Pareto optimal allocations and optimal risk sharing for quasiconvex risk measures

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- inf-convolution and quasiconvex inf-convolution
- characterization of Pareto optimal allocations: from convex risk measures to quasiconvex risk measures
- (weakly) optimal risk sharing (under cash-additivity of one risk measure or not)

Motivation I

Problem

Given

A1 - insurer: initial risky position X_1; premium π_1

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In other words

we want to find the "best way" for two (or more) agents to share the total (aggregate) risk.

Pareto optimal allocations

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$$\text{if } \exists (\eta_1, \eta_2) \in A(X) : \pi_1(\eta_1) \leq \pi_1(\xi_1) \text{ and } \pi_2(\eta_2) \leq \pi_2(\xi_2)$$
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- $(\xi_1, \xi_2) \in A(X)$ is weakly Pareto optimal if

$$\nexists (\eta_1, \eta_2) \in A(X) : \quad \pi_1(\eta_1) < \pi_1(\xi_1) \text{ and } \pi_2(\eta_2) < \pi_2(\xi_2)$$

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- $(\xi_1, \xi_2) \in A(X)$ is an optimal risk sharing (ORS) if it is Pareto optimal and $\pi_1(\xi_1) \leq \pi_1(X_1)$ and $\pi_2(\xi_2) \leq \pi_2(X_2)$ (Individual Rationality).

Quasiconvex risk measures

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- π_1, π_2 quasiconvex risk measures
- Comparison with the results established in the literature for convex risk measures

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Remind that $\pi : L^\infty \to \bar{\mathbb{R}}$ is quasiconvex if

$$\pi(\alpha X + (1-\alpha)Y) \leq \max\{\pi(X); \pi(Y)\}, \quad \forall \alpha \in (0,1), X, Y \in L^\infty.$$

Quasiconvex risk measures

Main reason of our interest for quasiconvex risk measures

The right formulation of diversification of risk is quasiconvexity:

$$\text{if } \pi(X), \pi(Y) \leq \pi(Z) \quad \Rightarrow \pi(\alpha X + (1-\alpha)Y) \leq \pi(Z), \forall \alpha \in (0,1)$$

(see Cerreia-Vioglio et al. (2011), Drapeau and Kupper (2013), Frittelli and Maggis (2011))

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For a monotone risk measure

- convexity $\Rightarrow$ quasi-convexity
- equivalence is true under cash-additivity of π

Convex case: existing literature and main tools

Several results:

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- firstly studied in the insurance literature: see Borch (1962), Bühlmann and Jewell (1979) and Deprez and Gerber (1985)
- more recently, studied for coherent and convex risk measures. See Barrieu and El Karoui (2005), Jouini, Schachermayer and Touzi (2008), Klöppel and Schweizer (2007), Filipovic and Kupper (2008), ...

Convex case: existing literature and main tools

Main tools

- Fenchel-Moreau representation of convex risk measures

$$\pi(X) = \max_{Q \in \mathcal{M}_1} \{E_Q[X] - F(Q)\},$$

where $\mathcal{M}_1$ denotes the set of all $Q \ll P$ and F is the convex conjugate of π.

- inf-convolution, where

$$(\pi_1 \Box \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{\pi_1(X - Y) + \pi_2(Y)\}.$$

- Fenchel-Moreau subdifferential "∂" of a convex function

Quasiconvex Risk measures: main tools

- Fenchel-Moreau biconjugate Theorem cannot be applied; any quasiconvex, monotone and continuous from above risk measure $\pi : L^\infty \to \bar{\mathbb{R}}$ can be represented as

$$\pi(X) = \max_{Q \in \mathcal{M}_1} R(E_Q[X], Q), \quad \forall X \in L^\infty$$

where $\mathcal{M}_1$ denotes the set of all $Q \ll P$ and

$$R(t, Q) = \inf\{\pi(Y) | E_Q[Y] = t\} \tag{1}$$

See Penot and Volle (1990), Cerreia-Vioglio et al. (2011), Drapeau and Kupper (2013) and Frittelli and Maggis (2011).

Main tools

- inf-convolution replaced by quasiconvex inf-convolution (more appropriate since stable wrt convex and quasiconvex functionals):

$$(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{\pi_1(X - Y) \vee \pi_2(Y)\}.$$

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- inf-convolution replaced by quasiconvex inf-convolution (more appropriate since stable wrt convex and quasiconvex functionals):

$$(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{\pi_1(X - Y) \vee \pi_2(Y)\}.$$

- a suitable subdifferential: more precisely, the Greenberg-Pierskalla subdifferential of π at $\bar{X}$ will be useful

$$\partial^{GP}\pi(\bar{X}) \triangleq \{Q : \ E_Q[X - \bar{X}] < 0, \forall X \text{ s.t. } \pi(X) < \pi(\bar{X})\}$$

(see Penot and Zalinescu (2003) and Penot (2003))

For convex risk measures

Theorem [Jouini, Schachermayer and Touzi (2008)]

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ be convex risk measures satisfying monotonicity, $\sigma(L^\infty, L^1)$-lsc, cash-additivity and $\pi_1(0) = \pi_2(0) = 0$ with convex conjugate F_1, F_2.
Let $X \in L^\infty$ be a given aggregate risk.

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The following conditions are equivalent:

(i) $(\xi_1, \xi_2) \in A(X)$ is Pareto optimal;

(ii) $(\pi_1 \square \pi_2)(X) = \pi_1(\xi_1) + \pi_2(\xi_2)$ (that is, exact);

(iii) $\pi_i(\xi_i) = E_{\bar{Q}}[\xi_i] - F_i(\bar{Q})$ for some $\bar{Q} \in \mathcal{M}_1$;

(iv) $\partial\pi_1(\xi_1) \cap \partial\pi_2(\xi_2) \neq \emptyset$.

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YES, as seen in a while

Quasiconvex inf-convolution of risk measures

Given an insurance premium $\pi_1 : L^\infty \to \bar{\mathbb{R}}$ and a reinsurance premium $\pi_2 : L^\infty \to \bar{\mathbb{R}}$,

$$(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y \in L^\infty} \{\pi_1(X - Y) \vee \pi_2(Y)\}. \tag{2}$$

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Interpretation

$\pi_1(X - Y) \vee \pi_2(Y)$ maximal premium to be paid for the insurance and reinsurance separately when Y is transferred by the insurer to the reinsurer.

$\rightsquigarrow$ $(\pi_1 \nabla \pi_2)(X)$ can be seen as minimization of the maximal premium to be paid for each insurance and reinsurance contract.

Quasiconvex inf-convolution: properties

Proposition

If $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ are quasiconvex and monotone,
then $(\pi_1 \nabla \pi_2)$ is quasiconvex and monotone.

Moreover:

(i) if at least one between π_1 and π_2 is continuous from above, then also $(\pi_1 \nabla \pi_2)$ is continuous from above.

(ii) if at least one between π_1 and π_2 is cash-subadditive, then also $(\pi_1 \nabla \pi_2)$ is cash-subadditive.

(iii) if $\pi_1(0) = \pi_2(0) = 0$, then $(\pi_1 \nabla \pi_2)(0) \leq 0$.

Representation of the quasiconvex inf-convolution

Remind that any quasiconvex, monotone and continuous from above risk measure $\pi : L^\infty \to \bar{\mathbb{R}}$ can be represented as

$$\pi(X) = \max_{Q \in \mathcal{M}_1} R(E_Q[X], Q), \quad \forall X \in L^\infty.$$

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Theorem

Let $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ be quasiconvex, monotone and continuous from above risk measures and let R_1, R_2 be their corresponding functionals.

Then $\pi^\nabla = \pi_1 \nabla \pi_2$ is quasiconvex, monotone and continuous from above with

$$R^\nabla(t, Q) \triangleq (R_1 \nabla_t R_2)(t, Q) \triangleq \inf_{t_1 + t_2 = t} \{R_1(t_1, Q) \vee R_2(t_2, Q)\}.$$

... coming back to Pareto optimal allocations ...

Problem

What about POA with quasiconvex risk measures?

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What about POA with quasiconvex risk measures?

is it possible to extend the characterization of Jouini, Schachermayer and Touzi (2008) by means of quasiconvex inf-convolution and GP-subdifferential?

Assumption (Aπ):

Let $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ be quasiconvex risk functionals satisfying monotonicity and continuity from above. Hence, π_1 and π_2 can be represented as

$$\pi(X) = \max_{Q \in \mathcal{M}_1} R(E_Q[X], Q), \quad \forall X \in L^\infty.$$

Theorem (Pareto optimal - quasiconvex case)

Let π_1, π_2 satisfy Assumption (Aπ) and let R_1, R_2 be the corresponding functionals. Let $X \in L^\infty$ be a given aggregate risk.

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(i) If $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ for some $(\xi_1, \xi_2) \in A(X)$, then (ξ_1, ξ_2) is a weakly Pareto optimal allocation.

(ii) Let $(\xi_1, \xi_2) \in A(X)$ and $\bar{Q}$, $Q_{1,2}$ be s.t. $(\pi_1 \nabla \pi_2)(X) = R^\nabla(E_{\bar{Q}}(X), \bar{Q})$ and $\pi_i(\xi_i) = R_i(E_{Q_i}(\xi_i), Q_i)$, $i = 1, 2$.
$(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ iff the following conditions are both satisfied:

(ii-r) $R^\nabla(E_{\bar{Q}}(X), \bar{Q}) = R_1(E_{Q_1}(\xi_1), Q_1) \vee R_2(E_{Q_2}(\xi_2), Q_2)$;

(ii-p) $\pi_i(\xi_i) = R_i(E_{\bar{Q}}(\xi_i), \bar{Q})$ whenever $\pi_i(\xi_i) > \pi_j(\xi_j)$ or $\pi_i(\xi_i) = \pi_j(\xi_j)$ and $R_i(E_{\bar{Q}}(\xi_i), \bar{Q}) > R_j(E_{\bar{Q}}(\xi_j), \bar{Q})$ (for $i, j = 1, 2$),

Theorem (Pareto optimal - quasiconvex case) - Continued

(iii) Let π_1 and π_2 be $\sigma(L^\infty, L^1)$-upper semi-continuous.

If $\pi_1 \nabla \pi_2(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ for $(\xi_1, \xi_2) \in A(X)$ and ξ_1, ξ_2 are not local minimizers of π_1, π_2, then

$$\partial^{GP}(\pi_1 \nabla \pi_2)(X) = \partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2).$$

Comparison with the convex case (see Jouini et al. (2008))

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    the converse is true under further continuity assumptions

Optimal risk sharing

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- for convex risk measures:
  $(\xi_1^*, \xi_2^*) \in A(X)$ optimal risk sharing iff $(\pi_1 \Box \pi_2)(X_1 + X_2)$ is exact at (ξ_1^*, ξ_2^*) (or, equivalently, Pareto optimal) and $\pi_i(\xi_i^*) \leq \pi_i(X_i)$ for $i = 1, 2$ (individual rationality) - see Jouini et al. (2008)

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- for convex risk measures (see JST):
  ORS may be obtained starting from a POA and taking into account a suitable price
- for quasiconvex risk measures: similar result?

Optimal risk sharing - Convex case

Following the approach of Jouini et al. (2008):

$$p_1(\eta) \triangleq \pi_1(X_1) - \pi_1(X_1 - \eta)$$
$$p_2(\eta) \triangleq \pi_2(X_2 + \eta) - \pi_2(X_2),$$

for any $\eta \in L^\infty$.

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For cash-additive risk measures:
$p_1(\eta)$ maximal price that agent 1 would pay because of the "risk exchange"; similarly $p_2(\eta)$ can be seen as the minimal amount that 2 would like to receive because of the additional risk η.

Optimal risk sharing - Convex cash-additive case

Given a POA $(X_1 - \xi^*, X_2 + \xi^*)$ we would like to find $p > 0$ st

$$\pi_1(X_1) - \pi_1(X_1 - \xi^* + p) \geq 0$$
$$\pi_2(X_2) - \pi_2(X_2 + \xi^* - p) \geq 0$$

Hence $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ (with $p \in \mathbb{R}$) is an ORS iff $p \in [p_2(\xi^*), p_1(\xi^*)]$

Problem

What about optimal risk sharing in the quasiconvex case???

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More difficult since cash-additivity does not hold in general!

Theorem (Optimal risk sharing - Quasiconvex case)

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i) If $\pi_1(X_1) = \pi_2(X_2)$, then $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule.

ii) If $\pi_1(X_1) > \pi_2(X_2)$, then either $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule or the following hold:

if $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly ORS for some $p > 0$, then $\pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*)$ and

$$p \geq \max \left\{ \pi_2(X_2 + \xi^*) - \pi_2(X_2); \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*) \right\}.$$

Theorem (Optimal risk sharing - Quasiconvex case)

Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ satisfy assumption $(A\pi)$ and cash-subadditivity and let $X = X_1 + X_2$ be the aggregate risk.

Assume that $\pi_1(X_1) \geq \pi_2(X_2)$ and that $(\pi_1 \nabla \pi_2)(X)$ is exact at $(X_1 - \xi^*, X_2 + \xi^*)$.

i) If $\pi_1(X_1) = \pi_2(X_2)$, then $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule.

ii) If $\pi_1(X_1) > \pi_2(X_2)$, then either $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly ORS rule or the following hold:

if $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly ORS for some $p > 0$, then $\pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*)$ and

$$p \geq \max\left\{\pi_2(X_2 + \xi^*) - \pi_2(X_2); \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*)\right\}.$$

If, in addition, π_2 is cash-additive, then also the converse holds true.

Differently from Jouini et al. (2008), in the quasiconvex case the constraint on p depends not only on $p_1(\xi^*)$ and $p_2(\xi^*)$, but also on the difference between $\pi_1(X_1)$ and $\pi_2(X_2)$.

Indeed it is equivalent to

$$p \geq \max\{p_2(\xi^*); p_2(\xi^*) + p_1(\xi^*) + \pi_2(X_2) - \pi_1(X_1)\}.$$

Thank you for your attention!!!

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