On solvability conditions of stochastic Riccati equaitons

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Overview

1. **Introduction**
   - Stochastic LQ optimal control
   - Formulation Riccati equations
2. **Solvability of SREs**
   - Definite case
   - Indefinite case
   - Main results
   - Outline of the proof

Starting point – Stochastic LQ optimal control

- State equation with Linear form:

$$x(0) = \xi \in \mathbb{R}^n$$

$$\mathrm{d}x = (Ax + Bu)\,\mathrm{d}t + \sum_{i=1}^{d}(C_i x + D_i u)\,\mathrm{d}W_t^i$$

- Cost functional with Quadratic form:

$$J(u) = \mathbb{E}|Hx(T) - h|^2 + \mathbb{E}\int_0^T |Qx(t) - q(t)|^2 + |Ru(t) - r(t)|^2\,\mathrm{d}t$$

- Objective:

$$J(u) \to \min!$$

Typical examples – Mean-variance hedging

- X: the wealth process
- Hedging a terminal cash-flow H:

$$J = \mathbb{E}(X(T) - H)^2$$

- Hedging a dynamical flow $H(t)$:

$$J = \mathbb{E}\int_0^T (X(t) - H(t))^2 \, \mathrm{d}t$$

- Other Markowitz-type problems ...

Value function

- Definition:

$$V(t,\xi) = \operatorname*{ess\,inf}_u J(u;t,\xi) = \operatorname*{ess\,inf}_u \mathbb{E}[\cdots|\mathcal{F}_t]$$

  (t,ξ): the starting position

- Typical form – a quadratic form:

$$V(t,\xi) = \xi^\top P(t)\xi + \xi^\top p(t) + p_0(t)$$

- A classical approach is to characterize the coefficients

$$P(t),\ p(t) \text{ and } p_0(t)$$

  by Differential Equations

Our focus – Second-order coefficient

- Coefficient of second-order term, $P(t)$
- Assumptions and Simplifications:
  - all coefficients are bounded
  - the dimension of Wiener process = 1
  - "pure" quadratic cost:

$$J(u) = \mathbb{E}\left\{\int_0^T \left[u^\top Ru + x^\top Qx\right](t)\,\mathrm{d}t + x(T)^\top Hx(T)\right\},$$

- In this case

$$V(t,\xi) = \xi^\top P(t)\xi$$

Markovian case — HJB equation

- Deterministic coefficients
- HJB equation:

$$0 = V_t' + \inf_u \Big\{ \tfrac{1}{2} \mathrm{trace}(C\xi + Du)^\top V_{\xi\xi}''(C\xi + Du) + (A\xi + Bu)V_\xi' + \xi^\top Q\xi + \xi^\top R\xi \Big\}$$

with terminal condition

$$V(T,\xi) = \xi^\top H\xi$$

Equation of $P(t)$ – Riccati equation

- Substituted by

$$V(t,\xi) = \xi^\top P(t)\xi$$

- Completing the square $\Rightarrow$
  - the "best" u – optimal feedback control:

$$u = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC)\xi$$

  - coercivity condition:

$$R + D^\top PD > 0$$

Equation of $P(t)$ – Riccati equation (ctd.)

- The formulation of Riccati equation of $P(t)$:

$$
\begin{aligned}
0 &= \dot{P} + A^\top P + PA + C^\top PC + Q \\
&\qquad - (PB + C^\top PD)(R + D^\top PD)^{-1}(B^\top P + D^\top PC) \\
P(T) &= H \\
R &+ D^\top PD > 0
\end{aligned}
$$

Non-Markovian case – Maximum principle

- Random coefficients
- In this case $V(\cdot,\cdot)$ is a random field
- Classical DPP does NOT work
- Using MP $\Rightarrow$ the optimal pair $(x^\star, u^\star)$ s.t.

$$\begin{aligned}
&x^\star(0) = \xi, \quad y(T) = Hx^\star(T) \\
&\mathrm{d}x^\star = (Ax^\star + Bu^\star)\,\mathrm{d}t + (Cx^\star + Du^\star)\,\mathrm{d}W \\
&\mathrm{d}y = -(A^\top y + C^\top z + Qx^\star)\,\mathrm{d}t + z\,\mathrm{d}W \\
&Ru^\star + B^\top y + D^\top z = 0
\end{aligned}$$

Formal computations

- A key fact:
$$V(0,\xi) = \xi^\top y(0) = x^\star(0)^\top y(0)$$
- $y(\cdot)$ is linear with respect to $x^\star(\cdot)$
- It is reasonable to assume
$$y(t) = P(t)x^\star(t)$$
where
$$\mathrm{d}P(t) = \mathring{P}(t)\,\mathrm{d}t + \breve{P}(t)\,\mathrm{d}W_t$$
- The value function is give by P:
$$V(0,\xi) = \xi^\top P(0)\xi$$

Formal computations (ctd.)

- Itô form:

$$\mathrm{d}P(t) = \mathring{P}(t)\,\mathrm{d}t + \breve{P}(t)\,\mathrm{d}W_t, \quad P(T) = H$$

- Compare equations of $P(t)x^\star(t)$ and $y(t) \Rightarrow$

$$\mathring{P} + (A^\top P + PA + C^\top PC + C^\top \breve{P} + \breve{P}C + Q)$$
$$= (PB + C^\top PD + \breve{P}D)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top \breve{P})$$

- $(P, \breve{P})$ satisfies a matrix-valued BSDE
- In particular, deterministic coefficients $\Rightarrow \breve{P}(t) = 0$
  $\Longrightarrow$ Markovian case

Formal computations (ctd.)

- Let us derive (formally) the optmal feedback control
- Itô's formula to $x(t)^\top P(t)x(t) \Rightarrow$

$$J(u) = \mathbb{E}\int_0^T [u(t) - \Gamma(t)x(t)]^\top (R + D^\top PD)^{-1}[u(t) - \Gamma(t)x(t)]\,\mathrm{d}t$$

where

$$\Gamma = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top \breve{P})$$

$\Rightarrow$ optimal control

$$u^\star(t) = \Gamma(t)x^\star(t)$$

$\Rightarrow$ coercivity condition:

$$R + D^\top PD > 0$$

Stochastic Riccati equation (SRE)

- Complete formulation of SRE (Bismut '76):

$$
\begin{aligned}
&P(T) = H \\
&\mathrm{d}P(t) = \mathring{P}(t)\,\mathrm{d}t + \breve{P}(t)\,\mathrm{d}W_t \\
&\mathring{P} + A^\top P + PA + C^\top PC + C^\top \breve{P} + \breve{P}C + Q \\
&\quad = (PB + C^\top PD + \breve{P}D)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top \breve{P}) \\
&R + D^\top PD > 0
\end{aligned}
$$

- Under Markovian condition, the red terms do NOT appear, and

$$\mathring{P} = \dot{P}$$

Wellposedness and Solvability

Relation

Solability of SRE: $P \in L^\infty$ and $\breve{P} \in L^2$ $\Longrightarrow$ Wellposedness of LQ: $V(0,\xi) > -\infty$

- Markovian case:

$$V(0,\xi) = \xi^\top P(0)\xi$$

- non-Markovian case:

$$V(0,\xi) \geq \xi^\top P(0)\xi$$

"=" holds if the optimal feedback control

$$u^\star = \Gamma x^\star \in L^2$$

Classical setting – $Q, H \geq 0$, $R \gg 0$

- "Definite" SRE — [Won68], [Bis76], [Pen92], [Tan03]
- $Q, H \geq 0$, $R \gg 0 \Longrightarrow$

$$R + D^\top P D \gg 0$$

- Markovian case
  — Well Solved by Bellman's Quasilinearization method
  cf. Wonham ('68), Peng ('92)
- non-Markovian case
  — LQ problem is well-posed, cf. Bismut ('76)
  — Solvability of SRE was an open problem, cf. Bismut ('76)
  — Solved by Tang ('03) by "stochastic-flow approach"
  — Quasilinearization method seems not to work

Bellman's Quasilinearization Method

Markovian Case.

- An iteration approach — based on the control motivation
- Rewriting equation as

$$\dot{P} + \tilde{A}^\top P + P\tilde{A} + \tilde{C}^\top P\tilde{C} + Q + U(P)^\top RU(P) = 0$$

where

$$U(P) = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC)$$
$$\tilde{A} = A + BU(P)$$
$$\tilde{C} = C + DU(P)$$

note: P – old one, P – new one

- **Key fact:** $0 \leq P \leq P$

Bellman's Quasilinearization Method (ctd.)

Non-Markovian Case.

- Rewriting equation as $\mathrm{d}P(t) = \mathring{P}(t)\,\mathrm{d}t + \breve{P}(t)\,\mathrm{d}W_t$,

$$\mathring{P} + \tilde{A}^\top P + P\tilde{A} + \tilde{C}^\top P\tilde{C} \underline{+ \tilde{C}^\top \breve{P} + \breve{P}\tilde{C}} + Q + U(P)^\top RU(P) = 0$$

where

$$U(P) = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC \underline{+ D^\top \breve{P}})$$
$$\tilde{A} = A + BU(P)$$
$$\tilde{C} = C + DU(P)$$

- **Trouble.** $\breve{P} \in L^2$ unbounded $\Rightarrow U(P)$ unbounded
  $\Rightarrow$ BSDE with unbounded coefficients (unknown solvability)

Indefinite SRE — R is indefinite

- In most finance models, $R = 0$
- In some pollution models, $R < 0$ – cf. [CLZ98]
- LQ problem does NOT always well-posed when $R < 0$
  — recalling the cost functional

$$J(u) = \mathbb{E}\left\{\int_0^T \left[u^\top R u + x^\top Q x\right](t)\,\mathrm{d}t + x(T)^\top H x(T)\right\}$$

- Intuitively, the constraint

$$R + D^\top P D > 0$$

in SRE requires R not to be "too negative"

Example (Chen, et al. ('98))

Minimize

$$J = \mathbb{E}\int_0^1 [x(t)^2 + r(t)u(t)^2]\,\mathrm{d}t + x(1)^2$$

subject to

$$\mathrm{d}x(t) = u(t)\,\mathrm{d}W_t, \quad x(0) = 0$$

A simple calculation yields

$$J = \mathbb{E}\int_0^1 [r(t) + (2-t)]u(t)^2\,\mathrm{d}t$$

obviously, when $r(t) < -2$ the problem is ill-posed; in other words, $r(t)$ cannot be "too negative"

Solvability condition — Singular case $R = 0$

- for Markovian case

Theorem (Chen, et al. ('98))

If $R = 0$ *and*

1. $Q \geq 0$ *and* $H \gg 0$
2. $D^\top D \gg 0$

Then SRE admits a unique solution

- for non-Markovian case

Theorem (Tang ('03), D. ('14))

Under the same condition, SRE admits a solution

Solvability condition — Indefinite R

Markovian case

Theorem (Rami & Zhou [RZ00])

If the linear matrix inequality (LMI) admits a solution:

$$
\begin{aligned}
&0 \le \dot{P} + A^\top P + PA + C^\top PC + Q \\
&\qquad - (PB + C^\top PD)(R + D^\top PD)^{-1}(B^\top P + D^\top PC) \\
&P(T) \le H \\
&R + D^\top PD > 0
\end{aligned}
$$

Then SRE admits a unique (continuous) solution

— Unfortunately this is NOT always true

— Solvability of LMI yields Wellposedness of LQ problem
but NOT always implies Solvability of SRE

Example (D. '14)

Consider the following ODE over the time interval $[0,2]$:

$$\dot{P}(t) = \frac{P(t)^2}{R(t)}, \quad P(2) = 1.$$

where

$$R(t) = (1-t)^2 \chi_{[0,1)}(t) + \chi_{[1,2]}(t) > 0$$

Clearly, $P = 0$ is a solution to the corresponding LMI.
However, from the point of view of LQ problem, the solution (if exists) must equal

$$P(t) = \begin{cases} 0, & 0 \leq t < 1 \\ (3-t)^{-1}, & 1 \leq t \leq 2 \end{cases}$$

which is discontinuous!

Solvability condition — Indefinite R (ctd.)

non-Markovian case

Definition (D. '14)

A bounded matrix process F is called a subsolution to SRE, if

$$\begin{aligned} & \mathrm{d}F(t) = \mathring{F}(t)\,\mathrm{d}t + \breve{F}(t)\,\mathrm{d}W_t \\ & F(T) \leq H \\ & \mathring{F} + A^\top F + FA + C^\top FC + C^\top \breve{F} + \breve{F}C + Q \\ & \geq (FB + C^\top FD + \breve{F}D)(R + D^\top FD)^{-1}(B^\top F + D^\top FC + D^\top \breve{F}) \\ & R + D^\top FD > 0 \end{aligned}$$

— A stochastic counterpart of LMI

Main Results

Theorem (D. '14)

If there is a constant $\varepsilon > 0$ such that

- *SRE $(R - \varepsilon)$ has a bounded subsolution F*

Then SRE (R) admits a bounded solution P and

$$P(t) \geq F(t)$$

in this case,

$$V(0, \xi) = \xi^\top P(0)\xi$$

Corollary (D. '14)

Let $D^\top D \gg 0$*, and* $\alpha : (0,T] \to [0,1)$*. Let* $\varphi > 0$ *satisfy the following ODE:*

$$\dot{\varphi} + \lambda_*(\Upsilon(\alpha))\varphi + \lambda_*(Q) = 0, \quad \varphi(T) = \lambda_*(H),$$

where $\lambda_*(M) := \inf_\omega\{$*the minimal eigenvalue of matrix* $M(\omega)\}$*,*

$$\Upsilon(\alpha) := A^\top + A + C^\top C - \frac{1}{1-\alpha}(B + C^\top D) \cdot (D^\top D)^{-1}(B + C^\top D)^\top.$$

Then, SRE (R) *admits a bounded solution if*

$$R \gg -\alpha\varphi D^\top D$$

An example

Consider the following equation

$$\dot{P}(t) = \frac{P(t)^2}{r + P(t)}, \quad r + P(t) > 0, \quad P(1) = 1. \tag{1}$$

Take $\alpha : (0,1] \to [0,1)$. By the previous result, if

$$r > -r_0(t) := -\alpha(t)\varphi(t) = -\alpha(t) \exp\left(-\int_t^1 \frac{1}{1-\alpha(s)}\,\mathrm{d}s \right),$$

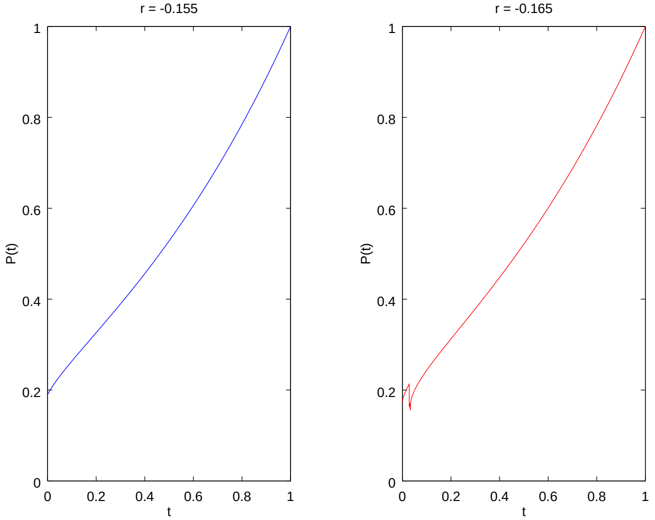
(1) admits a solution. Let us choose $\alpha(\cdot)$ such that r_0 is a constant, i.e.,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}(t) = -\frac{\alpha(t)}{1-\alpha(t)}.$$

By some arguement one can show that the lowest r_0 satisfying

$$1 = r_0 - \ln r_0 - 1$$

approximately, $r_0 \approx 0.15859$


r = -0.155
r = -0.165
P(t)
P(t)
t
t
0
0.2
0.4
0.6
0.8
1

Main idea of the proof

- Solving the matrix-valued FBSDE with unknown (X, Y, Z, U):

$$\begin{cases} \mathrm{d}X = (AX + BU)\,\mathrm{d}t + (CX + DU)\,\mathrm{d}W \\ \mathrm{d}Y = -(A^\top Y + C^\top Z + QX)\,\mathrm{d}t + Z\,\mathrm{d}W \\ 0 = RU + B^\top Y + D^\top Z, \\ X(0) = I_n, \quad Y(T) = HX(T) \end{cases}$$

- Verifying $X^{-1} = \{X(t)^{-1}; t \in [0, T]\}$ is a continuous process
- Verifying the following processes solve the SRE

$$P = YX^{-1}$$
$$\breve{P} = ZX^{-1} - YX^{-1}(C + DUX^{-1})$$

Step #1

Lemma

If SRE $(R - \varepsilon I)$ admits a subsolution F, then FBSDE admits a unique solution and

$$X^\top F X \leq X^\top Y$$

- **Approach #1.** The assumption $\Rightarrow$ LQ problem is solvable
  $+$ Maximum principle $\Rightarrow$ the result
- **Approach #2.** by the Method of Continuation, cf. [PW99]

Step #2

- Stopping technique:

$$\tau_m := \inf\{t : \det(X(t)) \leq 1/m\} \wedge T$$
$$\tau := \inf\{t : \det(X(t)) \leq 0\} \wedge T$$

  clearly $\tau_m \uparrow \tau$
- Consider the SRE on $[[0, \tau))$
  - $X(t)^{-1}$ exits and is continuous on $[[0, \tau))$
  - $P = YX^{-1}$ solves SRE on $[[0, \tau))$
- Equation of X can be written as (on $[[0, \tau))$)

$$\mathrm{d}X = [A + B\Gamma(P)]X \,\mathrm{d}t + [C + D\Gamma(P)]X \,\mathrm{d}W, \quad X(0) = I_n$$

  where

$$\Gamma(P) = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top \breve{P})$$

Step #2 — Two Lemmas

Lemma (Tang '03, D. '14)

$$\breve{P},\ \Gamma(P) \in L^2(0,\tau)$$

Lemma (Gal'chuk '79, Tang '03)

Let $\tilde{A}, \tilde{C}$ *be* $\mathbb{R}^{n\times n}$*-valued adapted processes such that*

$$\int_0^\infty \left(|\tilde{A}(t)| + |\tilde{C}(t)|^2\right) \mathrm{d}t < \infty \quad a.s.$$

Then, the following SDE

$$\mathrm{d}\tilde{X} = \tilde{A}\tilde{X}\,\mathrm{d}t + \tilde{C}\tilde{X}\,\mathrm{d}W_t, \quad \tilde{X}(0) = I_n \in \mathbb{R}^{n\times n}$$

has a unique strong solution. Moreover, $\tilde{X}^{-1} = \{\tilde{X}(t)^{-1};\, t \geq 0\}$ *exists and is a continuous process.*

Steps 2 & 3

- By the method of zero-expansion, one has

$$X(\tau)^{-1} \text{ exists almost surely}$$

$\Rightarrow \tau = T$ a.s. $\Rightarrow X(t)^{-1}$ exits a.s. for each $t \in [0, T]$

- Itô's formula to $P(t) = Y(t)X(t)^{-1}$
- DONE!

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Thank you!