# Numerical simulation of multi-scale SDEs

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# Motivation: some models for asset prices



- Stochastic price models: The price of a risky asset is described as an stochastic process.
  Black Scholes model:

$$dS_t = \alpha S_t dt + \sigma S_t dB_t$$

- Stochastic volatility models: $\sigma$ is itself an stochastic process. Typically, $\sigma_t = f(\nu_t)$ where $\nu_t$ is a driving stochastic process.
  Heston model:

$$dS_t = \alpha S_t dt + \sqrt{\nu_t} S_t dB_t^S$$
$$d\nu_t = \kappa(\theta - \nu_t)dt + \sqrt{\nu_t} dB_t^\nu$$

- Multi-scale stochastic volatility models $\sigma$ contains components that evolve much faster than the price. $\sigma = f(\nu_t, \eta_t)$
  Fouque, Papanicolaou, Sircar and Solna:

$$dS_t = \alpha S_t dt + f(\nu_t, \eta_t) S_t dB_t^S$$
$$d\nu_t = \delta c(\nu_t)dt + \sqrt{\delta} g(\nu_t) dB_t^\nu$$
$$d\eta_t = \epsilon^{-1} \alpha(\eta_t)dt + \epsilon^{-1/2} \beta(\eta_t) dB_t^\eta$$

# Multi-scale Stochastic Differential Equations



Let us consider the following system

$$\begin{cases} X_t^\epsilon = x_0 + \int_0^t f(X_s^\epsilon, Y_s^\epsilon)ds + \int_0^t g(X_s^\epsilon, Y_s^\epsilon)dB_s \\ Y_t^\epsilon = y_0 + \epsilon^{-1} \int_0^t b(X_s^\epsilon, Y_s^\epsilon)ds + \epsilon^{-1/2} \int_0^t \sigma(X_s^\epsilon, Y_s^\epsilon)d\tilde{B}_s \end{cases}$$

- We are interested in the regime $\epsilon \ll 1$
- Besides the financial application we just gave, such system is well adapted to model several natural, economic and financial phenomena.
- We are interested in the numerical solution of such systems.
- $\epsilon \ll 1$ implies that a classical discretization scheme could be both inefficient and unstable.
- Our approach: profit from the scale difference
  - Homogenization technique: instead of $X^\epsilon$, consider $X$, its limit when $\epsilon \to 0$ (provided it exists)
  - Solve the effective equation $X$. Study numerical approximation of the homogenized equation

# Homogenization [Pardoux and Veretennikov, 2003]

## Tightness

## Identification of the limit.

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Tightness $f, g$ are at most linear in $x$, polynomial in $y$. Then, $T > 0$, $\{X_t^\epsilon, 0 \leq t \leq T\}_{0<\epsilon\leq 1}$ is weakly relatively compact in $C([0,T];\mathbb{R}^l)$

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Identification of the limit. Let $a := \sigma\sigma^*$. Take $a, b \in C_b^{2,1+\alpha}$ , and $a$ elliptic in $y$ uniformly in $x$. Assume $\lim_{|y|\to\infty} b(x,y) \cdot y = -\infty$ for all $x$.

- The rescaled fixed parameter fast variable:

$$Y_t^x = y_0 + \int_0^t b(x, Y_s^x)ds + \int_0^t \sigma(x, Y_s^x)d\tilde{B}_s$$

is ergodic with i.m. $\mu^x$.

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- Heuristics: If $\epsilon \ll \Delta t \ll 1$, $X_{t+\Delta t} \approx X_t$ while $Y_{t+\Delta t}^\epsilon$ will already attain its stable value.
- Then $X^\epsilon \Rightarrow X$ weakly

$$X_t = x_0 + \int_0^t F(X_s)ds + \int_0^t \sqrt{H(X_s)}dB_s, \quad \text{with } \sqrt{H}(\sqrt{H})^* = H$$

  where

$$F(x) := \int_{\mathbb{R}} f(x,y)\mu^x(dy); \quad H(x) := \int_{\mathbb{R}} h(x,y)\mu^x(dy).$$

# Numeric approximation

- Following our program, instead of approximating $X^\epsilon$, we solve the effective equation

$$X_t = x_0 + \int_0^t F(X_s)ds + \int_0^t \sqrt{H(X_s)}dB_s$$

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- Uniform step [Talay, 1990] :

$$\bar{F}^M(x) := \frac{1}{N}\sum_{k=0}^{N} f(x, \bar{Y}_k^x)$$

where $\bar{Y}_k^x$ is the outcome of a weak approximation scheme taken at times $k\Delta$ for some small $\Delta$ and large $N$

- Decreasing step [Lamberton and Pagès, 2002] :

$$\tilde{F}^{(M)}(x) := \frac{1}{\Gamma_M}\sum_{k=1}^{M} \gamma_k f\left(x, \tilde{Y}_{k-1}^x\right)$$

where

  - $\{\gamma_k\}_{k\in\mathbb{N}^*}$ ; decreasing sequence of positive reals tending to zero
  - $\Gamma_M := \sum_{k=1}^M \gamma_k$ diverges
  - Here we take: $\gamma_k = k^{-\theta}$ for $0 < \theta < 1$
  - $\tilde{Y}_{k+1}^x = \bar{Y}_k^x + \gamma_{k+1} b\left(x, \tilde{Y}_k^x\right) + \sqrt{\gamma_{k+1}}\sigma\left(x, \tilde{Y}_k^x\right)\bar{U}_{k+1}$

# MsDS algorithm

Let $t_k = kT/n$ for $0 \le k \le n$ and $\underline{t} = \lfloor nt \rfloor / n$

The Multi-scale decreasing step algorithm (MsDs) is given by

$$\tilde{X}_t = x_0 + \int_0^t \tilde{F}^{(M)}(\tilde{X}_{\underline{s}})ds + \int_0^t \sqrt{\tilde{H}^{(M)}(\tilde{X}_{\underline{s}})}dB_s$$

- Euler scheme
- At each step, independent realization of decreasing step estimator $\tilde{F}^{(M)} \approx F$ and $\tilde{H}^{(M)}$
- Cholesky decomposition

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We study its strong convergence towards the effective equation its normalized path-wise error behavior and its weak convergence.

## Strong convergence

$$X_t - \tilde{X}_t = \int_0^t \left[ F(X_s) - \tilde{F}^{(M)}(\tilde{X}_{\underline{s}}) \right] ds + \int_0^t \left[ \sqrt{H(X_s)} - \sqrt{\tilde{H}^{(M)}(\tilde{X}_{\underline{s}})} \right] dB_s$$

# Strong convergence

$$X_t - \tilde{X}_t = \int_0^t \Big( \overbrace{[F(X_s) - F(\tilde{X}_s)]}^{\text{Propagation error}} + \overbrace{[F(\tilde{X}_s) - F(\tilde{X}_{\underline{s}})]}^{\text{Euler error}} + \overbrace{[F(\tilde{X}_{\underline{s}}) - \tilde{F}^{(M)}(\tilde{X}_{\underline{s}})]}^{\text{Estimation error}} \Big) ds$$

$$+ \int_0^t \Big( \underbrace{[\sqrt{H(X_s)} - \sqrt{\tilde{H}^{(M)}(\tilde{X}_{\underline{s}})}]}_{\text{Cholesky stability}} \Big) dB_s$$

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**Cholesky factorization:** $[\sqrt{H(X_s)} - \sqrt{\tilde{H}^{(M)}(\tilde{X}_{\underline{s}})}]$:

- The Cholesky factorization is locally Lipschitz away from singular matrices [Sun, 1991]
- Moreover, we can give an explicit second order development

# Strong convergence

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**Decreasing step estimator** $F(\tilde{X}_{\underline{s}}) - \tilde{F}^{(M)}(\tilde{X}_{\underline{s}})$ :

- The estimator converges almost surely $\tilde{F}^{(M)} \xrightarrow{a.s.} \int f(x,z)\mu^x(dz)$

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**Decreasing step estimator** $F(\tilde{X}_{\underline{s}}) - \tilde{F}^{(M)}(\tilde{X}_{\underline{s}})$ :

- The estimator converges almost surely $\tilde{F}^{(M)} \xrightarrow{a.s.} \int f(x,z)\mu^x(dz)$
- Rates of convergence available for ergodic averages of functions written as $\mathcal{L}_y^x \phi$ (it holds for both $f, h$ under homogenization assumptions)
- The $L_2$ error is bounded by $K||\nabla^3 \phi_f||_\infty M^{-(\theta \wedge (1-\theta))}$ (the same for $H$)
- Moreover, we have an explicit expansion for the appropriately normalized terms [Lemaire, 2005]

# Strong convergence



$$X_t - \tilde{X}_t = \int_0^t \Big( \overbrace{[F(X_s) - F(\tilde{X}_s)]}^{\text{Propagation error}} + \overbrace{[F(\tilde{X}_s) - F(\tilde{X}_{\underline{s}})]}^{\text{Euler error}} + \overbrace{[F(\tilde{X}_{\underline{s}}) - \tilde{F}^{(M)}(\tilde{X}_{\underline{s}})]}^{\text{Estimation error}} \Big) ds$$

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$$\mathbb{E}|X_t - \tilde{X}_t|^2 = K \int_0^t \left[ \mathbb{E}(|X_s - \tilde{X}_s|^2) + O\left( n^{-1} + M^{-2\theta} \vee M^{-(1-\theta)} \right) \right] ds$$

and conclude with a Gronwall lemma

- Optimal relation between parameters is obtained when $M^{(1-\theta)} = O(n)$

# Normalized error distribution



We study a normalized error distribution to obtain bounds for confidence intervals

$$\zeta^n := n^{1/2}\left(X - \tilde{X}^n\right).$$

It converges weakly to the solution of an SDE with explicit coefficients.

- Same idea that the limit result for the Euler scheme [Jacod and Protter, 1998]
- The analysis uses tightness results ( [Jakubowski et al., 1989] , [Kurtz and Protter, 1991] ) and a second order expansion of the error terms

# Weak convergence of the scheme



For a measurable and bounded function $\phi$, we study

$$\mathbb{E}[\phi(\tilde{X}_T^n)] - \mathbb{E}[\phi(X_T)]$$

- The weak convergence can be analyzed using the technique introduced in [Talay and Tubaro, 1990] . Let $u$ be solution of

$$\frac{\partial u}{\partial t} + \mathcal{L}u = 0$$

with boundary condition $u(T,x) = \phi(x)$. Then,

$$\begin{aligned} \mathbb{E}[\phi(\tilde{X}_T^n)] - \mathbb{E}[\phi(X_T)] &= \mathbb{E}u(T, \tilde{X}_t^n) - u(0, x_0) \\ &= \sum_{i=0}^{n-1} \mathbb{E}\left[u(T_{i+1}, \tilde{X}_{T_{i+1}}^n) - u(T_i, \tilde{X}_{T_i}^n)\right] \\ &= \sum_{i=0}^{n-1} \int_{T_i}^{T_{i+1}} \mathbb{E}\left[\tilde{\mathcal{L}}_{\underline{s}} u(s, \tilde{X}_s^n) + \partial_t u(s, \tilde{X}_s^n)\right] dt \end{aligned}$$

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&= \sum_{i=0}^{n-1} \int_{T_i}^{T_{i+1}} \mathbb{E}\left[\tilde{\mathcal{L}}_{\underline{s}} u(s, \tilde{X}_s^n) - \mathcal{L}_s u(s, \tilde{X}_s^n)\right] dt
\end{aligned}$$

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- Under the assumed ellipticity assumption for the slow scale,

$$|\mathbb{E}[\phi(\tilde{X}_T^n)] - \mathbb{E}[\phi(X_T)]| \leq C(n^{-1} + M^{-[\theta \wedge (1-\theta)]})$$

# Main results - MsDS



## Theorem

*Let* $0 < \theta < 1$; $\gamma_1, M_1 \in \mathbb{R}^+$ *and* $\gamma_k = \gamma_1 k^{-\theta}$ *for* $k \in \mathbb{N}^*$. *Set* $M(n) = \left\lceil M_1 n^{\frac{1}{1-\theta}} \right\rceil$. *Assume*

- $f, g$ *Lipschitz in* $x$ *uniformly in* $y$, *and* $f, h \in C^{2,r^y}_{b,p}$ *for some* $r^y > 3$ , $h$ *uniformly elliptic*
- $a := \sigma\sigma^*$ *is uniformly elliptic and bounded.* $a, b \in C^{2,0}_{b,l}$, *and* $\sup_x b(x,y) \cdot y \leq -c_1|y|^2 + c_2$, *for some* $c_1 \in \mathbb{R}^*_+, c_2 \in \mathbb{R}$

*then,*

i) $\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t - \tilde{X}^n_t\right|^2\right] \leq K n^{-[(1-\theta)\wedge 2\theta]/(1-\theta)}$

ii) *If in addition* $f, h = gg^*$ *are* $C^7_p$ *in* $y$, $\theta \geq 1/3$. *Then*

$$n^{1/2}\left(X - \tilde{X}^n\right) =: \zeta^n \Rightarrow \zeta^\infty$$

*for* $\zeta^\infty$ *given as the solution of an SDE with explicitly known coefficients depending on the invariant law.*

iii) $|\mathbb{E}[\phi(\tilde{X}^n_T)] - \mathbb{E}[\phi(X_T)]| \leq C n^{-[\theta\wedge(1-\theta)]/(1-\theta)}$

# EMsDS



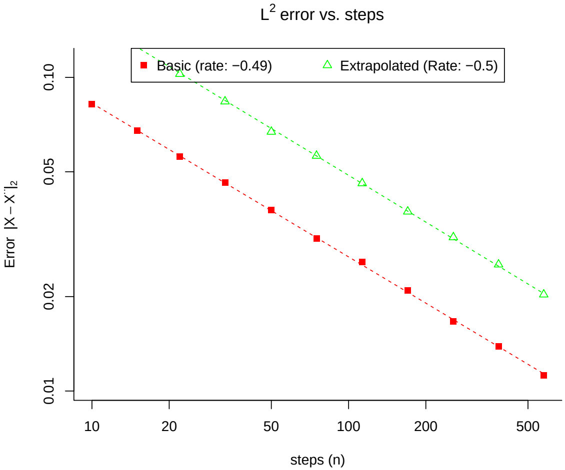
Romberg extrapolation:

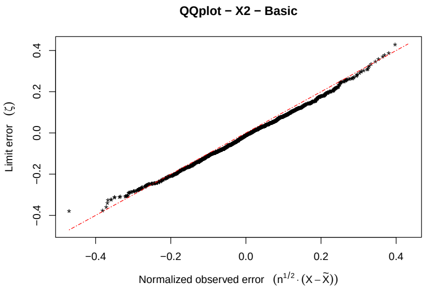
- We would like to profit from the explicit expansions available for the decreasing error step algorithm [Lemaire, 2005]
- Construct from linear combinations of methods of a given order an approximation of higher order. Several methods
- Fix $\tilde{\gamma}_{2k+1} = \tilde{\gamma}_{2k} := \frac{\gamma_k}{2}$ and let $\tilde{F}'$ be its associated diffusion.
- Extrapolated estimator: $\hat{F} := 2\tilde{F}' - \tilde{F}$.
- We call EMsDS the same algorithm as before but using the extrapolated estimator.

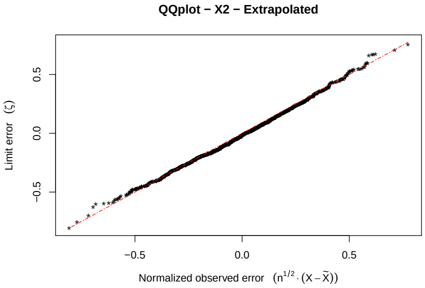
Extrapolation allows for an asymptotic complexity reduction while preserving the convergence rate and the limit results.

## Test problem:

$$dX_t^\epsilon =$$
$$\begin{pmatrix} 1 + Y_t^\epsilon - (|X_t^\epsilon|^2 + 1)^{-1/2} \\ 1 \end{pmatrix} + \sqrt{\frac{|X_t^\epsilon|^2+1}{2|X_t^\epsilon|^2+3}((Y_t^\epsilon)^2 + 1)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} dW_t,$$
$$dY_t^\epsilon = \left(\sqrt{\frac{1}{|X_t^\epsilon|^2+1}}dt - Y_t^\epsilon\right) + \sqrt{2}d\tilde{W}_t$$



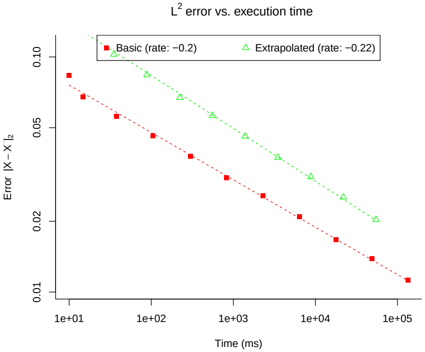

QQplot − X2 − Basic
Limit error (ζ)
Normalized observed error $(n^{1/2} \cdot (X - \tilde{X}))$
−0.4
−0.2
0.0
0.2
0.4


QQplot − X2 − Extrapolated
Limit error (ζ)
Normalized observed error $(n^{1/2} \cdot (X - \tilde{X}))$
−0.5
0.0
0.5

| | MsDS | EMsDs |
|---|---|---|
| $\theta_{\min}$ | $1/3$ | $1/5$ |
| $\tau_{\min}(\Delta)$(*) | $O([d_x^2 d_y + d_x^3]\Delta^{-5})$ | $O([d_x^2 d_y + d_x^3]\Delta^{-4.5})$ |
| Regime of interest(**) | $\epsilon < \Delta^3$ | $\epsilon < \Delta^{2.5}$ |

(*)$\tau(\Delta)$ : # of operations to attain tolerance $\Delta$

(**) Direct Euler scheme: $\tau_{\min}(\Delta) = O((d_x + d_y)\epsilon^{-1}\Delta^2)$



# Final remarks and perspectives:

- In the case $g \equiv 0$, we proved better rates of convergence
- Numerical simulations suggest that a mixed-step estimator converges faster
- It is interesting to study some grid approach to obtain the weak approximation
- We would like to extend this work to controlled multi-scaled BSDEs [Bardi, Cessaroni and Manca, 2010].

Thank you for your attention

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