BSEs, BSDEs, and fixed points

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Outline I

Backward Stochastic Differential Equation

Let $(W, \mathbb{F}^W := (\mathcal{F}_t^W)_{t\in[0,T]}, \mathbb{P})$ be a n-dim. Brownian motion. The most classical form of backward stochastic differential equation (BSDE) is

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \qquad \text{(BSDE)}$$

Input	Output
$\xi \in \mathcal{F}_T^W$	Y
$f \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+d\times n})$	Z

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Remark

BSDE can be generalized to backward stochastic equation (BSE): For a driver $F : \mathbb{S}^p \times \mathbb{M}_0^p \to \mathbb{S}_0^p$, we want to find $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ such that

$$Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T \qquad \text{(BSE)}$$

Brief History

	Linear Growth	Quadratic Growth
$d = 1$	Bismut (1973) Pardoux and Peng (1990) El Karoui et al. (1997) Hamadène (2003) $\vdots$	Kobylanski (2000) Briand and Hu (2006, 2008) Delbaen et al. (2011) $\vdots$
$d \geq 1$		Tevzadze (2008) Frei and Dos Reis (2011)

Liang, Lyons, and Qian (2011): BSDE without martingale representation theorem
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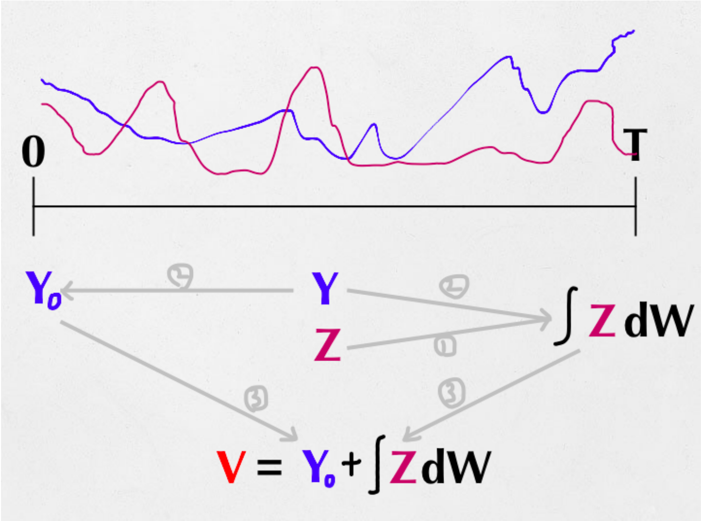
Compactness Issue

Key Idea: FPT on $\mathbb{L}^2$

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t; \quad Y_T = \xi$$

1. Martingale representation
2. Forward SDE
3. $Y_0 = \mathbb{E}_0 V$ and $\int_0^T Z_s dW_s = V - \mathbb{E}_0 V$
4. BSDE becomes random variable mapping (RVM) $V = \xi + \int_0^T f(t, Y_s^V, Z_s)ds$
5. Use FPT for V: Banach-type, Schauder-type



BSDE of $(Y, Z) <--- >$ Fixed point V of RVM $V = \xi + \int_0^T f(s, Y_s^V, Z_s)ds$

BSDE to RVM: Obvious.
RVM to BSDE: Assume that V satisfies RVM. Then,

$$Y_0 = Y_t^V + \int_0^t f(s, Y_s^V, Z_s)ds - \int_0^t Z_s dW_s$$

$$\int_0^T Z_s dW_s = V - \mathbb{E}_0 V = \xi + \int_0^T f(s, Y_s^V, Z_s)ds - Y_0$$

$$= \xi + \int_0^T f(s, Y_s^V, Z_s)d - Y_t^V - \int_0^t f(s, Y_s^V, Z_s)ds + \int_0^t Z_s dW_s$$

$$Y_t^V = \xi + \int_t^T f(s, Y_s^V, Z_s)ds - \int_t^T Z_s dW_s$$

(Y^V, Z) is a solution of BSDE.

Fixed Point Theorems

Theorem (Banach FPT)

For $p > 1$, let $G : \mathbb{L}^p \to \mathbb{L}^p$ be a contraction mapping. Then there exists a unique fixed point of G.

Theorem (Krasnoselskii FPT in Smart (1974))

Assume that $\mathcal{C} \subset \mathbb{L}^2(\mathcal{F})$ is a closed convex nonempty set. Assume that $G^1, G^2 : \mathcal{C} \to \mathbb{L}^2(\mathcal{F})$ satisfy the following conditions

- $G^1(v) + G^2(v') \in \mathcal{C}$ *for all* $v, v' \in \mathcal{C}$.
- G^1 *is a contraction.*
- G^2 *is continuous and* $G^2(\mathcal{C})$ *is contained in a compact set.*

Then, $G^1 + G^2$ *has a fixed point in* $\mathcal{C}$.

Applying Our Lemma

- Lipschitz $\Rightarrow$ Use Banach FPT:
  Generalizes the results by Pardoux and Peng (1990) and Buckdahn et al. (2009). $\triangleright$
- Lipschitz+compact $\Rightarrow$ Use Krasnoselskii FPT

Applying Our Lemma

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Compact set in $\mathbb{L}^2$?

Krasnoselskii FPT and Compact Sets in $\mathbb{L}^2$

We assume the following conditions on our probability space.

- $\Omega := L^2([0,T];\mathbb{R}^n)$ be the Hilbert space of functions from $[0,T]$ to $\mathbb{R}^n$ endowed with the inner product $\langle x, y\rangle := \int_0^T x_t \cdot y_t dt$ for $x, y \in \Omega$. We let $(e_k)_{k=1,2,\cdots}$ to be a complete orthonormal basis of Ω.
- Borel algebra $\mathcal{F}$ and nondegenerate Gaussian measure $\mathbb{P} = N_Q$ on Ω where we define Q by $Qe_k = \lambda_k e_k$. We assume $\lambda_k > 0$ and $\sum_k \lambda_k < \infty$.

A n-dimensional Brownian motion W is well-defined by white noise mapping (▷). We can define a Sobolev space $\mathbb{W}^{1,2}$ and use the following lemma in Da Prato (2006) holds.

Lemma (Compact Embedding Theorem in Da Prato (2006))

$\mathbb{W}^{1,2}$ *is compactly embedded to* $\mathbb{L}^2$. $(\mathbb{W}^{1,2} \subset\subset \mathbb{L}^2)$

$(\iff)$ For any $C \in \mathbb{R}_+$, $\{V \in \mathbb{W}^{1,2} : \|V\|_{\mathbb{W}^{1,2}} \leq C\}$ is compact.

Lipschitz Random Variable

Definition

We call a random variable φ is L-Lipschitz in ω if

$$|\varphi(\omega) - \varphi(\omega')| \leq L\sqrt{\langle \omega - \omega', \omega - \omega' \rangle}$$

for all $\omega, \omega' \in \Omega$.

Lemma

If φ is a L-Lipschitz random variable, then φ is in $\mathbb{W}^{1,2}$ with $\mathbb{E}\langle \mathbf{D}\varphi, \mathbf{D}\varphi \rangle \leq L^2$.

Corollary

The set of L-Lipschitz random variables with mean zero is relatively compact in $\mathbb{L}^2$.

These lemmas are proved in Da Prato (2006).

Example

Assume the Brownian motion is one dimensional. Let $h : \mathbb{R}^n \to \mathbb{R}^d$ be a Lipschitz function and assume that a random variable X is of form of

$$X = h(W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}})$$

for $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$. If the orthonormal basis is

$$e_k := \frac{1}{t_k - t_{k-1}} 1_{(t_{k-1}, t_k]}$$

for $k = 1, 2, \cdots, n$,

$$(W_{t_k} - W_{t_{k-1}})(\omega) = \left\langle \omega, Q^{-1/2} 1_{(t_{k-1}, t_k]} \right\rangle = \left\langle \omega, \lambda_k^{-1/2} 1_{(t_{k-1}, t_k]} \right\rangle.$$

Since $\lambda_k > 0$, $(W_{t_k} - W_{t_{k-1}})$ are Lipschitz for $k = 1, 2, \cdots, n$. Therefore, X is Lipschitz random variable.

Applications of Krasnoselskii FPT

Case	Type	Driver	Solution
3	BSDE	$f_1(s, Y_0, Z) + f_2(s, Y_0, Z)$	$\exists$
4	BSDE	$f_1(s, Z) + f_2(s, Z)$	$\exists$
5	BSDE	$f(s, Z)$	$\exists!$
6	BSDE	$f_1(s, Z_s) + f_2(s, Z_s)$	$\exists$

We assume that f_1 and f are Lipschitz and that f_2 and f are compact and continuous. The cases 3 – 4 require the Lipschitz constant of f_1 to be small enough while cases 5 – 6 do not have such condition. We remark the followings.

- The cases 3 – 5 are path-dependent BSEs or BSDEs.
- In the case 5, the uniqueness is shown because f is Lipschitz as well.
- In the case 6, we use iteration scheme by localizing f_1 only.

Krasnoselskii FPT: Case 6

Let $\mathbb{F} = \mathbb{F}^W$ and assume the following conditions.

- $g_1 : \Omega \times [0, T] \times \mathbb{L}^2(\mathcal{F}_T) \to \mathbb{R}^d$ satisfies

$$\|g_1(s, v) - g_1(s, v')\|_2 \le C \|v - v'\|_2 ,$$

- $g_2 : (\omega, s, Z_s) \in \Omega \times [0, T] \times \mathbb{L}^2(\mathcal{F}_s) \mapsto g_2(\omega, s, Z_s) \in \mathbb{R}^d$ is uniformly L-Lipschitz in ω for all (s, Z_s), and for all $u, v \in \mathbb{H}^2$,

$$\left\| \int_0^T |g_2(s, u_s) - g_2(s, v_s)| \, ds \right\|_2 \le \rho(\|u\|_{\mathbb{H}^2} + \|v\|_{\mathbb{H}^2}) \|u - v\|_{\mathbb{H}^2}$$

  for some increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$.

Then, there is a solution (Y, Z) to the following BSDE.

$$Y_t = \xi + \int_t^T (g_1(s, Z_s) + g_2(s, Z_s)) \, ds - \int_t^T Z_s dW_s .$$

Examples using Krasnoselskii FPT: Case 6

Let $\mathbb{F} = \mathbb{F}^W$ and assume the following conditions.

- $h : (\omega, s, u, v) \in \Omega \times [0, T] \times \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n} \mapsto h(\omega, s, u, v) \in \mathbb{R}^d$ is uniformly Lipschitz in (u, v) with coefficient C, $h(\cdot, u, v)$ is predictable for any (u, v), and $h(\cdot, 0, 0) \in \mathbb{H}^2$.
- Let $G : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \Omega \times [0, T] \times \mathbb{R}^{d\times n} \to \mathbb{R}^m$ where

$$|g(s,a) - g(s,b)| \leq C(1 + |a| + |b|)|a - b|$$
$$|G(s,x) - G(s,y)| \leq C|x - y|, \quad \text{and} \quad |G(s,0)|, |g(s,0)| \leq C$$

  for all $a, b \in \mathbb{R}^{d\times n}, x, y \in \mathbb{R}^m$. In addition, we assume that $G(s, x)$ is uniformly L-Lipschitz in ω for any given $(s, x) \in [0, T] \times \mathbb{R}^m$ and that $g(\cdot, x)$ and $G(\cdot, x)$ are predictable for any given $x \in \mathbb{R}^m$.

Then, there exists a solution to the following BSDE

$$Y_t = \xi + \int_t^T \left(\mathbb{E}'h(s, Z_s, Z_s') + G(s, \mathbb{E}g(s, Z_s))\right) ds - \int_t^T Z_s dW_s$$

where

$$\mathbb{E}'h(s, Z_s, Z_s')(\omega) := \int_\Omega h(\omega, s, Z_s(\omega), Z_s(\omega'))\mathbb{P}(d\omega')$$

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