Denoting the one-step discretization errors for $i = 0, \cdots, N-1$ by

$$\begin{cases} \tau_i(Y) := \mathbb{E}[|Y_{t_i} - Y_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})}|^2] \\ \tau_i(Z) := \mathbb{E}[|\bar{Z}_{t_i} - Z_{i,(Y_{t_{i+1}},\bar{Z}_{t_{i+1}})}|^2 h], \end{cases}$$

there exists a constant $C = C(\rho, T, c)$ such that

$$\left(\mathrm{ERR}_\pi(Y,Z)\right)^2$$

$$\leq C\Big\{\mathbb{E}[\,|Y_{t_N} - Y_N|^2] + \mathbb{E}[|\bar{Z}_{t_N} - Z_N|^2]h + \sum_{i=0}^{N-1}\Big(\frac{\tau_i(Y)}{h} + \tau_i(Z)\Big)\Big\} + (1+h)\mathcal{R}^{\mathcal{S}}(H).$$

Sketch of the proof - Tamed scheme

We consider FBSDE

$$Y'_t = T_{L_h}\big(g(X_T)\big) + \int_t^T f\big(Y'_u, Z'_u\big)\,du - \int_t^T Z'_u dW_u, \qquad t \in [0,T].$$

The difference between this BSDE and original can be estimated using

Lemma

Let ξ be a random variable in L^q for some $q > 2$, and $L > 0$. Then we have

$$\mathbb{E}[\,|\xi - T_L(\xi)|^2] \leq 4\mathbb{E}[\,|\xi|^q]\Big(\frac{1}{L}\Big)^{q-2}$$

- $+$ Comparison Theorem

Sketch of the proof- Stability for tamed scheme

Lemma

Assume (HX0), (HY0) and that $h \leq 1/(32dL_z^2)$. If for a given $i \in \{0, \ldots, N-1\}$ one has $|Y_{i+1}| \leq h^{-1/(2m-2)}$, then one also has

$$|Y_i|^2 + \frac{1}{d}|Z_i|^2 h \leq (1 + c_1 h)\,\mathbb{E}_i\Big[|Y_{i+1}|^2 + \frac{1}{4d}|Z_{i+1}|^2 h\Big]$$

Lemma

Let (HX0) and (HY0) hold. For any $i \in \{0, \cdots, N-1\}$,

$$|Y_i|^2 + \frac{1}{4d}|Z_i|^2 h + \frac{3}{4d}\mathbb{E}_i\Big[\sum_{j=i}^{N-1}|Z_j|^2 h\Big] \leq e^{c_1(N-i)h}\mathbb{E}_i\big[|Y_N|^2\big]$$

This implies in particular that $|Y_i| \leq h^{-1/(2m-2)}$.

$$|\delta f_{i+1}|^2 h \quad = L_y^2\big(h + |Y'_{t_{i+1}}|^{2(m-1)}h + |Y_{i+1}|^{2(m-1)}h\big)|Y'_{t_{i+1}} - Y_{i+1}|^2 \leq 3L_y^2|\delta Y_{i+1}|^2.$$

$f(y) = -y^3$, X-GBM, $g(x) = x$

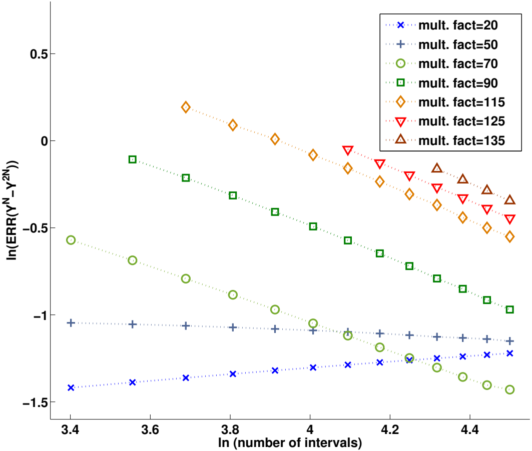

Figure: Convergence of $e(N)$ for the tamed explicit scheme and various values of the multiplying factor, computed for $N \in \{5i : i = 7, \cdots, 18\}$, in log-log scale. This used also $K = 4$, $M = 10^5$, and 10 simulations for each point.

Definition (Tamed Euler scheme)

Define $(Y_N^N, Z_N^N) := (\xi_N, 0)$ and for $0 \leq i \leq N-1$:

$$
\begin{aligned}
Y_i^R &:= \mathbb{E}_i\big[Y_{i+1}^R + f_i^h(Y_{i+1}^R, Z_i^R)h\big], \\
Z_i^R &:= \mathbb{E}_i\big[H_i^R Y_{i+1}^R\big].
\end{aligned}
$$

- Preserves Positivity
- Satisfy Comparison Theorem
- Converges without "complicated" truncations.

Stability of the scheme

Assumption (HY0-T) Let $D \subseteq \mathbb{R}^k$ $f^h : [0,T] \times \mathbb{R}^d \times D \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ is a continuous function such that for some $L, L_y, L_z > 0$ for all z, z' and all $y, y' \in D$,

$$
\begin{aligned}
&|f^h(y,z)| \leq L + L_y|y|^m + L_z\|z\|, \quad m \geq 1, \\
&\langle y' - y, f^h(y',z) - f^h(y,z)\rangle \leq L_y|y' - y|^2 + H(y',y,z,h), \\
&|f^h(y,z) - f^h(y',z)| \leq L_y(1 + |y|^{m-1} + |y'|^{m-1})|y - y'| + \hat{H}(y',y,z,h)
\end{aligned}
$$

Recall $f^h(y,z) \approx f(y,z) + R(y,z)$. Example:

$$
\frac{f(y)}{1 + |f(y)|h^\alpha} = f(y) - \frac{f(y)|f(y)|h^\alpha}{1 + |f(y)|h^\alpha}
$$

In 1-d case :

$$
\begin{aligned}
&\langle y' - y, f^h(y',z) - f^h(y,z)\rangle \leq L_y|y' - y|^2 \\
&|f^h(y,z) - f^h(y',z)| \leq L_y h^{-\alpha}|y - y'|
\end{aligned}
$$

Comparison and Positivity in 1-D in Y

Corollary (Comparison theorem)

Assume that $f^{1,h}$ satisfies (HY0). If for all $0 \leq i \leq N-1$

$$Y_N^1 \geq Y_N^2 \quad \text{and} \quad f_i^{1,h}(Y_i^2, Z_i^2) \geq f_i^{2,h}(Y_i^2, Z_i^2).$$

Then for any $i = 0, \cdots, N$ we have that

$$Y_i^1 \geq Y_i^2.$$

In particular, if $\xi_N^2 \geq 0$ and $f_i^{2,h}(0,0) \geq 0$ for all $0 \leq i \leq N-1$ then $Y_i \geq 0$ for any $1 \leq i \leq N$, in other words is positivity preserving.

- See similar result for qBSDE by J.F Chassagneux and A Richou.

Message to take home

- ▷ Standard explicit schemes have very limited scope of applications.
- ▷ Suitably tamed explicit schemes can offer very good stability results, cover very wide class of SDEs and BSDEs, cheap to simulate