Supersolutions of BSDEs: Minimality, Constraints, Duality

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based on joint works with

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Outline

Minimal Supersolutions of BSDEs

Supersolutions of BSDEs under Constraints

Duality under Constraints

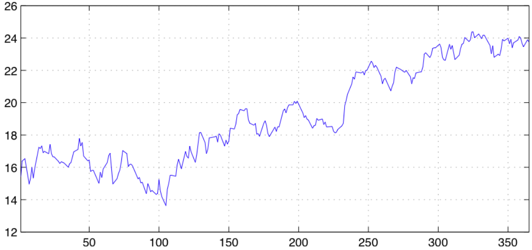
Minimal Supersolutions of BSDEs

Supersolutions of BSDEs

Motivation

Superhedging

$S_t(\omega)$ = price process of financial asset (stock)



Contingent claim ξ, for instance a call option $\xi = (S_T - K)^+$

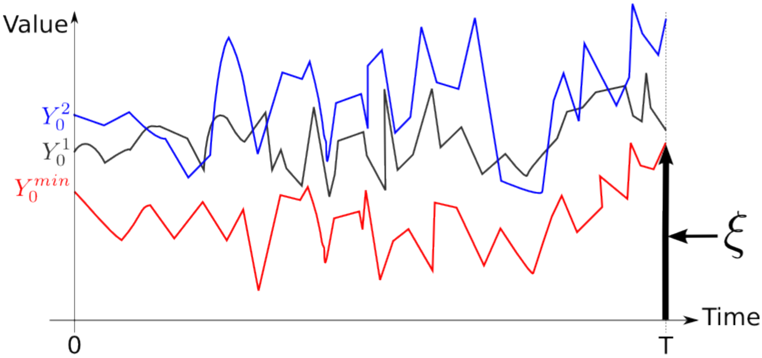
Goal: Find a (super-)hedging strategy for ξ.

Supersolutions of BSDEs

Motivation

$$Y_0 + \underbrace{\int_0^T Z_u dS_u}_{\text{trading gains}} \geq \xi$$

- Y_0= superhedging price of ξ
- Z= superhedging strategy



Supersolutions of BSDEs
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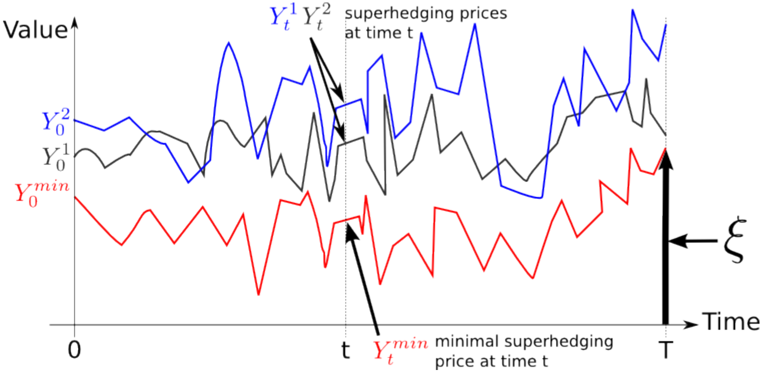
$$Y_t + \underbrace{\int_t^T Z_u dS_u}_{\text{trading gains}} \geq \xi$$

- Y_t= superhedging price of ξ at t
- Z= superhedging strategy



Supersolutions of BSDEs

Definition

Brownian filtered probability space $(\Omega, (\mathcal{F}_t), P, W)$

Definition

A *value process* Y together with a *control process* Z is supersolution of the Backward Stochastic Differential Equation with *generator* g and *terminal condition* ξ if

$$\begin{cases} Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t, \quad s \leq t \\ Y_T \geq \xi \end{cases} \qquad (*)$$

generator: values in $[0, \infty]$, jointly lower semicontinuous

terminal condition: $\mathcal{F}_T$-measurable.

value process: adapted and càdlàg $\rightsquigarrow \mathcal{S}$

control process: progressive, $\int_0^T Z_u^2 du < \infty$ and $\int ZdW$ supermartingale $\rightsquigarrow \mathcal{L}$

$$\mathcal{A} = \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : (Y, Z) \text{ fulfills } (*)\}$$

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Supersolutions of BSDEs

Minimality, principal aim

Supersolutions are not unique! $\rightsquigarrow$ find the minimal one:

A supersolution $(Y^{min}, Z^{min}) \in \mathcal{A}$ is called a Minimal Supersolution if $Y_t^{min} \leq Y_t$, $t \in [0, T]$, for any other supersolution $(Y, Z) \in \mathcal{A}$.

Theorem (D.,H.,K., 2013, AoP)

Assume that $\xi^- \in L^1$ and $\mathcal{A} \neq \emptyset$. If the generator g is

- *convex in z*
- *monotone in y*

then a unique minimal supersolution exists.

Proof of the theorem below strongly relies on compactness results.

$\rightarrow$ convexity is indispensable!

Aim:

Obtain existence and uniqueness of minimal supersolution under weakest possible assumptions on g. In particular:

drop the convexity!

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Minimal Supersolutions

Assumptions on the generator

A generator is a measurable function $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to [0, +\infty]$ such that

(LSC) $(y, z) \mapsto g(\omega, t, y, z)$ is lower semicontinuous for all (ω, t).

(NOR) $g(y, 0) = 0$ for all y.

Supersolutions of BSDEs

Main theorem

A natural candidate for the value process of a minimal supersolution:

$$\hat{\mathcal{E}}_t = \operatorname{ess\,inf}\{Y_t : (Y,Z) \in \mathcal{A}\}, \quad t \in [0,T].$$

Question: Does there exist a càdlàg modification $\mathcal{E}$ of $\hat{\mathcal{E}}$ and a control process $Z \in \mathcal{L}$ such that $(\mathcal{E}, Z)$ is a supersolution ?

Theorem

Assume g satisfies (LSC) and (NOR). Suppose $\xi^- \in L^1$ and $\mathcal{A} \neq \emptyset$. Then

$$\mathcal{E}_t := \hat{\mathcal{E}}_t^+ = \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s$$

is the value process of the unique minimal supersolution, that is, there exists a unique control process Z such that $(\mathcal{E}, Z) \in \mathcal{A}$.

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Supersolutions of BSDEs

Idea of the proof

Step 1: Uniform Approximation

- Suppose we find a sequence $((Y^n, Z^n)) \subset \mathcal{A}$ such that

$$\lim_{n\to\infty} \|\mathcal{E} - Y^n\|_{\mathcal{R}^\infty} = 0\,.$$

- A result by [BARLOW,PROTTER] yields

$$\lim_{n\to\infty} \left\| \int Z^n dW - M \right\|_{\mathcal{H}^1} = 0\,,$$

  where $\mathcal{E} = \mathcal{E}_0 + M - A$.

- By martingale representation we know that $M = \int ZdW$. Verification of $(\mathcal{E}, Z)$ belonging to $\mathcal{A}$ follows from (LSC).

Supersolutions of BSDEs

Idea of the proof

Step 2: A preorder on $\mathcal{A}$ and Zorn's Lemma

- For two supersolutions (Y^1, Z^1) and (Y^2, Z^2) in $\mathcal{A}$ we define the preorder $\preceq$ by

$$(Y^1, Z^1) \preceq (Y^2, Z^2) \quad :\Longleftrightarrow \begin{cases} \tau_1 \leq \tau_2 \\ (Y^1, Z^1) 1_{[0,\tau_1[} = (Y^2, Z^2) 1_{[0,\tau_1[} \end{cases}$$

  for the stopping time $\tau_i = \inf \{ t \geq 0 \, : \, Y^i_t > \mathcal{E}_t + \varepsilon \}$.

- For $((Y^i, Z^i))_{i \in I}$ a totally ordered chain we consider

$$\tau^* := \operatorname*{ess\,sup}_{i \in I} \tau_i \, .$$

- By monotonicity we find (τ_k) such that $\tau^* = \lim_k \tau_k$.

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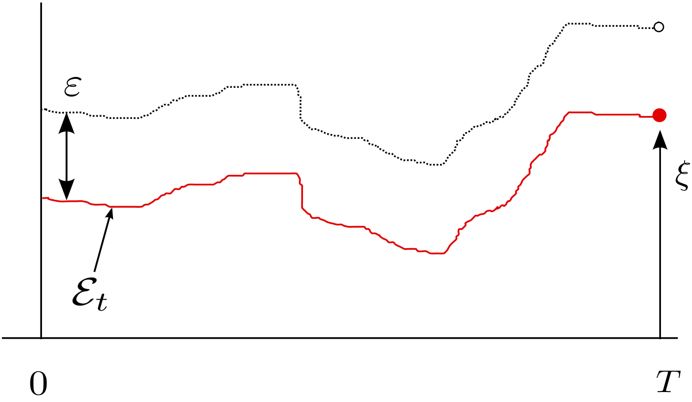
Supersolutions of BSDEs

Idea of the proof: Step 3: A candidate upper bound $(\overline{Y}, \overline{Z})$



Crucial part: construct upper bound $(\overline{Y}, \overline{Z})$ for the chain $((Y^i, Z^i))_{i \in I}$

Supersolutions of BSDEs

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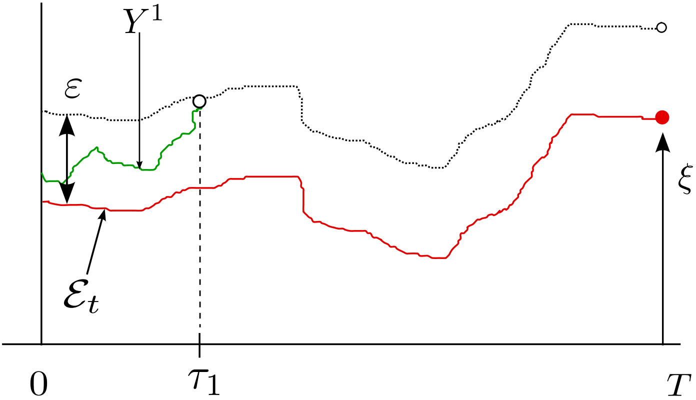

Paste corresponding supersolutions (Y^k, Z^k) at times τ_k up to τ^*

Supersolutions of BSDEs

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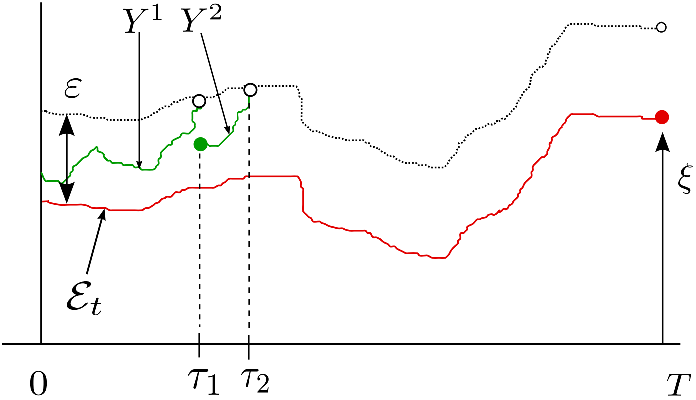

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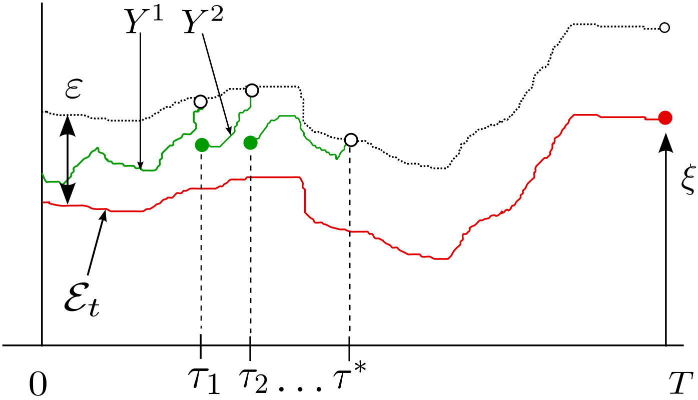

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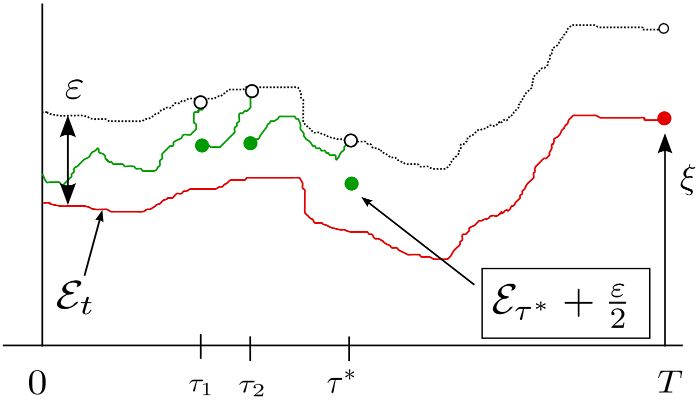
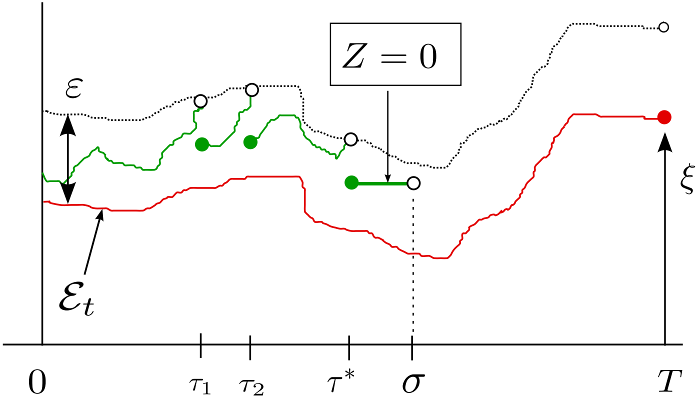
Idea of the proof



Jump down to $\mathcal{E}_{\tau^*} + \frac{\varepsilon}{2}$

Supersolutions of BSDEs

Idea of the proof



(NOR) allows for $Z = 0$ on short time interval $[\tau^*, \sigma[$ without leaving ε-nbh of $\mathcal{E}$

Supersolutions of BSDEs
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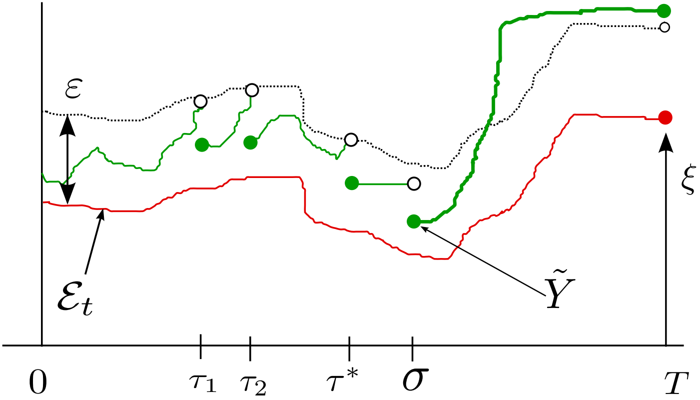
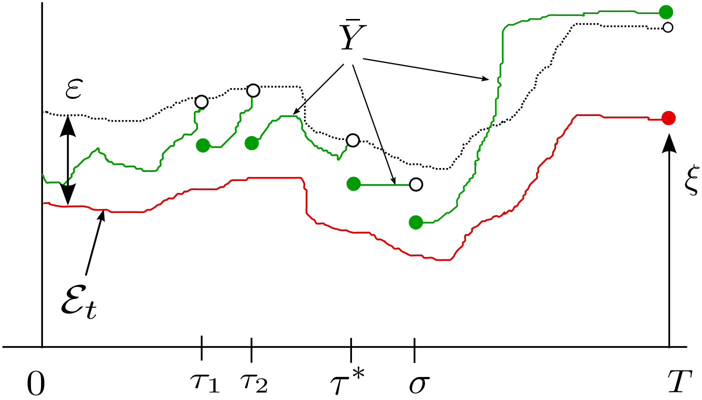


At time σ, there is $(\widetilde{Y}, \widetilde{Z}) \in \mathcal{A}$ lying below; Concatenate with it on $[\sigma, T]$

Supersolutions of BSDEs

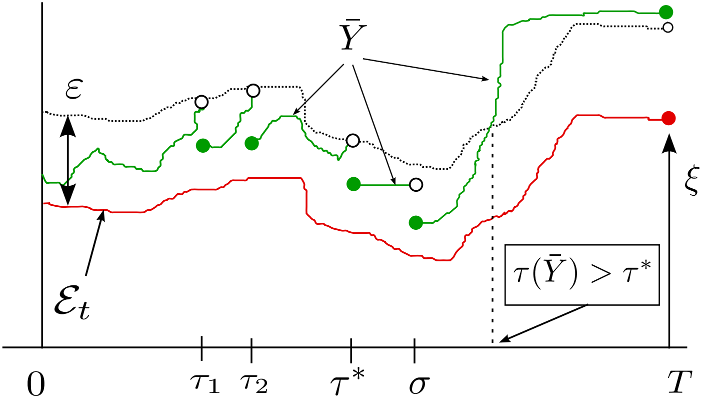
Idea of the proof



We have constructed a supersolution $(\overline{Y}, \overline{Z}) \in \mathcal{A}$

Supersolutions of BSDEs

Idea of the proof



$(\overline{Y}, \overline{Z})$ is an upper bound since it stays longer in ε-neighborhood than τ^*

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Final step: Zorn yields a maximal element (Y^M, Z^M).

- Verifying $\overline{Z} \in \mathcal{L}$ and that $(\overline{Y}, \overline{Z})$ satisfies $(*)$ yields that $(\overline{Y}, \overline{Z}) \in \mathcal{A}$ and we have thus constructed an upper bound. Zorn's lemma ensures the existence of a maximal element (Y^M, Z^M) with respect to $\preceq$.

- Finally, prove that the corresponding stopping time τ^M satisfies $\tau^M = T$ to conclude that

$$\left\| \mathcal{E} - Y^M \right\|_{\mathcal{R}^\infty} \leq \varepsilon ,$$

which finishes the proof.

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Supersolutions of BSDEs under Constraints

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Motivation

Being a priori only progressive, controls $Z \in \mathcal{L}$ exhibit in general no path regularities.

More structure $\rightsquigarrow$ constrain admissible controls to the specific set

$$\Theta := \left\{ Z \in \mathcal{L} : Z = z + \int \Delta du + \int \Gamma dW \right\}$$

$$\begin{cases} Y_s - \int_s^t g(Y_u, Z_u, \Delta_u, \Gamma_u) du + \int_s^t Z_u dW_u \geq Y_t, & 0 \leq s \leq t \leq T \\ Y_T \geq \xi \end{cases} \quad (*)$$

Set of constrained supersolutions with generator g and terminal condition ξ:

$$\mathcal{A} := \{(Y, Z) \in \mathcal{S} \times \Theta : (Y, Z) \text{ fulfills } (*)\}$$

Incorporated: Gamma Constraints, short-selling,...

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Supersolutions of BSDEs under Constraints

Generator, Notion of minimality

1. Introducing constraints comes at a cost. Generators need to satisfy

(CON) $(y, z, \delta, \gamma) \mapsto g(y, z, \delta, \gamma)$ is jointly convex

(DGC) $g(y, z, \delta, \gamma) \geq c_1 + c_2 \left(|\delta|^2 + |\gamma|^2\right)$ for $c_1 \in \mathbb{R}, c_2 > 0$.

2. Furthermore, we introduce a specific notion of minimality.

Definition

Fix a time $t \in [0, T]$. A supersolution (Y^{min}, Z^{min}) is said to be minimal at time t if it holds

$$Y_t^{min} \leq Y_t \quad \text{for all } (Y, Z) \in \mathcal{A} \quad \text{satisfying} \quad Z_{[0,t]} = Z_{[0,t]}^{min}.$$

Justification: pasting arbitrary supersolutions violates the constraints!

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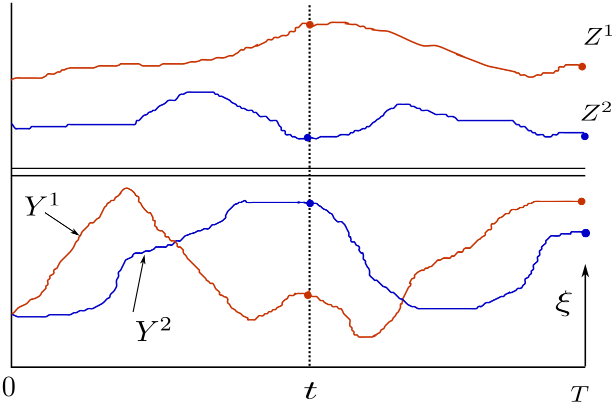
Supersolutions of BSDEs under Constraints

Pasting without constraints

Illustration: Pasting without constraints



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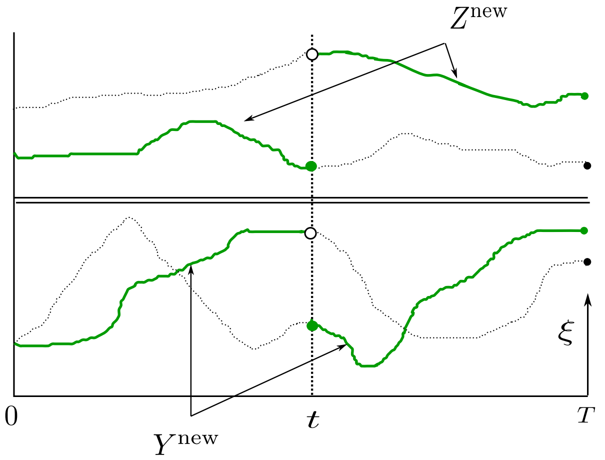

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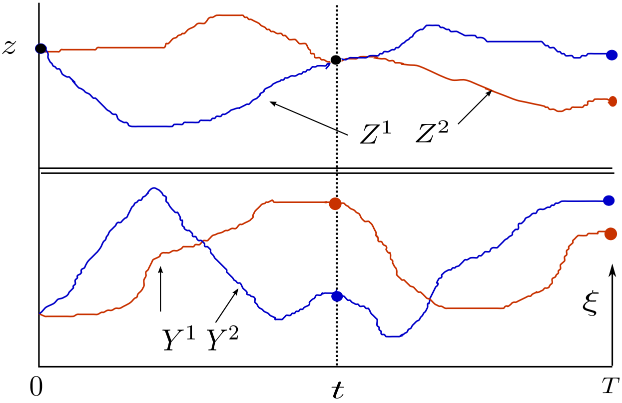

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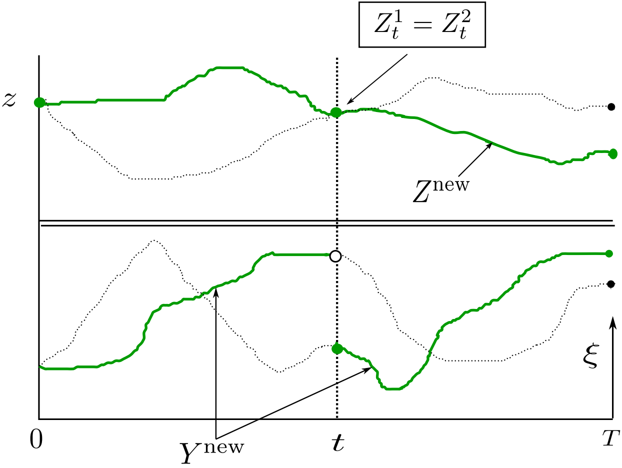

Supersolutions of BSDEs under Constraints

Existence of supersolution minimal at time t

At $t \in [0, T]$ $\rightsquigarrow$ candidate for the value process of a minimal supersolution given $Z^*_{[0,t]}$:

$$\mathcal{E}_t\left(Z^*_{[0,t]}\right) = \operatorname{ess\,inf}\left\{Y_t : (Y, Z) \in \mathcal{A} \quad \text{fulfilling } Z_{[0,t]} = Z^*_{[0,t]}\right\}.$$

Theorem

Assume a positive lsc generator g fulfils (CON) and (DGC). Suppose $\xi^- \in L^1$ and $\mathcal{A} \neq \emptyset$. Then for each attainable control $Z^*_{[0,t]}$ the set

$$\left\{(Y, Z) \in \mathcal{A} : Y_t = \mathcal{E}_t\left(Z^*_{[0,t]}\right) \quad \text{and} \quad Z_{[0,t]} = Z^*_{[0,t]}\right\}$$

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Supersolutions of BSDEs under Constraints

Idea of the proof

Idea of the proof for $t = 0$

- Choose minimizing sequence $((Y^n, Z^n))$ such that $Y_0^n \downarrow \mathcal{E}_0^g$.
- Compactness arguments yield $((\tilde{Y}^n, \tilde{Z}^n))$ satisfying $\int Z^n dW \to \int Z dW$ in $\mathcal{H}^2$, compare [DS] and [DHK].
- $Z \in \Theta$ and $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ by means of (DGC) and (CON).
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Supersolutions of BSDEs under Constraints

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Supersolutions of BSDEs under Constraints

Existence of supersolution minimal at finitely many times

The preceding result may be extended to finitely many times.

Theorem

Assume a positive lsc generator g fulfils (CON) and (DGC). Suppose $\xi^- \in L^1$ and $\mathcal{A} \neq \emptyset$. Then for each finite subset $\{t_1, \ldots, t_n\}$ of $[0, T]$ the set

$$\{(Y, Z) \in \mathcal{A} \,:\, Y_{t_i} = \mathcal{E}_{t_i}\left(Z_{[0,t_i]}\right) \quad \text{for all } i = 1, \ldots, n\}$$

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Duality under Constraints

Duality under Constraints

Nonlinear operator $\mathcal{E}_0(\cdot, z)$

Of particular interest are the properties of the nonlinear operator $\xi \mapsto \mathcal{E}_0(\xi) = \mathcal{E}_0(\xi, z)$

- Monotone convergence: $(\xi^n) \uparrow \xi$ implies $\mathcal{E}_0(\xi) = \lim_n \mathcal{E}_0(\xi^n)$.
- Fatou's lemma: $\mathcal{E}_0(\liminf_n \xi^n) \leq \liminf_n \mathcal{E}_0(\xi^n)$.
- $\sigma(L^1, L^\infty)$-lower semicontinuity
- convexity

Let us consider generators independent of y, that is $g(y, z, \delta, \gamma) = g(z, \delta, \gamma)$.
$\rightsquigarrow$ the last two points above give way to convex duality of the form

$$\mathcal{E}_0(\xi) = \sup_{\frac{dQ}{dP} \in L^\infty} \left\{ E_Q[\xi] - \mathcal{E}_0^*(Q) \right\} \qquad \text{where} \qquad \mathcal{E}_0^*(Q) = \sup_{\xi \in L^1} \left\{ E_Q[\xi] - \mathcal{E}_0(\xi) \right\} .$$

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Dual representation

1. Of which structure is $\mathcal{E}_0^*(Q)$ and is it always attained?

Theorem

For $Q \sim P$ with $\frac{dQ}{dP} = \exp(\int q dW - \frac{1}{2}\int |q|^2 du)$, the dual operator $\mathcal{E}_0^*(Q)$ is given by

$$\mathcal{E}_0^*(Q) = \sup_{(\Delta,\Gamma)} \left\{ E_Q\left[\int_0^T -g_u(Z_u,\Delta_u,\Gamma_u) + q_u\left(\int_0^u (\Delta_s + q_s\Gamma_s)ds\right) du\right]\right\}$$

There exist (Δ^Q, Γ^Q) attaining $\mathcal{E}_0^*(Q)$, they are unique if the convexity of g is strict.

2. How may we compute $\mathcal{E}_0^*(Q)$? Consider $g(\delta,\gamma) = |\delta|^2 + |\gamma|^2$.

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Duality $\leftrightarrow$ Solutions of constrained BSDEs

Existence of solutions of BSDEs under constraints $\rightsquigarrow$ connected to optimal measure $\hat{Q}$

Theorem

Assume that

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Then there exists a solution of the constrained BSDE with terminal condition ξ and generator g.

$\rightsquigarrow$ Extends results of [DELBAEN ET AL.] and [DRAPEAU ET AL.] to the constrained case.

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Supersolutions of BSDEs

Summary

Framework	Results
$(\mathcal{A}, \xi, g)$ (LSC),(POS),(NOR) unconstrained	existence and uniqueness of M.S.S. relaxations: (POS) and (NOR) $\rightsquigarrow$ $g \geq az + b$ and $\int g_u(y, 0)du$
$(\mathcal{A}, \Theta, \xi, g(\Delta, \Gamma, \cdot))$ (LSC),(CON),(DGC) constrained: $Z \in \Theta$	existence of supersolutions minimal at $\{t_0, \ldots, t_n\}$ stability of $\xi \mapsto \mathcal{E}_t(\xi, Z_{[0,t]})$ duality: characterisation of $\mathcal{E}_0^*(Q)$ in terms of (Δ, Γ)

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Thank you