Optimal Switching at Poisson Random Intervention Times

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Outline

Introduction to Optimal Switching

Introduction to Optimal Switching

- Optimal switching is the problem of determining an optimal sequence of stopping times for a switching system.
- Main ingredients.
  - Profit/cost functional: running payoff + switching cost;
  - Switching system: ODE, PDE, SDE, BSDE;
  - Stopping times: accessible, totally inaccessible.
- Financial applications.
  - Firm's investment, real options, trend following trading.
  - The player (the manager of a power plant) can enter or exit an economic activity, deciding when to produce electricity (if the profit generated from operation is high), and when to close the power station (if the profit generated from operation is low) in an optimal way.
  - Optimal switching is dubbed the starting and stopping problem, the reversible investment problem.

Literature Review: Incomprehensive

- How to model the underlying switching system?
  - ODE. Dolcetta and Evans (1984).
  - PDE. Stojanovic and Yong (1989).
  - SDE. Impulse control and quasi-variational inequality: Bensoussan and Lions (1984), Tang and Yong (1993).
    The structure of switching regions: Brekke and Oksendal (1994), Duckworth and Zervos (2001), Ly Vath and Pham (2007), Pham et al (2009), Bayraktar and Egami (2010).
  - BSDE. Characterization by multidimensional oblique reflected BSDE: Hamadène and Jeanblanc (2007), Hu and Tang (2010), Hamadène and Zhang (2010), Chassagneux et al (2013).
- Financial applications.
  - Firm's investment: Dixit (1989), Brekke and Oksendal (1994), Duckworth and Zervos (2001).
  - Real options: Hamadène and Jeanblanc (2007), Carmona and Ludkovski (2008), Porchet et al (2009).
  - Trend following trading: Dai et al (2010).

Optimal Switching at Poisson Random Intervention Times

Motivation

- Most of existing results on optimal switching consider accessible switching times (stopping times w.r.t. BM filtration).
- We consider the player is allowed to switch at a sequence of Poisson arrival times (totally inaccessible) instead of any stopping times.
  - The Poisson process can be regarded as an exogenous constraint on the player's ability to switch, reflecting the liquidity effect.
  - The Poisson process can also be seen as an information constraint. The player is only able to observe the switching system at Poisson arrival times.
  - Optimal switching at Poisson arrival times can also be seen as a randomized version of a discrete optimal switching problem.
- In optimal stopping time setting: Dupuis and Wang (2002) American options, Liang et al (2012) dynamic bank run problems.

Some Notations

- Let W be a BM with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$.
- For any fixed time $t \geq 0$, let $\{T_n\}_{n\geq 0}$ be the arrival times of a Poisson process $(N_s)_{s\geq t}$ with intensity λ and filtration $\{\mathcal{H}_s^{(t,\lambda)}\}_{s\geq t}$.
- The Brownian motion and the Poisson process are independent.
- Given the parameter set (t,λ), let $\mathcal{G}_s^{(t,\lambda)} = \mathcal{F}_s \vee \mathcal{H}_s^{(t,\lambda)}$ so that $\mathcal{G}_t^{(t,\lambda)} = \mathcal{F}_t$, and $\mathbb{G}^{(t,\lambda)} = \{\mathcal{G}_s^{(t,\lambda)}\}_{s\geq t}$.
- Given the Poisson arrival time T_n, define pre-T_n σ-field:

$$\mathcal{G}_{T_n}^{(t,\lambda)} = \left\{ A \in \bigvee_{s\geq t} \mathcal{G}_s^{(t,\lambda)} : A \cap \{T_n \leq s\} \in \mathcal{G}_s^{(t,\lambda)} \text{ for } s \geq t. \right\}$$

for $n \geq 0$, and denote $\tilde{\mathbb{G}}^{(t,\lambda)} = \{\mathcal{G}_{T_n}^{(t,\lambda)}\}_{n\geq 0}$.

The Switching System: Infinite Horizon BSDE System

- The switching system: For $0 \leq t \leq T < \infty$ and $1 \leq i \leq d$,

$$\begin{cases} Y_t^i = Y_T^i + \int_t^T f_s^i(Y_s, Z_s) + \lambda \max\left\{0, \mathcal{M}Y_s^i - Y_s^i\right\} ds - \int_t^T Z_s^i dW_s, \\ \lim_{T\uparrow\infty} \mathbf{E}\left[e^{2aT}|Y_T|^2\right] = 0. \end{cases} \tag{1}$$

- The driver $f_s = (f_s^1 \cdots, f_s^d)^*$ and the parameter a are the given data, and the impulse term $\mathcal{M}Y_t^i$ is defined as

$$\mathcal{M}Y_t^i = \max_{j \neq i}\left\{Y_t^j - g_t^{ij}\right\}.$$

- A solution to (1) is a pair of $\mathbb{F}$-progressively measurable processes (Y, Z).

The Corresponding Oblique Reflected BSDE

- The corresponding oblique reflected BSDE: For $0 \leq t \leq T < \infty$ and $1 \leq i \leq d$,

$$\begin{cases} Y_t^i = Y_T^i + \int_t^T f_s^i(Y_s, Z_s)ds + \int_t^T dK_s^i - \int_t^T Z_s^i dW_s, \\ Y_t^i \geq \mathcal{M}Y_t^i, \quad \text{with } \int_0^T (Y_s^i - \mathcal{M}Y_s^i)dK_s^i = 0, \\ \lim_{T\uparrow\infty} \mathbf{E}\left[e^{2aT}|Y_T|^2\right] = 0. \end{cases} \tag{2}$$

- A solution to (2) is a triple of $\mathbb{F}$-progressively measurable processes (Y, Z, K).
- (1) is called the penalized equation of (2). The solution of (1) will converge to the solution of (2) when $\lambda \uparrow \infty$ under certain condition.

Assumptions on the Switching System

- The driver $f_s(y, z)$ is monotone in y and Lipschitz continuous in z:

$$
\begin{aligned}
(y - \bar{y})^*(f_s(y, z) - f_s(\bar{y}, z)) &\leq -a_1|y - \bar{y}|^2, \\
|f_s(y, z) - f_s(y, \bar{z})| &\leq a_2||z - \bar{z}||,
\end{aligned}
$$

  and it has linear growth in both components (y, z).

- The parameter a satisfies the structure condition (Darling and Pardoux (1997)):

$$
a = -a_1 + \frac{\delta}{2}a_2^2 + (\frac{d+3}{2})\lambda
$$

  for $\delta > 1$ such that

$$
\mathbf{E}\left[\int_0^\infty |f_s(0,0)|^2 e^{2as} ds\right] < \infty.
$$

- If $a \geq 0$, small class of $f_s(0, 0)$ and Y goes to zero exponentially fast; If $a < 0$, large class of $f_s(0, 0)$, and Y does not necessarily go to zero.

Assumptions on the Switching System Cont.

- The switching cost g^{ij} is a bounded $\mathbb{F}$-progressively measurable process valued in $\mathbb{R}$, and satisfies
  - $g_t^{ii} = 0$;
  - $\inf_{t \geq 0} g_t^{ij} + g_t^{ji} \geq C > 0$ for $i \neq j$;
  - $\inf_{t \geq 0} g_t^{ij} + g_t^{jl} - g_t^{il} \geq C > 0$ for $i \neq j \neq l$.
- Under the above assumptions, the switching system admits a unique solution pair (Y, Z).
- However, the oblique reflected BSDE (2) not necessarily admits a solution.

The Optimal Switching Model

- Given d switching regimes, a player starts in regime i at any fixed time $t \geq 0$, and makes her switching decisions sequentially at a sequence of Poisson arrival times $\{T_n\}_{n\geq 0}$.
- The switching decision at any time $s \geq t$ is

$$u_s = \alpha_0 \mathbf{1}_{\{t\}}(s) + \sum_{k\geq 0} \alpha_k \mathbf{1}_{(T_k, T_{k+1}]}(s), \tag{3}$$

where $(\alpha_k)_{k\geq 0} \in \mathcal{G}_{T_k}^{(t,\lambda)}$, valued in $\{1, \cdots, d\}$.

$$\mathcal{K}_i(t,\lambda) = \Big\{ \mathbb{G}^{(t,\lambda)}\text{-progressively measurable process } (u_s)_{s\geq t} : \\ u \text{ has the form (3) with } \alpha_0 = i \Big\}.$$

- For any $r \leq a$, $y_t^{i,(t,\lambda)}$ is the value of the optimal switching problem

$$\operatorname*{ess\,sup}_{u\in\mathcal{K}_i(t,\lambda)} \mathbf{E}\left[\int_t^\infty e^{r(s-t)}[f_s^{u_s}(Y_s, Z_s) - rY_s^i]ds - \sum_{k\geq 1} e^{r(T_k - t)} g_{T_k}^{\alpha_{k-1},\alpha_k} | \mathcal{G}_t^{(t,\lambda)} \right] \tag{4}$$

The Optimal Switching Model Cont.

- If $a \geq 0$, then by choosing $r = 0$, we obtain an optimal switching problem without discounting:

$$y_t^{i,(t,\lambda)} = \operatorname*{ess\,sup}_{u \in \mathcal{K}_i(t,s)} \mathbf{E}\left[\int_t^\infty f_s^{u_s}(Y_s, Z_s)ds - \sum_{k \geq 1} g_{T_k}^{\alpha_{k-1},\alpha_k} | \mathcal{G}_t^{(t,\lambda)}\right].$$

- If $a < 0$, then discounting by rate $r \leq a$ in the optimal switching problem (4) is necessary.

Optimal Switching Representation

Theorem

Let (Y, Z) be the unique solution to the infinite horizon BSDE system (1). Then the value of the optimal switching problem (4) is given by

$$y_t^{i,(t,\lambda)} = Y_t^i, \text{ a.s. for } t \geq 0,$$

and the optimal switching strategy is $\tau_0^* = t$ and $\alpha_0^* = i$,

$$\tau_{k+1}^* = \inf\left\{T_N > \tau_k^* : Y_{T_N}^{\alpha_k^*} \leq \mathcal{M}Y_{T_N}^{\alpha_k^*}\right\}$$

where

$$\alpha_{k+1}^* = \arg\max_{j \neq \alpha_k^*}\left\{Y_{\tau_{k+1}^*}^j - g_{\tau_{k+1}^*}^{\alpha_k^*, j}\right\}$$

for $k \geq 0$. Hence, the optimal switching strategy at any time $s \geq t$ is

$$u_s^* = i\mathbf{1}_{\{t\}}(s) + \sum_{k \geq 0} \alpha_k^* \mathbf{1}_{(\tau_k^*, \tau_{k+1}^*]}(s).$$

The Structure of Switching Regions

Markovian Assumptions

- Assume there are two regimes, and 1-dim BM.
- The driver $f_s(y,z)$ has the form: $f_s(y,z) = h(X_s) - a_1 y$, where

$$dX_s = bX_s ds + \sigma X_s dW_s,$$

  $h = (h^1, h^2)^*$ is nonnegative and Lipschitz continuous, and $a_1 > \max\{b, 0\}$ is large enough so that for $a = -a_1 + 2.5\lambda$,

$$\mathbf{E}\left[\int_0^\infty |h(X_s)|^2 e^{2as} ds\right] < \infty.$$

- The switching cost g^{ij} is a constant, and satisfies (1) $g^{ii} = 0$; and (2) $g^{ij} + g^{ji} > 0$ for $i \neq j$.

The Optimal Switching Problem

- There exist measurable functions $v = (v^1, v^2)$ such that $Y_t = v(X_t)$.
- Choosing $r = -a_1$, we know that $v^i(X_0) = v^i(x)$ is the value of the optimal switching problem

$$v^i(x) = \sup_{u \in \mathcal{K}_i(0,\lambda)} \mathbf{E}\left[\int_0^\infty e^{-a_1 s} h^{u_s}(X_s) ds - \sum_{k \geq 1} e^{-a_1 T_k} g^{\alpha_{k-1},\alpha_k}\right].$$

  Moreover, the optimal switching strategy is $\tau_0^* = 0$ and $\alpha_0^* = i$,

$$\tau_{k+1}^* = \inf\left\{T_N > \tau_k^* : v^{\alpha_k^*}(X_{T_N}) \leq v^{\alpha_{k+1}^*}(X_{T_N}) - g^{\alpha_k^*,\alpha_{k+1}^*}\right\}$$

  where $\alpha_{k+1}^* = 3 - \alpha_k^*$.
- The switching regions: For $i = 1, 2$ and $j = 3 - i$,

$$\mathcal{S}^i = \{x \in (0, \infty) : v^i(x) \leq v^j(x) - g^{ij}\}.$$

The Structure of Switching Regions

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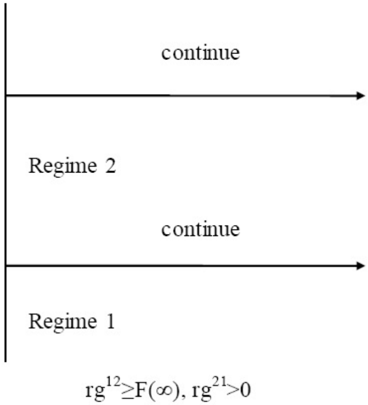
Theorem

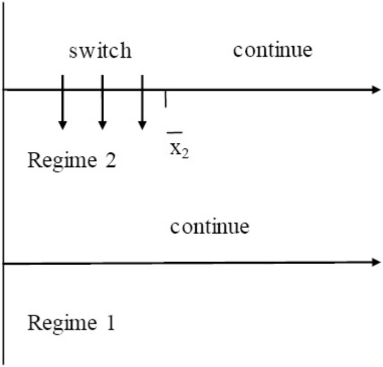
Suppose that $F(x) = h^2(x) - h^1(x) \geq 0$, strictly increasing on $(0, \infty)$, and moreover, the switching cost $g^{12} > 0$. Then we have the following five cases for the switching regions $\mathcal{S}^1$ and $\mathcal{S}^2$:

1. *If $a_1 g^{12} \geq F(\infty)$ and $a_1 g^{21} \geq 0$, then, $\mathcal{S}^1 = \phi, \mathcal{S}^2 = \phi$;*
2. *If $a_1 g^{12} \geq F(\infty)$ and $-F(\infty) < a_1 g^{21} < 0$, then, $\mathcal{S}^1 = \phi, \mathcal{S}^2 = (0, \overline{x}^2]$, where $\overline{x}^2 \in (0, \infty)$;*
3. *If $a_1 g^{12} \geq F(\infty)$ and $a_1 g^{21} \leq -F(\infty)$, then, $\mathcal{S}^1 = \phi, \mathcal{S}^2 = (0, \infty)$;*
4. *If $a_1 g^{12} < F(\infty)$ and $a_1 g^{21} \geq 0$, then, $\mathcal{S}^1 = [\underline{x}^1, \infty), \mathcal{S}^2 = \phi$, where $\underline{x}^1 \in (0, \infty)$.*
5. *If $a_1 g^{12} < F(\infty)$ and $-F(\infty) < a_1 g^{21} < 0$, then, $\mathcal{S}^1 = [\underline{x}^1, \infty), \mathcal{S}^2 = (0, \overline{x}^2]$, where $\underline{x}^1, \overline{x}^2 \in (0, \infty)$.*

The Structure of Switching Regions

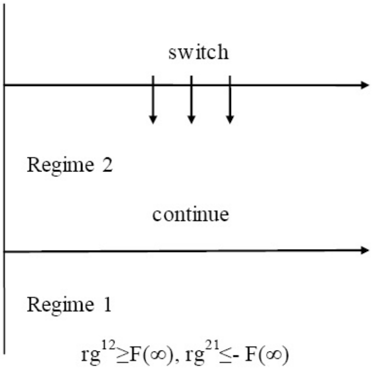


The Structure of Switching Regions



$$rg^{12} \geq F(\infty), \; -F(\infty) < rg^{21} < 0$$

The Structure of Switching Regions



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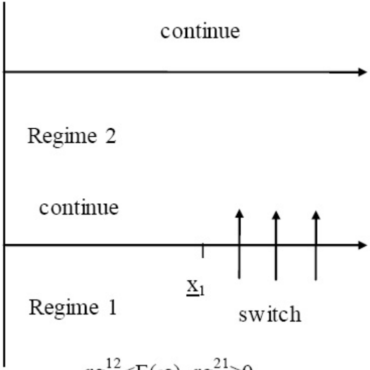
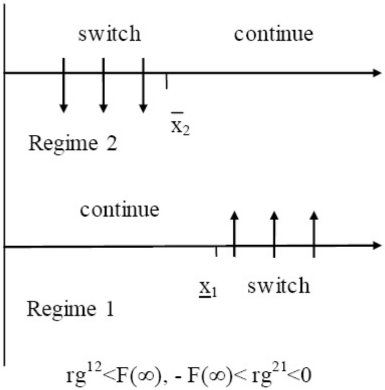


$rg^{12} < F(\infty)$, $rg^{21} \geq 0$

The Structure of Switching Regions



The Proof of Optimal Switching Representation

Reformulation of the Switching System

- Observe (Y, Z) solve (1), if and only if $(U_t^i, V_t^i) = (e^{rt} Y_t^i, e^{rt} Z_t^i)$ for $t \geq 0$, $1 \leq i \leq d$ and $r \leq a$ solve the following infinite horizon BSDE system:

$$\begin{cases} U_t^i = U_T^i + \int_t^T \tilde{f}_s^i(U_s, V_s) + \lambda \max\left\{0, \tilde{\mathcal{M}} U_t^i - U_t^i\right\} ds - \int_t^T V_s^i dW_s, \\ \lim_{T\uparrow\infty} \mathbf{E}\left[e^{2(a-r)T} |U_T|^2\right] = 0. \end{cases} \quad (5)$$

- The driver $\tilde{f}_s = (\tilde{f}_s^1, \cdots, \tilde{f}_s^d)^*$ is given by

$$\tilde{f}_s^i(y, z) = e^{rs} f_s^i(e^{-rs} y, e^{-rs} z) - ry^i$$

for $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$, and the impulse term $\tilde{\mathcal{M}} U_t^i$ is defined as

$$\tilde{\mathcal{M}} U_t^i = \max_{j\neq i} \left\{ U_t^j - e^{rt} g_t^{ij} \right\}.$$

Reformulation of the Optimal Switching Model

- Observe the solution Y_t^i to (1) is the value of the optimal switching problem (4) with the optimal switching strategy u^*, if and only if the solution $U_t^i = e^{rt}Y_t^i$ to (5) is the value of the following optimal switching problem (without discounting):

$$e^{rt}y_t^{i,(t,\lambda)} = \operatorname*{ess\,sup}_{u\in\mathcal{K}_i(t,\lambda)} \mathbf{E}\left[\int_t^{\infty} \tilde{f}_s^{u_s}(U_s,V_s)ds - \sum_{k\geq 1} e^{rT_k} g_{T_k}^{\alpha_{k-1},\alpha_k} | \mathcal{G}_t^{(t,\lambda)}\right]. \tag{6}$$

- The optimal switching strategy is $\tau_0^* = t$ and $\alpha_0^* = i$,

$$\tau_{k+1}^* = \inf\left\{T_N > \tau_k^* : U_{T_N}^{\alpha_k^*} \leq \tilde{\mathcal{M}} U_{T_N}^{\alpha_k^*}\right\}$$

where

$$\alpha_{k+1}^* = \operatorname*{arg\,max}_{j\neq\alpha_k^*}\left\{U^j_{\tau^*_{k+1}} - e^{r\tau^*_{k+1}} g_{\tau^*_{k+1}}^{\alpha_k^*,j}\right\}.$$

Optimal Stopping Representation

Lemma

Let (U,V) be the unique solution to the infinite horizon BSDE system (5). For $n\geq 0$ and $1\leq i\leq d$, consider the following auxiliary optimal stopping problem:

$$\tilde{y}_{T_n}^{i,(t,\lambda)}=\operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_{n+1}}(t,\lambda)}\mathbf{E}\left[\int_{T_n}^{\tau}\tilde{f}_s^i(U_s,V_s)ds+\tilde{\mathcal{M}}U_\tau^i|\mathcal{G}_{T_n}^{(t,\lambda)}\right],\qquad(7)$$

where the control set $\mathcal{R}_{T_{n+1}}(t,\lambda)$ is defined as

$$\mathcal{R}_{T_{n+1}}(t,\lambda)=\left\{\mathbb{G}^{(t,\lambda)}\text{-stopping time }\tau\text{ for }\tau(\omega)=T_k(\omega)\text{ where }k\geq n+1\right\}.$$

Then its value is given by $\tilde{y}_{T_n}^{i,(t,\lambda)}=U_{T_n}^i$, a.s. for $n\geq 0$, and in particular, $\tilde{y}_t^{i,(t,\lambda)}=U_t^i$, a.s. for $t\geq 0$. The optimal stopping time is given by

$$\tau_{T_{n+1}}^*=\inf\left\{T_k\geq T_{n+1}:U_{T_k}^i\leq\tilde{\mathcal{M}}U_{T_k}^i\right\}.$$

The Proof of Optimal Switching Representation

- Take any switching strategy $u \in \mathcal{K}_i(t, \lambda)$ with the form:

$$u_s = i\mathbf{1}_{\{t\}}(s) + \sum_{k \geq 0} \alpha_k \mathbf{1}_{(T_k, T_{k+1}]}(s).$$

- Consider the optimal stopping time problem (7) starting from $T_0 = t$, stopping at the first Poisson arrival time T_1, and switching to α_1:

$$\tilde{y}_t^{i,(t,\lambda)} \geq \mathbf{E}\left[\int_t^{T_1} \tilde{f}_s^i(U_s, V_s)ds + U_{T_1}^{\alpha_1} - e^{rT_1} g_{T_1}^{i,\alpha_1} | \mathcal{G}_t^{(t,\lambda)}\right]. \quad (8)$$

- The value of the optimal stopping time problem (7) starting from T_1 is given by $\tilde{y}_{T_1}^{\alpha_1,(t,\lambda)} = U_{T_1}^{\alpha_1}$. Consider such an optimal stopping time problem stopping at the Poisson arrival time T_2, and switching to α_2:

$$U_{T_1}^{\alpha_1} = \tilde{y}_{T_1}^{\alpha_1,(t,\lambda)} \geq \mathbf{E}\left[\int_{T_1}^{T_2} \tilde{f}_s^{\alpha_1}(U_s, V_s)ds + U_{T_2}^{\alpha_2} - e^{rT_2} g_{T_2}^{\alpha_1,\alpha_2} | \mathcal{G}_{T_1}^{(t,\lambda)}\right]. \quad (9)$$

The Proof of Optimal Switching Representation Cont.

- Plugging (9) into (8), we obtain that $\tilde{y}_t^{i,(t,\lambda)} \geq$

$$\mathbf{E}\left[\int_t^{T_1} \tilde{f}_s^i(U_s, V_s)ds + \int_{T_1}^{T_2} \tilde{f}_s^{\alpha_1}(U_s, V_s)ds - e^{rT_1} g_{T_1}^{i,\alpha_1} - e^{rT_2} g_{T_2}^{\alpha_1,\alpha_2} + U_{T_2}^{\alpha_2} | \mathcal{G}_t^{(t,\lambda)}\right]$$

- Repeat the above procedure M times:

$$\tilde{y}_t^{i,(t,\lambda)} \geq \mathbf{E}\left[\int_t^{T_{M+1}} \tilde{f}_s^{u_s}(U_s, V_s)ds - \sum_{k=1}^{M+1} e^{rT_k} g_{T_k}^{\alpha_{k-1},\alpha_k} + U_{T_{M+1}}^{\alpha_{M+1}} | \mathcal{G}_t^{(t,\lambda)}\right].$$

- Recall that the solution U_T converges to zero in L^2 as $r \leq a$:

$$\lim_{T\uparrow\infty} \mathbf{E}[|U_T|^2] \leq \lim_{T\uparrow\infty} \mathbf{E}[e^{2(a-r)T}|U_T|^2] = \lim_{T\uparrow\infty} \mathbf{E}[e^{2aT}|Y_T|^2] = 0.$$

  Hence, letting $M \uparrow \infty$, we get

$$\tilde{y}_t^{i\,(t,\lambda)} \geq \mathbf{E}\left[\int_t^{\infty} \tilde{f}_s^{u_s}(U_s, V_s)ds - \sum_{k\geq 1} e^{rT_k} g_{T_k}^{\alpha_{k-1},\alpha_k} | \mathcal{G}_t^{(t,\lambda)}\right].$$

- Taking the supremum over $u \in \mathcal{K}_i(t, \lambda)$,

$$\tilde{y}_t^{i\,(t,\lambda)} \geq \sup_{u \in \mathcal{K}_i(t,\lambda)} \mathbf{E}\left[\int_t^{\infty} \tilde{f}_s^{u_s}(U_s, V_s)ds - \sum_{k \geq 1} e^{rT_k} g_{T_k}^{\alpha_{k-1},\alpha_k} | \mathcal{G}_t^{(t,\lambda)}\right].$$

- Since $U_t^i = \tilde{y}_t^{i\,(t,\lambda)}$, we obtain that

$$U_t^i \geq e^{rt} y_t^{i\,(t,\lambda)}.$$

- The reverse inequality is obtained by considering the optimal switching strategy $u = u^*$.

Reformulation of the Auxiliary Optimal Stopping Problem

- An equivalent formulation of the optimal stopping time problem (7):

$$\tilde{y}_{T_n}^{i,(t,\lambda)} = \operatorname*{ess\,sup}_{N \in \mathcal{N}_{n+1}(t,\lambda)} \mathbf{E}\left[\int_{T_n}^{T_N} \tilde{f}_s^i(U_s, V_s)ds + \tilde{\mathcal{M}} U_{T_N}^i | \mathcal{G}_{T_n}^{(t,\lambda)} \right], \quad (10)$$

  where the control set $\mathcal{N}_{T_{n+1}}(t,\lambda)$ is defined as

$$\mathcal{N}_{n+1}(t,\lambda) = \left\{ \tilde{\mathbb{G}}^{(t,\lambda)}\text{-stopping time } N \text{ for } N \geq n+1 \right\}.$$

- (10) is a discrete optimal stopping problem, as the player is allowed to stop at a sequence of integers $n+1, n+2, \cdots$.
- The optimal stopping time is then some integer-valued random variable N_{n+1}^*:

$$N_{n+1}^* = \inf \left\{ k \geq n+1 : U_{T_k}^i \leq \tilde{\mathcal{M}} U_{T_k}^i \right\}.$$

The Proof of Optimal Stopping Representation

- The first observation: the solution to the infinite horizon BSDE system (5) on the Poisson arrival time T_n can be calculated recursively as follows:

$$U^i_{T_n} = \mathbf{E}\left[\int_{T_n}^{T_{n+1}} \tilde{f}^i_s(U_s, V_s)ds + \max\{\tilde{\mathcal{M}}U^i_{T_{n+1}}, U^i_{T_{n+1}}\}|\mathcal{G}^{(t,\lambda)}_{T_n}\right].$$

- The second observation: If we define $\widehat{U}^i = \max\left\{\tilde{\mathcal{M}}U^i, U^i\right\}$, then $\widehat{U}^i$ satisfies the following recursive equation:

$$\widehat{U}^i_{T_n} = \max\left\{\tilde{\mathcal{M}}U^i_{T_n}, \mathbf{E}\left[\int_{T_n}^{T_{n+1}} \tilde{f}^i_s(U_s, V_s)ds + \widehat{U}^i_{T_{n+1}}|\mathcal{G}^{(t,\lambda)}_{T_n}\right]\right\}.$$

From the snell envelop theory,

$$\hat{U}^i_{T_n} = \operatorname*{ess\,sup}_{N\in\mathcal{N}_n(t,\lambda)} \mathbf{E}\left[\int_{T_n}^{T_N} \tilde{f}^i_s(U_s, V_s)ds + \tilde{\mathcal{M}}U^i_{T_N}|\mathcal{G}^{(t,\lambda)}_{T_n}\right].$$

The Proof of Optimal Stopping Representation Cont.

- From the first observation:

$$U^i_{T_n} = \mathbf{E}\left[\int_{T_n}^{T_{n+1}} \tilde{f}^i_s(U_s, V_s)ds + \widehat{U}^i_{T_{n+1}} | \mathcal{G}^{(t,\lambda)}_{T_n}\right]. \qquad (11)$$

- From the second observation: for any $N \in \mathcal{N}_{n+1}(t,\lambda)$,

$$\widehat{U}^i_{T_{n+1}} \geq \mathbf{E}\left[\int_{T_{n+1}}^{T_N} \tilde{f}^i_s(U_s, V_s)ds + \tilde{\mathcal{M}} U^i_{T_N} | \mathcal{G}^{(t,\lambda)}_{T_{n+1}}\right]. \qquad (12)$$

- Plugging (12) into (11), we obtain that

$$U^i_{T_n} \geq \mathbf{E}\left[\int_{T_n}^{T_{n+1}} \tilde{f}^i_s(U_s, V_s)ds + \int_{T_{n+1}}^{T_N} \tilde{f}^i_s(U_s, V_s)ds + \tilde{\mathcal{M}} U^i_{T_N} | \mathcal{G}^{(t,\lambda)}_{T_n}\right].$$

  Taking the supremum over $N \in \mathcal{N}_{n+1}(t,\lambda)$ gives us $U^i_{T_n} \geq \tilde{y}^{i,(t,\lambda)}_{T_n}$.
- The reverse inequality is obtained by considering the optimal stopping time $N = N^*_{n+1}$.

Some Future Works

- Relating to piecewise-deterministic Markov process: Davis (1984).
- Relating to randomized stopping time technique: Krylov (1980), Buckdahn and Engelbert (1984).
- More applications, such as
  - Optimal switching of rough differential equation,
  - Optimal investment switching,
  - Numerical solution to (oblique) reflected BSDE,
  - $\cdots$.

Thank You