Pareto optimal allocations and optimal risk sharing for quasiconvex risk measures

Elisa Mastrogiacomo

University of Milano-Bicocca, Italy

(joint work with Prof. Emanuela Rosazza Gianin)

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Agenda

- Pareto optimal allocations and optimal risk sharing: motivation and basic definitions
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- inf-convolution and quasiconvex inf-convolution
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- Inf-convolution and quasiconvex inf-convolution
- Characterization of Pareto optimal allocations: from convex risk measures to quasiconvex risk measures
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- inf-convolution and quasiconvex inf-convolution
- characterization of Pareto optimal allocations: from convex risk measures to quasiconvex risk measures
- (weakly) optimal risk sharing (under cash-additivity of one risk measure or not)
Motivation I

Problem
Given
A1 - insurer: initial risky position $X_1$; premium $\pi_1$
A2 - reinsurer: initial risky position $X_2$; premium $\pi_2$
and $X = X_1 + X_2$ aggregate risk
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find $Y$ and $(X - Y)$ such that insurer and reinsurer optimally share the risk $X$
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\[ \rightsquigarrow \text{find } Y \text{ and } (X - Y) \text{ such that insurer and reinsurer optimally share the risk } X \]

In other words

we want to find the “best way” for two (or more) agents to share the total (aggregate) risk.
Pareto optimal allocations

Let $\pi_1, \pi_2 : L^\infty \rightarrow \overline{\mathbb{R}}$.

- $A(X) \triangleq \{(\xi_1, \xi_2) : \xi_1, \xi_2 \in L^\infty \text{ and } \xi_1 + \xi_2 = X\}$
  \text{set of attainable allocations}
Pareto optimal allocations

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- $(\xi_1, \xi_2) \in A(X)$ is **Pareto optimal** (POA) when:
  
  if $\exists (\eta_1, \eta_2) \in A(X) : \pi_1(\eta_1) \leq \pi_1(\xi_1)$ and $\pi_2(\eta_2) \leq \pi_2(\xi_2)$
  
  $\Rightarrow \pi_1(\eta_1) = \pi_1(\xi_1)$ and $\pi_2(\eta_2) = \pi_2(\xi_2)$
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  \]

- $(\xi_1, \xi_2) \in A(X)$ is **weakly Pareto optimal** if
  
  \[
  \not\exists (\eta_1, \eta_2) \in A(X) : \pi_1(\eta_1) < \pi_1(\xi_1) \text{ and } \pi_2(\eta_2) < \pi_2(\xi_2)
  \]

\[\text{(Individual Rationality)}\]
Pareto optimal allocations

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  \]

- $(\xi_1, \xi_2) \in A(X)$ is weakly Pareto optimal if
  
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  \]

- $(\xi_1, \xi_2) \in A(X)$ is an optimal risk sharing (ORS) if it is Pareto optimal and $\pi_1(\xi_1) \leq \pi_1(X_1)$ and $\pi_2(\xi_2) \leq \pi_2(X_2)$ (Individual Rationality).
Quasiconvex risk measures

Our contribution:

- $\pi_1, \pi_2$ quasiconvex risk measures
- Comparison with the results established in the literature for convex risk measures
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Remind that $\pi : L^{\infty} \to \bar{\mathbb{R}}$ is quasiconvex if

$$\pi(\alpha X + (1 - \alpha) Y) \leq \max\{\pi(X); \pi(Y)\}, \quad \forall \alpha \in (0, 1), X, Y \in L^{\infty}.$$
Quasiconvex risk measures

Main reason of our interest for quasiconvex risk measures
The right formulation of diversification of risk is quasiconvexity:

if $\pi(X), \pi(Y) \leq \pi(Z) \Rightarrow \pi(\alpha X + (1-\alpha)Y) \leq \pi(Z)$, $\forall \alpha \in (0, 1)$

(see Cerreia-Vioglio et al. (2011), Drapeau and Kupper (2013), Frittelli and Maggis (2011))
Main reason of our interest for quasiconvex risk measures
The right formulation of diversification of risk is quasiconvexity:

if $\pi(X), \pi(Y) \leq \pi(Z)$ \implies $\pi(\alpha X + (1 - \alpha) Y) \leq \pi(Z)$, $\forall \alpha \in (0, 1)$

(see Cerreia-Vioglio et al. (2011), Drapeau and Kupper (2013), Frittelli and Maggis (2011))

For a monotone risk measure
- convexity $\Rightarrow$ quasi-convexity
- equivalence is true under cash-additivity of $\pi$
Several results:

- firstly studied in the insurance literature: see Borch (1962), Bühlmann and Jewell (1979) and Deprez and Gerber (1985)
Convex case: existing literature and main tools

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- firstly studied in the insurance literature: see Borch (1962), Bühlmann and Jewell (1979) and Deprez and Gerber (1985)

- more recently, studied for coherent and convex risk measures. See Barrieu and El Karoui (2005), Jouini, Schachermayer and Touzi (2008), Klöppel and Schweizer (2007), Filipovic and Kupper (2008), ...
Convex case: existing literature and main tools

Main tools

- Fenchel-Moreau representation of convex risk measures

\[ \pi(X) = \max_{Q \in \mathcal{M}_1} \{ E_Q[X] - F(Q) \}, \]

where \( \mathcal{M}_1 \) denotes the set of all \( Q \ll P \) and \( F \) is the convex conjugate of \( \pi \).

- inf-convolution, where

\[ (\pi_1 \square \pi_2)(X) \triangleq \inf_{Y \in L_\infty} \{ \pi_1(X - Y) + \pi_2(Y) \}. \]

- Fenchel-Moreau subdifferential “\( \partial \)” of a convex function
Fenchel-Moreau biconjugate Theorem cannot be applied; any quasiconvex, monotone and continuous from above risk measure $\pi : L^\infty \to \bar{\mathbb{R}}$ can be represented as

$$\pi (X) = \max_{Q \in \mathcal{M}_1} R(E_Q [X], Q), \quad \forall X \in L^\infty$$

where $\mathcal{M}_1$ denotes the set of all $Q \ll P$ and

$$R(t, Q) = \inf \{ \pi (Y) | E_Q [Y] = t \}$$

(1)

Main tools

- inf-convolution replaced by quasiconvex inf-convolution (more appropriate since stable wrt convex and quasiconvex functionals):

\[(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{\pi_1(X - Y) \vee \pi_2(Y)\}.\]
Main tools

- inf-convolution replaced by quasiconvex inf-convolution (more appropriate since stable wrt convex and quasiconvex functionals):

\[(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{\pi_1(X - Y) \vee \pi_2(Y)\}.\]

- a suitable subdifferential: more precisely, the Greenberg-Pierskalla subdifferential of \( \pi \) at \( \tilde{X} \) will be useful

\[\partial^{GP} \pi(\tilde{X}) \triangleq \{Q : E_Q[X - \tilde{X}] < 0, \forall X \text{ s.t. } \pi(X) < \pi(\tilde{X})\}\]

(see Penot and Zalinescu (2003) and Penot (2003))
For convex risk measures

Theorem [Jouini, Schachermayer and Touzi (2008)]

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ be convex risk measures satisfying monotonicity, $\sigma(L^\infty, L^1)$-lsc, cash-additivity and $\pi_1(0) = \pi_2(0) = 0$ with convex conjugate $F_1, F_2$.

Let $X \in L^\infty$ be a given aggregate risk.
Theorem [Jouini, Schachermayer and Touzi (2008)]

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ be convex risk measures satisfying monotonicity, $\sigma(L^\infty, L^1)$-lsc, cash-additivity and $\pi_1(0) = \pi_2(0) = 0$ with convex conjugate $F_1, F_2$.

Let $X \in L^\infty$ be a given aggregate risk.

The following conditions are equivalent:

(i) $(\xi_1, \xi_2) \in A(X)$ is Pareto optimal;

(ii) $(\pi_1 \square \pi_2)(X) = \pi_1(\xi_1) + \pi_2(\xi_2)$ (that is, exact);

(iii) $\pi_i(\xi_i) = E_{\tilde{Q}}[\xi_i] - F_i(\tilde{Q})$ for some $\tilde{Q} \in \mathcal{M}_1$;

(iv) $\partial \pi_1(\xi_1) \cap \partial \pi_2(\xi_2) \neq \emptyset$. 
Can Pareto optimal allocations be characterized also in the quasiconvex framework????
Can Pareto optimal allocations be characterized also in the quasiconvex framework???

YES, as seen in a while
Quasiconvex inf-convolution of risk measures

Given an insurance premium $\pi_1 : L^\infty \to \bar{\mathbb{R}}$ and a reinsurance premium $\pi_2 : L^\infty \to \bar{\mathbb{R}}$, $$(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{ \pi_1(X - Y) \lor \pi_2(Y) \}. \quad (2)$$

Interpretation $\pi_1(X - Y) \lor \pi_2(Y)$ maximal premium to be paid for the insurance and reinsurance separately when $Y$ is transferred by the insurer to the reinsurer.

$(\pi_1 \nabla \pi_2)(X)$ can be seen as minimization of the maximal premium to be paid for each insurance and reinsurance contract.
Quasiconvex inf-convolution of risk measures

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Quasiconvex inf-convolution: properties

Proposition

If $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ are quasiconvex and monotone, then $(\pi_1 \nabla \pi_2)$ is quasiconvex and monotone.

Moreover:

(i) if at least one between $\pi_1$ and $\pi_2$ is continuous from above, then also $(\pi_1 \nabla \pi_2)$ is continuous from above.

(ii) if at least one between $\pi_1$ and $\pi_2$ is cash-subadditive, then also $(\pi_1 \nabla \pi_2)$ is cash-subadditive.

(iii) if $\pi_1(0) = \pi_2(0) = 0$, then $(\pi_1 \nabla \pi_2)(0) \leq 0$. 
Representation of the quasiconvex inf-convolution

Remind that any quasiconvex, monotone and continuous from above risk measure $\pi : L^\infty \to \overline{\mathbb{R}}$ can be represented as

$$\pi (X) = \max_{Q \in \mathcal{M}_1} R (E_Q [X], Q), \quad \forall X \in L^\infty.$$
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**Theorem**

Let $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ be quasiconvex, monotone and continuous from above risk measures and let $R_1, R_2$ be their corresponding functionals.

Then $\pi \triangledown = \pi_1 \triangledown \pi_2$ is quasiconvex, monotone and continuous from above with

$$R \triangledown (t, Q) \triangleq (R_1 \triangledown_t R_2)(t, Q) \triangleq \inf_{t_1 + t_2 = t} \{ R_1 (t_1, Q) \vee R_2 (t_2, Q) \}.$$
... coming back to Pareto optimal allocations ...

Problem
What about POA with quasiconvex risk measures?
... coming back to Pareto optimal allocations ...

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What about POA with quasiconvex risk measures?

is it possible to extend the characterization of Jouini, Schachermayer and Touzi (2008) by means of quasiconvex inf-convolution and GP-subdifferential?
Assumption (A_π):

Let \( \pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}} \) be quasiconvex risk functionals satisfying monotonicity and continuity from above. Hence, \( \pi_1 \) and \( \pi_2 \) can be represented as

\[
\pi (X) = \max_{Q \in \mathcal{M}_1} R (E_Q [X], Q), \quad \forall X \in L^\infty.
\]
Theorem (Pareto optimal - quasiconvex case)

Let $\pi_1, \pi_2$ satisfy Assumption $(A_{\pi})$ and let $R_1, R_2$ be the corresponding functionals. Let $X \in L^\infty$ be a given aggregate risk.

(i) If $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \lor \pi_2(\xi_2)$ for some $(\xi_1, \xi_2) \in A(X)$, then $(\xi_1, \xi_2)$ is a weakly Pareto optimal allocation.

(ii) Let $(\xi_1, \xi_2) \in A(X)$ and $\bar{Q}, Q_1, Q_2$ be s.t. $(\pi_1 \nabla \pi_2)(X) = R(\nabla (E\bar{Q}(X), \bar{Q})))$ and $\pi_i(\xi_i) = R_i(EQ_i(\xi_i), Q_i)$, $i = 1, 2$.

(ii-r) $R(\nabla (E\bar{Q}(X), \bar{Q})) = R_1(EQ_1(\xi_1), Q_1) \lor R_2(EQ_2(\xi_2), Q_2)$;

(ii-p) $\pi_i(\xi_i) = R_i(E\bar{Q}(\xi_i), \bar{Q})$ whenever $\pi_i(\xi_i) > \pi_j(\xi_j)$ or $\pi_i(\xi_i) = \pi_j(\xi_j)$ and $R_i(E\bar{Q}(\xi_i), \bar{Q}) > R_j(E\bar{Q}(\xi_j), \bar{Q})$ (for $i, j = 1, 2$).
Theorem (Pareto optimal - quasiconvex case)

Let $\pi_1, \pi_2$ satisfy Assumption (A$\pi$) and let $R_1, R_2$ be the corresponding functionals. Let $X \in L^\infty$ be a given aggregate risk.

(i) If $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \lor \pi_2(\xi_2)$ for some $(\xi_1, \xi_2) \in A(X)$, then $(\xi_1, \xi_2)$ is a weakly Pareto optimal allocation.

(ii) Let $(\xi_1, \xi_2) \in A(X)$ and $\bar{Q}, Q_{1,2}$ be s.t.

$(\pi_1 \nabla \pi_2)(X) = R^\nabla (E\bar{Q}(X), \bar{Q})$ and $\pi_i(\xi_i) = R_i(E_{Q_i}(\xi_i), Q_i)$, $i = 1, 2$.

$(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \lor \pi_2(\xi_2)$ iff the following conditions are both satisfied:

(ii-r) $R^\nabla (E\bar{Q}(X), \bar{Q}) = R_1(E_{Q_1}(\xi_1), Q_1) \lor R_2(E_{Q_2}(\xi_2), Q_2)$;

(ii-p) $\pi_i(\xi_i) = R_i(E\bar{Q}(\xi_i), \bar{Q})$ whenever $\pi_i(\xi_i) > \pi_j(\xi_j)$ or $\pi_i(\xi_i) = \pi_j(\xi_j)$ and $R_i(E\bar{Q}(\xi_i), \bar{Q}) > R_j(E\bar{Q}(\xi_j), \bar{Q})$ (for $i, j = 1, 2$),
(iii) Let \( \pi_1 \) and \( \pi_2 \) be \( \sigma(L^\infty, L^1) \)-upper semi-continuous.

If \( \pi_1 \nabla \pi_2(X) = \pi_1(\xi_1) \lor \pi_2(\xi_2) \) for \( (\xi_1, \xi_2) \in A(X) \) and \( \xi_1, \xi_2 \) are not local minimizers of \( \pi_1, \pi_2 \), then

\[
\partial^{GP}(\pi_1 \nabla \pi_2)(X) = \partial^{GP} \pi_1(\xi_1) \cap \partial^{GP} \pi_2(\xi_2).
\]
Comparison with the convex case (see Jouini et al. (2008))

- Link between weakly Pareto, exactness of qco inf-convolution, representation of quasiconvex risk measures and Greenberg-Pierskalla subdifferential.
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- weakly Pareto $\not\Rightarrow$ exactness of qco inf-convolution
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- weakly Pareto $\not\Rightarrow$ exactness of qco inf-convolution
- exactness $\not\Rightarrow \partial^{GP} \pi_1(\xi_1) \cap \partial^{GP} \pi_2(\xi_2) \neq \emptyset$
Comparison with the convex case (see Jouini et al. (2008))

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- weakly Pareto $\nRightarrow$ exactness of qco inf-convolution
- exactness $\nRightarrow$ $\partial^{GP} \pi_1(\xi_1) \cap \partial^{GP} \pi_2(\xi_2) \neq \emptyset$
  the converse is true under further continuity assumptions
Optimal risk sharing
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- for convex risk measures:
  \((\xi_1^*, \xi_2^*) \in A(X)\) optimal risk sharing iff \((\pi_1 \Box \pi_2)(X_1 + X_2)\) is exact at \((\xi_1^*, \xi_2^*)\) (or, equivalently, Pareto optimal) and \(\pi_i(\xi_i^*) \leq \pi_i(X_i)\) for \(i = 1, 2\) (individual rationality) - see Jouini et al. (2008)
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\(\rightsquigarrow\) for qco risk measures we will look for a \textit{weakly optimal risk sharing}, i.e. \((\xi_1^*, \xi_2^*) \in A(X)\) at which \((\pi_1 \nabla \pi_2)(X)\) is exact and satisfying individual rationality.
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  \((\xi_1^*, \xi_2^*) \in A(X)\) optimal risk sharing \(\text{iff } (\pi_1 \Box \pi_2)(X_1 + X_2)\) is exact at \((\xi_1^*, \xi_2^*)\) (or, equivalently, Pareto optimal) and \(\pi_i(\xi_i^*) \leq \pi_i(X_i)\) for \(i = 1, 2\) (individual rationality) - see Jouini et al. (2008)

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- For convex risk measures (see JST):
  ORS may be obtained starting from a POA and taking into account a suitable price

- For quasiconvex risk measures: similar result?
Following the approach of Jouini et al. (2008):

\[
p_1(\eta) \triangleq \pi_1(X_1) - \pi_1(X_1 - \eta)
\]
\[
p_2(\eta) \triangleq \pi_2(X_2 + \eta) - \pi_2(X_2),
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for any \(\eta \in L^\infty\).
Following the approach of Jouini et al. (2008):

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\[ p_2(\eta) \triangleq \pi_2(X_2 + \eta) - \pi_2(X_2), \]

for any \( \eta \in L^\infty \).

\( \eta \) can be seen as the risk transferred from insurer to reinsurer.
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p_2(\eta) \triangleq \pi_2(X_2 + \eta) - \pi_2(X_2),
\]

for any \( \eta \in L^\infty \).

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**For cash-additive risk measures:**

\( p_1(\eta) \) maximal price that agent 1 would pay because of the “risk exchange”; similarly \( p_2(\eta) \) can be seen as the minimal amount that 2 would like to receive because of the additional risk \( \eta \).
Given a POA \( (X_1 - \xi^*, X_2 + \xi^*) \) we would like to find \( p > 0 \) st
\[
\pi_1(X_1) - \pi_1(X_1 - \xi^* + p) \geq 0 \\
\pi_2(X_2) - \pi_2(X_2 + \xi^* - p) \geq 0
\]
Hence \( (X_1 - \xi^* + p, X_2 + \xi^* - p) \) (with \( p \in \mathbb{R} \)) is an ORS iff \( p \in [p_2(\xi^*), p_1(\xi^*)] \)
Problem
What about optimal risk sharing in the quasiconvex case???
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What about optimal risk sharing in the quasiconvex case???

More difficult since cash-additivity does not hold in general!
Theorem (Optimal risk sharing - Quasiconvex case)

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ satisfy assumption $(A\pi)$ and cash-subadditivity and let $X = X_1 + X_2$ be the aggregate risk.

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Let \( \pi_1, \pi_2 : L^\infty \to \mathbb{R} \) satisfy assumption \((A\pi)\) and cash-subadditivity and let \( X = X_1 + X_2 \) be the aggregate risk. Assume that \( \pi_1(X_1) \geq \pi_2(X_2) \) and that \( (\pi_1 \nabla \pi_2)(X) \) is exact at \( (X_1 - \xi^*, X_2 + \xi^*) \).

i) If \( \pi_1(X_1) = \pi_2(X_2) \), then \( (X_1 - \xi^*, X_2 + \xi^*) \) is a weakly ORS rule.

ii) If \( \pi_1(X_1) > \pi_2(X_2) \), then neither \( (X_1 - \xi^*, X_2 + \xi^*) \) is a weakly ORS rule or the following hold:

if \( (X_1 - \xi^* + p, X_2 + \xi^* - p) \) is a weakly ORS for some \( p > 0 \), then \( \pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) \lor \pi_2(X_2 + \xi^*) \) and \( p \geq \max\{\pi_2(X_2 + \xi^*), \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*)\} \).

If, in addition, \( \pi_2 \) is cash-additive, then also the converse holds true.
Theorem (Optimal risk sharing - Quasiconvex case)

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ satisfy assumption $(A_{\pi})$ and cash-subadditivity and let $X = X_1 + X_2$ be the aggregate risk. Assume that $\pi_1(X_1) \geq \pi_2(X_2)$ and that $(\pi_1 \triangledown \pi_2)(X)$ is exact at $(X_1 - \xi^*, X_2 + \xi^*)$.

i) If $\pi_1(X_1) = \pi_2(X_2)$, then $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule.
Theorem (Optimal risk sharing - Quasiconvex case)

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ satisfy assumption $(A\pi)$ and cash-subadditivity and let $X = X_1 + X_2$ be the aggregate risk.

Assume that $\pi_1(X_1) \geq \pi_2(X_2)$ and that $(\pi_1 \nabla \pi_2)(X)$ is exact at $(X_1 - \xi^*, X_2 + \xi^*)$.

i) If $\pi_1(X_1) = \pi_2(X_2)$, then $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule.

ii) If $\pi_1(X_1) > \pi_2(X_2)$, then either $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule or the following hold:

if $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly ORS for some $p > 0$, then

$\pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) \lor \pi_2(X_2 + \xi^*)$ and

$p \geq \max \{\pi_2(X_2 + \xi^*) - \pi_2(X_2); \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*)\}$.
Theorem (Optimal risk sharing - Quasiconvex case)

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ satisfy assumption $(A\pi)$ and cash-subadditivity and let $X = X_1 + X_2$ be the aggregate risk. Assume that $\pi_1(X_1) \geq \pi_2(X_2)$ and that $(\pi_1 \nabla \pi_2)(X)$ is exact at $(X_1 - \xi^*, X_2 + \xi^*)$.

i) If $\pi_1(X_1) = \pi_2(X_2)$, then $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule.

ii) If $\pi_1(X_1) > \pi_2(X_2)$, then either $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule or the following hold:

if $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly ORS for some $p > 0$, then

$p \geq \max \{ \pi_2(X_2 + \xi^*) - \pi_2(X_2); \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*) \}$.

If, in addition, $\pi_2$ is cash-additive, then also the converse holds true.
Differently from Jouini et al. (2008), in the quasiconvex case the constraint on $p$ depends not only on $p_1(\xi^*)$ and $p_2(\xi^*)$, but also on the difference between $\pi_1(X_1)$ and $\pi_2(X_2)$.

Indeed it is equivalent to

$$p \geq \max\{p_2(\xi^*); p_2(\xi^*) + p_1(\xi^*) + \pi_2(X_2) - \pi_1(X_1)\}.$$
Thank you for your attention!!!
Basic references, I

Basic references, II