Quadratic Backward Stochastic Differential Equations Driven by G-Brownian Motion

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Bordeaux, 7-9 July 2014

The G-BSDE

$$Y_t = \xi + \int_t^T g\left(s, Y_s, Z_s\right) d\left\langle B \right\rangle_s - \int_t^T Z_s dB_s - \left(K_T - K_t\right), \qquad (1)$$

where $g\left(t, \omega, y, z\right) : [0, T] \times \Omega_T \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

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Lipschitz case

Lipschitz assumptions (Hu et al 2012):

(**H1**) There exists a constant $\beta > 1$ such that

$$\forall\ y, z, \quad g(.,.,y,z) \in M_G^{\beta}(0,T);$$

(**H2**) There exists a constant $L > 0$ such that, for all y, z, y', z' we have :

$$\left|g(t,\omega,y,z) - g(t,\omega,y',z')\right| \leq L\left(|y-y'| + |z-z'|\right).$$

Definition

[**Hu et al 2012**] Let $\xi \in L_G^{\beta}(\Omega_T)$ and g satisfy **(H1)** and (**H2**) for some $\beta > 1$. A triple of processes (Y, Z, K) is a solution of the Lipschitz G-BSDE (1) if for some $1 < \alpha \leq \beta$ the following properties hold:
(a) $Y \in S_G^{\alpha}(0,T)$, $\ Z \in H_G^{\alpha}(0,T)$, $\ K$ is a decreasing G-martingale satisfying: $K_0 = 0$ and $K_T \in L_G^{\alpha}(\Omega_T)$;
(b) $Y_t = \xi + \int_t^T g(s, Y_s, Z_s) d\left\langle B\right\rangle_s - \int_s^T Z_s dB_s - (K_T - K_t)$.

Theorem

*[**Hu et al 2012**] Suppose that $\xi \in L_G^{\beta}(\Omega_T)$ for $\beta > 1$ and g satisfy (**H1**) and (**H2**). Then the G-BSDE (1) has a unique solution (Y, Z, K) such as, for all $1 < \alpha < \beta$, $Y \in S_G^{\alpha}(0, T)$, $Z \in H_G^{\alpha}(0, T)$, K is a decreasing G-martingale satisfying $K_0 = 0$ and $K_T \in L_G^{\alpha}(\Omega_T)$.*

As in the classical framework, one can obtain an upper bound for the process Y and Z solution of the G-BSDE (1), where the generator g is Lipschitz and the terminal condition ξ is bounded. To this end, we assume that the generator g satisfies the following Lipschitz condition.
(H0) There exists a constant $L_0 > 0$ such that, for each (t, ω) :

$$|g(t, \omega, 0, 0)| + |\xi| \leq L_0, \ \ q.s.;$$

(**Hlip**) There are two constants $L_y > 0$ and $L_z > 0$ such that for all $t \in [0, T]$, $y, y', z, z' \in \mathbb{R}$:

$$\left|g\left(t, y, z\right) - g\left(t, y', z'\right)\right| \leq L_y \left|y - y'\right| + L_z \left|z - z'\right|.$$

Lemma

*If (**H0**), **(H1)** and (**Hlip**) hold, then the unique solution (Y, Z, K) of the Lipschitz G-BSDE (1) satisfies the following properties:*

1. *$Y \in S_G^\infty(0, T)$ and $\|Y\|_{S_G^\infty(0,T)}$ admits an upper bound which is not dependent of L_z:*

$$\|Y\|_{S_G^\alpha(0,T)} \leq e^{\bar{\sigma}^2 L_y T} L_0 \left(1 + \bar{\sigma}^2 T\right).$$

2. *If ξ_1 and ξ_2 are two terminal conditions of this G-BSDE, we have for all $t \in [0, T]$:*

$$\left|Y_t^1 - Y_t^2\right| \leq e^{\bar{\sigma}^2 L_y T} \left|\xi_1 - \xi_2\right|,$$

where for $i = 1, 2$, Y^i is the solution of the equation with terminal condition ξ_i.

Lemma

*Under the assumptions (**H0**), **(H1)** and (**Hlip**), if $\xi \in Lip(\Omega_T)$, then $Z \in M_G^\infty$ and $\|Z\|_{M_G^\infty}$ admits an upper bound which is not dependent of L_z:*

$$\|Z\|_{M_G^\infty} \leq Ce^{\bar{\sigma}^2 L_y T}.$$

For the proof we follow the proof of Theorem 4.1 in [**Hu et al 2012**] for the existence of the solution in the Lipschitz case. At each step we only show that the process Z is bounded under our assumptions.

Quadratic assumptions

(H1') $g(\cdot,\cdot,y,z) \in M_G^2([0,T])$ for any $y,\ z$;
(Hq) for each (t,ω), g is Lipschitz in y, and with quadratic growth in z: e.g. there exist some constants $L_y > 0$ and $L_q > 0$ such that

$$\left|g(t,\omega,y,z) - g(t,\omega,y',z')\right| \leq L_y|y-y'| + L_q(1+|z|+|z'|)|z-z'|.$$

Definition

Let ξ and g satisfy (**H0**), **(H1')** and (**Hq**). A triple of processes (Y,Z,K) is a solution of the quadratic G-BSDE (1) if the following properties are satisfied:
(a) $Y \in S_G^\infty(0,T)$, $\quad Z \in H_G^2(0,T)$, $\quad K$ is a decreasing G-martingale satisfying: $K_0 = 0$ and for any $p > 1$, $K_T \in L_G^p(\Omega_T)$;
(b) $Y_t = \xi + \int_t^T g(s,Y_s,Z_s)d\left<B\right>_s - \int_s^T Z_s dB_s - (K_T - K_t)$.

Proposition

*Suppose that (**H0**), **(H1')** and (**Hq**) hold. Suppose that*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $Y \in \mathbb{S}^\infty(0,T)$, $Z \in \mathbb{H}^2(0,T)$, K is a decreasing process such that $K_0 = 0$ and $K_T \in \mathbb{L}^p(\Omega_T)$, for all $p \geq 1$. Then, the process $\left(\int_0^t Z_s dB_s\right)_{0\leq t\leq T}$ is a G-BMO martingale and there exist some constants α and C which depend only on p, T, L_0, L_y, $L_q, \underline{\sigma}^2$ such that:

$$\left\|\int_0^\cdot Z_s dB_s\right\|_{BMO}^2 \leq C e^{\alpha L_q \|Y\|_\infty},$$

$$\mathbb{E}[|K_T|^p] \leq C\left(1 + \|Y\|_\infty^p + \|Z\|_{BMO}^p + \|Z\|_{BMO}^{2p}\right).$$

For $\mathbb{P} \in \mathcal{P}_1$, a set that represent the G-expectation $\mathbb{E}$, and $\tau \in \mathcal{T}_0^T$, the set of stopping times with values in $[0, T]$. By applying Itô's formula to $e^{\lambda L_q Y_t}$ under the probability $\mathbb{P} \in \mathcal{P}_1$, we get:

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T |Z_s|^2 \, ds\right] \leq Ce^{\alpha L_q \|Y\|_\infty}.$$

On the other hand, we have the equality:

$$K_T = \xi - Y_0 + \int_0^T g(s, Y_s, Z_s) \, d\langle B\rangle_s - \int_0^T Z_s dB_s,$$

then taking the power p, by inequalities of BDG type and Young's inequality, we obtain

$$\mathbb{E}[|K_T|^p] \leq C\left(1 + \|Y\|_\infty^p + \|Z\|_{BMO}^p + \|Z\|_{BMO}^{2p}\right).$$

For all triple (Y, Z, K) satisfying the quadratic G-BSDE (1), such that Z is a $G-$BMO martingale generator and the process K is a decreasing G-martingale such that for all $p > 1$, $K_t \in L^p_G(\Omega_t)$, the process Y is quasi surely bounded and we have an explicit upper bound of the $S^\infty_G(0, T)$ norm.

Proposition

*Suppose that assumptions (**H0**), **(H1')** and (**Hq**) hold and (Y, Z, K) a solution of the quadratic G-BSDE (1). Then $Y \in \mathcal{S}^\infty_G(0, T)$ and we have:*

$$\|Y\|_{\mathcal{S}^\infty_G(0,T)} \leq e^{\bar{\sigma}^2 L_y T} L_0 \left(1 + \bar{\sigma}^2 T\right).$$

The proof is based on the argument of linearization and G-BMO property of the process Z. The quadratic G-BSDE (1) can be rewritten in the linear form:

$$Y_t = \xi + \int_t^T \left(g_s + m_s^\varepsilon + a_s^\varepsilon Y_s + b_s^\varepsilon Z_s\right) d\left\langle B\right\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where for each given $\varepsilon > 0$, we choose a Lipschitz function l such that $\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) \leq l(x) \leq \mathbf{1}_{[-2\varepsilon,2\varepsilon]}(x)$ and the process a^ε, b^ε, m^ε and $(g_s)_{s\in[0,T]}$ are respectively defined by:

$$\begin{aligned}
a_s^\varepsilon &= (1 - l(Y_s)) \frac{g(s, Y_s, Z_s) - g(s, 0, Z_s)}{Y_s} \mathbf{1}_{\{Y_s \neq 0\}} \\
b_s^\varepsilon &= (1 - l(|Z_s|)) \frac{g(s, 0, Z_s) - g(s, 0, 0)}{|Z_s|^2} Z_s \mathbf{1}_{\{Z_s \neq 0\}}, \\
m_s^\varepsilon &= l(Y_s)\left[g(s, Y_s, Z_s) - g(s, 0, Z_s)\right] + l(Z_s)\left[g(s, 0, Z_s) - g(s, 0, 0)\right], \\
g_s &= g(s, 0, 0)
\end{aligned}$$

Uniqueness

Here we establish a stability property for the quadratic G-BSDEs, as for the classical case, whose proof is based on the BMO properties.
We have the result of stability for the process Y and Z :

Proposition

For $i = 1, 2$, let $\left(Y^i, Z^i, K^i\right)$ be a solution of the $G-$BSDE:

$$Y_t^i = \xi^i + \int_t^T g^i\left(s, Y_s^i, Z_s^i\right) d\left\langle B\right\rangle_s - \int_t^T Z_s^i dB_s - \left(K_T^i - K_t^i\right),$$

*where ξ^i and g^i satisfy assumptions (**H0**), **(H1')** and (**Hq**). Then there exists a positive constant $C = \left(T, L_y, L_q, \overline{\sigma}^2, \underline{\sigma}^2\right)$ such that:*

$$\|\hat{Y}\|_\infty + \mathbb{E}\left[\int_0^T \left|\hat{Z}\right|^2 ds\right] \leq Ce^{\bar{\sigma}^2 L_y T}\left(\|\hat{\xi}\|_\infty + \bar{\sigma}^2\widehat{\mathbb{E}}\left[\int_t^T \hat{g}_s ds\right]\right),$$

where $\hat{g}_s = \left|g^1\left(s, Y_s^2, Z_s^2\right) - g^2\left(s, Y_s^2, Z_s^2\right)\right|.$

Existence

From estimates of Lipschitz G-BSDE under assumption **(Hlip)**, we first show the existence of a solution to the quadratic G-BSDE when the terminal condition $\xi \in Lip(\Omega)$.

Theorem

*Assume that assumptions (**H0**), **(H1')** and (**Hq**) hold and the terminal condition $\xi \in Lip(\Omega)$. Then the quadratic G-BSDE admits a unique solution (Y, Z, K) such that $Z \in M_G^{\infty}(0, T)$.*

Existence

For $n \in \mathbb{N}$ define

$$g_n(t, y, z) = g\left(t, y, \frac{|z| \wedge n}{|z|} z\right),$$

The G-BSDE

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, Z_s^n) d\langle B\rangle_s - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n)$$

is Lipschitz with generator g_n satisfying the assumptions (**H0**) **(H1')**, (**Hlip**) (here $L_z = L_q(1 + 2n)$) and $\xi \in Lip(\Omega_T)$. So $Y^n \in S_G^\infty(0, T)$, $Z^n \in M_G^\infty(0, T)$, and for $N_0 \geq \sup_{n \in \mathbb{N}} \|Z^n\|_{M_G^\infty}$, we get that

$$g_{N_0}\left(s, Y_s^{N_0}, Z_s^{N_0}\right) = g\left(s, Y_s^{N_0}, Z_s^{N_0}\right), \quad q.s.,$$

therefore $\left(Y^{N_0}, Z^{N_0}, K^{N_0}\right)$, is a solution of the quadratic G-BSDE.

Existence

Theorem

*Assume that the assumptions (**H0**), (**H1'**) and (**Hq**) are verified. Then the quadratic G-BSDE has a unique solution (Y, Z, K) such that $Y \in S_G^\infty(0, T)$, $\quad Z \in H_G^2(0, T)$, $\quad K$ is a decreasing G-martingale satisfying $K_0 = 0$ and $K_T \in L_G^p(\Omega_T)$ for any $p \geq 1$.*

Existence

Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of elements of $Lip(\Omega_T)$, which converge to ξ in $L_G^\infty(\Omega_T)$. For $n \in \mathbb{N}$ note (Y^n, Z^n, K^n) the solution of the quadratic G-BSDEs with terminal condition $\xi^n \in Lip(\Omega_T)$:

$$Y_t^n = \xi^n + \int_t^T g(s, Y_s^n, Z_s^n) d\langle B\rangle_s - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n).$$

For $m,\ n \in \mathbb{N}$, by the the proposition of stability we get:

$$\|Y^m - Y^n\|_{S_G^\infty(0,T)} + \mathbb{E}\left[\int_0^T |Z_s^m - Z_s^n|^2 ds\right] \leq Ce^{\bar{\sigma}^2 L_y T} \|\xi^m - \xi^n\|_{L_G^\infty}^2 .$$

Since the sequence $(\xi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence under the norm $\|.\|_{L_G^\infty}$, we get that the sequence $\{Y^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence under the norm $\|.\|_{S_G^\infty}$ and $\{Z^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence under the norm $\|.\|_{H_G^2}$.

Thanks for your attention!