An explicit Euler scheme with strong rate of convergence for non-Lipschitz SDEs

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Content

Introduction

Continuous-time random dynamics on $\mathbb{R}$ for $t \in [0, T]$:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \qquad X_0 = x_0 \tag{1}$$

where

- drift and diffusion, $\mu, \sigma : \mathbb{R} \to \mathbb{R}$
- W is a Brownian motion
- Assumption: we have a unique strong solution.

Convergence results for discretisation schemes:
Maruyama [Mar55], Milstein [Mil75], Kloeden & Platen [KP92]

Explicit Euler Scheme I

μ and σ are globally Lipschitz: $\exists K > 0$ such that $\forall x, y \in \mathbb{R}$

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|$$

Fix $n \in \mathbb{N}^+$, consider partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$.

Definition (Explicit Euler Scheme)

Equidistant discretisation, $h = T/n$,

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \mu(\hat{X}_{t_i})h_{i+1} + \sigma(\hat{X}_{t_i})\Delta W_{i+1}, \qquad \hat{X}_0 = x_0,$$

where $h_{i+1} = t_{i+1} - t_i$ and $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$.

Explicit Euler Scheme II

Definition

For $p > 0$, define $\|Z\|_p := \mathbb{E}\left[|Z|^p\right]^{\frac{1}{p}}$.

Linear interpolation defines $\hat{X}$ for all $t \in [0, T]$.

Theorem (Lipschitz drift and diffusion [KP92])

$\max_{t\in[0,T]} \|X_t - \hat{X}_t\|_2 \leq Ch^{1/2}$.

Rate of strong convergence $1/2$.

Non-classical SDEs

Moving away from the classical setting:

- SDE solution taking values in some domain, D (typically we will consider $D = (0, \infty)$);
- drift or diffusion not globally Lipschitz continuous.

Definition (One-sided Lipschitz continuous)

A function f is one-sided Lipschitz continuous in D if for all $x, y \in D$, then $(x - y)(f(x) - f(y)) \leq K(x - y)^2$.

Approaches:

- **Localisation:** A modification of scheme, say, $|x|$ for $D = (0, \infty)$ and monotonic drift [Gyö98];
- **Implicit scheme:** Strong convergence rate, when drift is one-sided Lipschitz, locally Lipschitz, and we have finite moments for process [HMS02].

Some more Euler schemes

Definition (Symmetrised Euler scheme [BD04])

$$\hat{X}_{t_{i+1}} = |\hat{X}_{t_i} + \mu(\hat{X}_{t_i})h_{i+1} + \sigma(\hat{X}_{t_i})\Delta W_{i+1}|, \qquad \hat{X}_0 = x_0.$$

Definition (Implicit-Euler scheme [Alf13, NS12])

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \mu(\hat{X}_{t_{i+1}})h_{i+1} + \sigma(\hat{X}_{t_i})\Delta W_{i+1}, \qquad \hat{X}_0 = x_0.$$

Definition (Tamed-Euler scheme [HJK12])

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \frac{\mu(\hat{X}_{t_i})h_{i+1}}{1 + |\mu(\hat{X}_{t_i})|h_{i+1}} + \sigma(\hat{X}_{t_i})\Delta W_{i+1}, \qquad \hat{X}_0 = x_0.$$

Setting

- Shift non-Lipschitz behaviour from the diffusion to the drift.
- Apply Lamperti-style transformation $Y = F(X)$ to (1):

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}t + \gamma(Y_t)\mathrm{d}W_t, \qquad Y_0 = y_0 > 0.$$

Assumption (First assumptions)

- *The solution stays in domain $D = (0, \infty)$, almost surely;*
- *f is globally one-sided Lipschitz;*
- *f is locally Lipschitz: $x, y \in D$, then* $|f(x) - f(y)| \leq K(1 + |x|^\alpha + |y|^\alpha + \frac{1}{|x|^\beta} + \frac{1}{|y|^\beta})|x - y|$;
- *γ is Lipschitz continuous on D.*

Projection

- For a closed interval $\mathcal{C} \subset \mathbb{R}$, define $p_{\mathcal{C}} : \mathbb{R} \to \mathcal{C}$ as the projection operator onto $\mathcal{C}$;
- domain $D_n = [n^{-k}, n^{k'}] \subseteq D$ with strictly positive k, k' (possibly infinite);
- projection map $p_n := p_{D_n}$, such that $p_n : \mathbb{R} \to D_n$ where

$$p_n(x) \equiv n^{-k} \vee x \wedge n^{k'}, \qquad k, k' > 0. \tag{2}$$

  Clearly, p_n is one-Lipschitz;
- closure of domain D, $\bar{D} = [0, \infty)$ defines the projection $p_{\bar{D}}$.

Scheme

We now introduce our explicit scheme:

Definition (Explicit Euler scheme with projection map)

Set $\hat{Y}_0 = Y_0$ and for $i = 0, \ldots, n-1$,

$$\hat{Y}_{i+1} := \hat{Y}_i + f_n(\hat{Y}_i)h_{i+1} + \bar{\gamma}_n(\hat{Y}_i)\Delta W_{i+1},$$

with $f_n \equiv f \circ p_n$ and $\bar{\gamma}_n \equiv \gamma \circ p_{\bar{D}}$.

Remark

f_n is Lipschitz continuous with Lipschitz constant

$$L(n) = 2K(1 + n^{k\beta} + n^{k'\alpha}).$$

Regularity assumptions

Assumption (Scheme constants)

Set constants k, k' such that $2\beta k \leq 1$ and $2\alpha k' \leq 1$.

Assumption (Weaker conditions)

There exists $q' > 2(\alpha + 1)$ and $q > 2\beta$ such that $\mathbb{E}(|Y_t|^{q'})$ and $\mathbb{E}(|Y_t|^{-q})$ are finite for all $t \in [0, T]$.

Assumption (Stronger conditions)

Above hold; in addition drift function f is of class $\mathcal{C}^2(D)$, and

$$\sup_{t\in[0,T]} \mathbb{E}|\gamma(Y_t)f'(Y_t)|^2 + \sup_{t\in[0,T]} \mathbb{E}\left|f'(Y_t)f(Y_t) + \frac{\gamma^2(Y_t)}{2}f''(Y_t)\right|^2$$

is finite.

Preliminary bounds on process

Lemma (Bounds using Weaker assumptions)

For any $t \in [0, T]$, the following inequalities hold:

- $\mathbb{E}|Y_t - p_n(Y_t)|^2 \leq C\left(\frac{1}{n^{(q+2)k}} + \frac{1}{n^{(q'-2)k'}}\right) =: K_1(n);$
- $\mathbb{E}|f(Y_t) - f_n(Y_t)|^2 \leq C\left(\frac{1}{n^{k(q-2(\beta-1))}} + \frac{1}{n^{k'(q'-2(\alpha+1))}}\right) =: K_2(n).$

Regularity

Definition (Regularity of process X)

Given partition π, we define the regularity of a process as

$$\mathcal{R}_\pi[X] := \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|X_t - X_{t_i}|^2 \mathrm{d}t.$$

Lemma (Process regularity)

If our weaker assumptions hold, then

- $\mathcal{R}_\pi[Y] \leq Ch$ *and* $\mathcal{R}_\pi[f(Y)] \leq C(L(n)^2 h + K_2(n))$.

Furthermore, assume the stronger conditions hold, then

- $\mathcal{R}_\pi[f(Y)] \leq Ch$.

Define the discretisation error as $\delta Y_i := Y_{t_i} - \hat{Y}_{t_i}$. Combine the preliminary bounds and regularity:

Main theorem

Theorem (Convergence result)

Assume that the weaker assumptions hold. Then

$$\max_{i=0,\dots,n} \mathbb{E}|\delta Y_i|^2 \leq C\left(K_2(n) + \mathcal{R}_\pi[f(Y)] + \mathcal{R}_\pi[Y]\right).$$

Theorem (Convergence result (continued))

Furthermore, the following holds:

$$\max_{i=0,\dots,n} \|\delta Y_i\|_2 \leq C_{q,q'} h^r,$$

with

- $r = \min(\frac{1}{2} - \frac{\beta}{q+2}, \frac{1}{2} - \frac{\alpha}{q'-2}) > 0$ *under the weak assumptions by setting* $(k, k') = (\frac{1}{q+2}, \frac{1}{q'-2})$
- $r = \min(\frac{1}{2}, \frac{q+2}{4\beta} - \frac{1}{2}, \frac{q'-2}{4\alpha} - \frac{1}{2}) > 0$ *under the strong assumptions by setting* $(k, k') = (\frac{1}{2\beta}, \frac{1}{2\alpha})$.

Modifications of scheme

Definition (Domains $\bar{D}_\eta$ and $\breve{D}_\zeta$)

- Domain $\bar{D}_\eta := [\eta, \infty)$, and $p_{\bar{D}_\eta} : \mathbb{R} \to \bar{D}_\eta$;
- Interval $\breve{D}_\zeta := [0, \zeta]$, and $p_{\breve{D}_\zeta} : \mathbb{R} \to \breve{D}_\zeta$.

For all $i \le n$, we define $\bar{Y}_{t_i} := p_{\bar{D}}(\hat{Y}_{t_i})$, $\tilde{Y}_{t_i} := p_{\bar{D}_\eta}(\hat{Y}_{t_i})$ and $\breve{Y}_{t_i} := p_{\breve{D}_\zeta}(\hat{Y}_{t_i})$, for some $\eta, \zeta > 0$.

Corollary (Modified schemes)

In the setting of the main theorem, we have

$$\max_{i=0,\dots,n} \left(\| Y_{t_i} - \bar{Y}_{t_i} \|_2 + \| Y_{t_i} - \tilde{Y}_{t_i} \|_2 + \| Y_{t_i} - \breve{Y}_{t_i} \|_2 \right) \le C_{q,q'} h^r \ ,$$

where $(\tilde{Y}_{t_i})_{i \le n}$ *and* $(\breve{Y}_{t_i})_{i \le n}$, *we set* $\eta = h^{2r/q}$ *and* $\zeta = h^{-2r/(q'-2)}$.

First order convergence

Proposition (First order convergence for constant diffusion)

$\gamma(x) \equiv \gamma > 0$ for all $x \in D$, and stronger assumptions holds, with $q > 6\beta - 2$ and $q' > 6\alpha + 2$. Then, $\max_{i=0,\dots,n} \|\delta Y_i\|_2 \leq C_{q,q'} h$.

Moment properties of the schemes

Lemma (Modified scheme)

Under the weak assumptions, then $\max_{i=0,\dots,n} \mathbb{E}\left[|\hat{Y}_{t_i}|^2\right] \leq C_{q,q'}$.

Proposition

1. *If weak assumptions holds, then* $\max_{i=0,\dots,n} \mathbb{E}\left[\check{Y}_{t_i}^{p'}\right] \leq C_{p',q'}$ *for all* $p' \in [1, q'/2]$;
2. *if weak assumptions hold with* $q \geq 4$, *then* $\max_{i=0,\dots,n} \mathbb{E}\left[\tilde{Y}_{t_i}^{-p}\right] \leq C_{p,q}$ *for all* $p \in [1, q/2-1]$.

Applications

Convergence of scheme for SDEs widely used in literature:

- Locally smooth coefficients (CIR/CEV families included);
- 3/2 process;
- Ait-Sahalia process.

Strategy:

- Verify assumptions on true process after a Lamperti transformation
  1. process stays in domain D, a.s.;
  2. drift being one-sided Lipschitz and locally Lipschitz (α and β);
  3. diffusion being Lipschitz continuous.
- Conditions on the scheme to fix k and k';
- Verify additional assumptions (finite moment and inverse moments of the true process, smoothness of drift f, ...).

A) Locally smooth coefficients

Consider the stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \gamma x^\nu \mathrm{d}W_t \qquad X_0 = x_0 > 0$$

where

- drift $\mu(x) \equiv \mu_1(x) - \mu_2(x)x$ with $\mu_1, \mu_2 : D \to \mathbb{R}$;
- $\gamma > 0$ and $\nu \in [1/2, 1]$;
- Three distinct cases: $\nu = 1/2$, $\nu \in (1/2, 1)$ and $\nu = 1$;
- Locally smooth coefficients: CIR, CEV families included.

A) Assumptions and corollary

Assumption

Functions μ_1, μ_2 *are bounded and belong to the class* $\mathcal{C}_b^2(D)$.

1. *If* $\nu \in (1/2, 1)$, *then* $\mu_1(0) > 0$;
2. *If* $\nu = 1/2$, *then there exists* $\bar{x} > 0$ *such that* $\omega := 2\mu_1(x)/\gamma^2 \geq 1$ *for all* $0 < x < \bar{x}$.

In [DM11, Proposition 3.1] it is shown that the unique strong solution stays in $(0, \infty)$.

Corollary

For the corresponding assumptions above, then
$$\max_{i=0,\ldots,n} \|\delta Y_i\|_2 + \|\delta X_i\|_1 \leq Ch^r$$

1. *with* $r = 1/2$;
2. *with* $r = 1/2 - 1/\omega > 0$ *if* $3 < \omega \leq 4$ *and* $r = 1/2$ *if* $\omega > 4$.

B) Ait-Sahalia model

Consider the stochastic differential equation

$$dX_t = \left(\frac{a_{-1}}{X_t} - a_0 + a_1 X_t - a_2 X_t^r \right) dt + \gamma X_t^\rho dW_t \ , \quad X_0 = x_0 > 0,$$

Lamperti transformation yields process

$$dY_t = f(Y_t)dt + (1-\rho)\gamma dW_t \ , \quad Y_0 = x_0^{1-\rho} > 0 \ ,$$

where the drift function is

$$f(x) \equiv (1-\rho) \left(a_{-1} x^{\frac{-1-\rho}{1-\rho}} - a_0 x^{\frac{-\rho}{1-\rho}} + a_1 x - a_2 x^{\frac{-\rho+r}{1-\rho}} - \frac{\rho\gamma^2}{2} x^{-1} \right) \ .$$

Corollary

Suppose that $r + 1 > 2\rho$ *holds, then* $\max_{i=0,\dots,n} \|\delta Y_i\|_2 \leq Ch^{1/2}$.

Numerical Results

We implement scheme and study strong rates of convergence achieved:

1. CIR (compared to the implicit scheme in [DNS12, NS12]);
2. Ginzburg-Landau (convergence and divergence for E-M scheme [HJK11]);
3. Ait-Sahalia (compared to a reference solution).

Absolute difference over M paths with $h := T/2^N$

$$\frac{1}{M}\sum_{j=1}^{M}|X_T^{(j)} - \hat{X}_T^{(j)}|$$

where $\hat{X}_t^{(j)}$ is the E-M with projection approximation and $X_t^{(j)}$ is the true/reference solution (using the same Brownian motion path).

CIR Rates of convergence I

CIR process X with $\kappa, \theta, \xi > 0$ defined by

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \xi\sqrt{X_t}\mathrm{d}W_t, \qquad X_0 = x_0 > 0.$$

Lamperti-transformed process Y:

$$\mathrm{d}Y_t = \left(\frac{a}{Y_t} + bY_t\right)\mathrm{d}t + c\mathrm{d}W_t, \qquad Y_0 = \sqrt{x_0} > 0.$$

with $a = (4\kappa\theta - \xi^2)/8$, $b = -\kappa/2$, $c = \xi/2$.

CIR Rates of convergence II

Drift-implicit square-root Euler method [DNS12] has unique positive solution defined for $i = 0, \dots, n-1$ by

$$\hat{Y}_{t_{i+1}} = \frac{\hat{Y}_{t_i} + c\Delta W_{i+1}}{2(1 - bh_{i+1})} + \sqrt{\frac{(\hat{Y}_{t_i} + c\Delta W_{i+1})^2}{4(1 - bh_{i+1})^2} + \frac{ah_{i+1}}{1 - bh_{i+1}}},$$

We approximate the CIR process, X, by $\hat{X} = \hat{Y}^2$ and the discretisation error is

$$\delta X_i := X_{t_i} - \hat{X}_{t_i} = Y_{t_i}^2 - \hat{Y}_{t_i}^2.$$

CIR Rates of convergence III

- Parameters $(\kappa, \theta, \xi, T, x_0, M) = (0.125\omega, 1, 0.5, 1, 1, 10000)$;
- $\omega = (1, 1.5, 2, 2.5, 3, 3.5, 4)$, and $2\kappa\theta/\xi^2 = \omega$;
- step sizes 2^N, for $N = 1, \ldots, 10$;
- reference solution uses $N = 12$.
- $k = 1/4$;
- Strong convergence with rate 1, as the Lamperti transformed CIR process has a constant diffusion.

CIR Rates of convergence IV

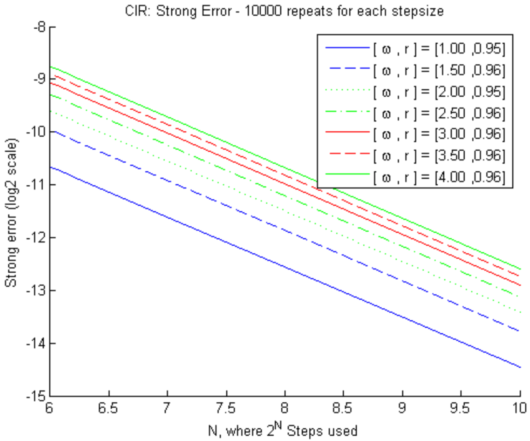

Figure: CIR model: $\mathcal{E}$ against number of steps ($\log_2$ scale).

Ginzburg-Landau - Strong Convergence I

1-d G-L SDE [KP92, Chapter 4]

$$\mathrm{d}X_t = \left[-X_t^3 + \left(\alpha + \frac{1}{2}\sigma^2\right) X_t\right] \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_0 = x_0 > 0,$$

with solution

$$X_t = \frac{X_0 \exp(\alpha t + \sigma W_t)}{\sqrt{1 + X_0^2 \int_0^t \exp(2\alpha s + 2\sigma W_s)\mathrm{d}s}}.$$

Special case of the Ait-Sahalia process with $(a_{-1}, a_0, a_1, a_2, r, \rho) = (0, 0, \alpha + 1/2\sigma^2, 1, 3, 1)$. Bounded positive moments and inverse moments since $r + 1 > 2\rho$ holds and the solution stays in the domain $D = (0, \infty)$ almost surely.

Ginzburg-Landau - Strong Convergence II

- Parameters $(\sigma, \lambda, T, x_0, 10000) = (1, 1/2, 1, 1, 10000)$.
- Strong convergence - rate $1/2$:

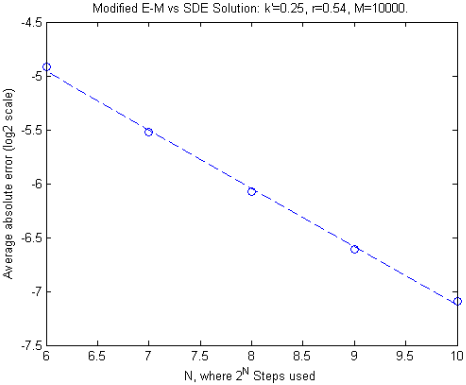

Figure: G-L model: average absolute error $\mathcal{E}$ vs N ($\log_2$ scale).

Ginzburg-Landau - E-M Divergence I

- Consider an example for which Euler-Maruyama scheme diverges;
- Compare it to our explicit scheme;
- Parameters $(\sigma, \alpha, T, x_0, M) = (7, 0, 3, 1, 10000)$ as in [HJK11].
- The authors prove moment explosion for the Euler-Maruyama scheme, see [HJK11, Table 1].

Ginzburg-Landau - E-M Divergence II

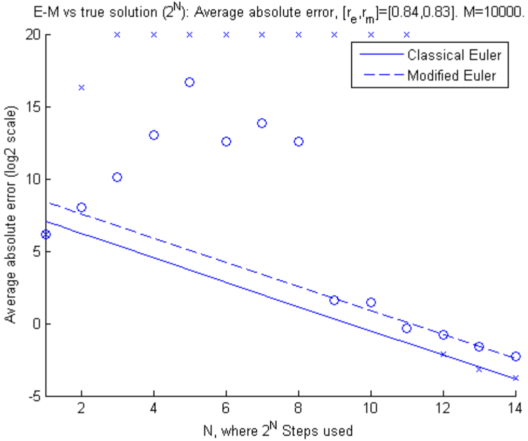

Figure: Average absolute error $\mathcal{E}$ vs number of steps ($\log_2$ scale).

Ginzburg-Landau - E-M Divergence III

- Both schemes eventually converge.
- However, for a range of step sizes, classical E-M scheme explodes [HJK11].
- Large errors and *NaN* are capped at 2^{20} for the E-M scheme.

Ait-Sahalia

Consider the Ait-Sahalia model, with parameters $(a_{-1}, a_0, a_1, a_2, \gamma, X_0) = (1, 1, 1, 1, 1, 1)$;

- $(r, \rho) = (2, 3/2)$;
- $\alpha = 4$ and $\beta = 2$;
- $k = 1/(2\beta)$ and $k' = 1/(2\alpha)$, such that assumptions hold.

Numerics: L^1 rate for X of 1.25

Further work

Extending our results to:

- discontinuous drift functions;
- multi-dimensional domains (e.g. $D = (0, \infty)^d$ or $D = \mathbb{R}^d$);
- singularities in the interior of D;
- Multilevel Monte Carlo.

Thank you for listening

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