Malliavin differentiation of random functions with applications to Lévy driven BSDEs

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joint work with Christel Geiss, University of Innsbruck

Let $X = (X_t)_{0\le t\le T}$ be a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \widehat{\mathcal{F}_T^X}$

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- N Poisson random measure: $A \in \mathcal{B}(\mathbb{R} \backslash \{0\})$

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$$\nu(A) := \mathbb{E} N([0,1] \times A)$$

- $\tilde{N}$ compensated Poisson random measure

$$\tilde{N}([0,t] \times A) \quad := \quad N([0,t] \times A) - t\nu(A)$$

- random measure M

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- For $B \in \mathcal{B}(\mathbb{R})$ and $s, t \in [0, T]$

$$\begin{aligned} \mathbb{E} M([0,t] \times A)\ M([0,s] \times B) \\ &= (s \wedge t)\,[\sigma^2 \delta_0(A \cap B) + \nu(A \cap B)] \\ &=: (s \wedge t)\,\mu(A \cap B) =: \mathrm{m}\,(([0,t] \times A) \cap ([0,s] \times B)) \end{aligned}$$

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- $I_n(f_n)$ multiple integrals (w.r.t. the random measure M)

Malliavin calculus for Lévy processes used here:

- for any $F \in L^2 = L^2(\Omega, \mathcal{F}, \mathbb{P})$ exists the chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n), \mathbb{P} - a.s.$$

  with symmetric functions $f_n \in L^2_n = L^2\left(([0, T] \times \mathbb{R})^n\right)$.
- $\mathbb{D}_{1,2}$ is the subspace of L^2 such that

$$\sum_{n=0}^{\infty} n n! \|f_n\|^2_{L^2_n} < \infty$$

- $\mathcal{D}F$ in $L^2(\Omega \times [0, T] \times \mathbb{R}; \mathbb{R})$ is defined by

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Formulation of the BSDE:

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, (X_r)_{0\leq r\leq T}, Y_s, Z_s, \int_{\mathbb{R}_0} g(U_s(x))g_1(x)\nu(dx)\right) ds \\ - \int_t^T Z_s dW(s) - \int_{(t,T]\times\mathbb{R}_0} U_s(x)\tilde{N}(ds, dx), \quad 0 \leq t \leq T \end{aligned}$$

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- f is $\mathbb{P}$-a.s. continuously differentiable in (y,z,u)
- g is $\mathcal{C}^1$ with bounded derivative
- $g_1 \in L_2(\mathbb{R},\nu)$

Questions:

- Explicit expression for the Z and U-processes?
- Can Malliavin differentiation be applied to the LHS and RHS of the BSDE to get a formula like

$$\begin{aligned}\mathcal{D}_{r,v}Y_t =& \mathcal{D}_{r,v}\xi + \int_t^T F_{r,v}\left(s, X, \mathcal{D}_{r,v}Y_s, \mathcal{D}_{r,v}Z_s, \mathcal{D}_{r,v}U_s(\cdot)\right) ds \\ & - \int_t^T \mathcal{D}_{r,v}Z_s dW_s - \int_{]t,T]\times\mathbb{R}} \mathcal{D}_{r,v}U_{s,x}\tilde{N}(ds,dx)?\end{aligned}$$

- How to find $\mathcal{D}_{r,v}f\left(s, (X_r)_{0\le r\le T}, Y_s, Z_s, \int_{\mathbb{R}_0} g(U_s(x))g_1(x)\nu(dx)\right)$?
- Conditions on f such that applying $\mathcal{D}_{r,v}$ is possible?

Theorems about Malliavin differentiation of BSDEs have been stated by:

Pardoux and Peng (1992): Brownian setting, Forward-Backward SDEs

El Karoui et al. (1997): Brownian setting

Elie (2006): FBSDEs with jumps

Ankirchner, dos Reis and Imkeller (2007): QBSDEs in the Brownian setting

Delong and Imkeller (2010): Time delayed BSDEs with jumps on the canonical probability space of a Lévy process, sufficiently small Lipschitz constant or time horizon

Require: Differentiability of the generator, $f(s, X, \Phi_s)$, here $\Phi_s = \left(Y_s, Z_s, \int_{\mathbb{R}_0} g(U_s(x)) g_1(x) \nu(dx) \right)$.
What does the differentiated object look like?

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Theorem

Let $F\colon D([0,T]) \to \mathbb{R}$. Assume that $F(X) \in \mathbb{D}_{1,2}$. Then for $v \neq 0$,

$$\mathcal{D}_{r,v} F(X) = F(X + v \mathbb{1}_{[r,T]}) - F(X), \mathbb{P} \otimes \mathrm{m}\text{-}a.e.$$

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$\mathbb{P} \otimes \mathrm{m}$-*a.e.*

What conditions hold for $f\left(s, X + v\mathbb{1}_{[r,T]}, y, z, u\right)$ (Lipschitz, differentiability, quadratic growth etc.)?

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Let $\Lambda \subseteq D([0,T])$ be the set, for which a path property for f in (y, z, u) holds.

Theorem

If $\mathbb{P}(X \notin \Lambda) = 0$, *then*

$$\mathbb{P} \otimes \mathrm{m}\left(\left\{(\omega, r, v) : X(\omega) + v\mathbb{1}_{[r,T]} \notin \Lambda\right\}\right) = 0.$$

The case $v = 0$:
Analyze the situation on the Wiener space of continuous functions:

The Levy-Itô decomposition implies that the Brownian and the pure-jump part of the process are independent,

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We may consider random variables (w.r.t. to σ-algebra generated by X) as function-valued random variables on the Wiener space

$$\xi \colon \Omega \to \mathbb{R} \quad \leftrightarrow \quad \tilde{\xi} \colon \Omega^W \times \Omega^J \to \mathbb{R} \quad \leftrightarrow \quad \tilde{\tilde{\xi}} \colon \Omega^W \to L_0(\Omega^J),$$

$$L_2(\Omega^W \times \Omega^J) \cong L_2(\Omega^W; L_2(\Omega^J)).$$

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$$t \mapsto \rho_{th}(\xi^h)(\omega) := \xi^h\left(\omega + t\int_0^{\cdot} h(s)ds\right) \text{ is absolutely continuous}$$

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- $\exists\ \nabla\xi \in L_p(\Omega^W; L_2([0,1]; E))$ such that for all $h \in L_2([0,1])$:

  $$\frac{\rho_{th}(\xi) - \xi}{t} \overset{\mathbb{P}^W}{\to} \langle \nabla\xi, h \rangle_{L_2([0,1])}$$

  ξ is stochastically Gateaux differentiable

Theorem (Sugita '85)

The Malliavin Sobolev spaces $\mathbb{D}_{1,p}(E)$ on the Wiener space equal the Kusuoka-Stroock Sobolev spaces.

Moreover, for the Malliavin derivative on the Wiener space $\mathcal{D}^W$ it holds that

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One identifies $\mathcal{D}^W$ and $\mathcal{D}_{\cdot,0}$ (up to the multiplicative constant σ).

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Suppose $G = (G_1, \dots, G_d) \in (\mathbb{D}_{1,2}(E))^d$ *and*

1. $f(\omega, \cdot) \in \mathcal{C}^1(\mathbb{R}^d)$ *for a.a.* $\omega \in \Omega$,
2. $\forall \eta \in \mathbb{R}^d : f(\cdot, \eta) \in \mathbb{D}^W_{1,2}(E)$,
3. $\forall N \in \mathbb{N} \; \exists K_N \in \bigcup_{p>0} L_p(\mathbb{P}) : \eta, \tilde{\eta} \in B_N(0)$ *and for a.a.* ω

   $$\|(D^W f(\cdot,\eta))(\omega) - (D^W f(\cdot,\tilde{\eta}))(\omega)\|_{L_2[0,T]} \le K_N(\omega)|\eta - \tilde{\eta}|,$$

   'locally Lipschitz'
4. $D^W f(\cdot,\eta)|_{\eta=G} \in L_2(\Omega^W; L_2(L_2[0,T]; E))$ *and*

   $$\sum_{k=1}^d \partial_{\eta_k} f(\cdot, G_1, \dots, G_d) D^W G_k \in L_2(\Omega^W; L_2([0,T]; E)).$$

Theorem

Then

$$f(\cdot, G_1, ..., G_d) \in \mathbb{D}^W_{1,2}(E)$$

and

$$\begin{aligned} D^W f(\cdot, G_1, ..., G_d) = & D^W f(\cdot, \eta)|_{\eta=G} \\ & + \sum_{k=1}^{d} \partial_{\eta_k} f(\cdot, G_1, ..., G_d) D^W G_k \\ & \in L_2(\Omega^W; L_2(L_2[0, T]; E)). \end{aligned}$$

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Theorem

Under assumptions given below the following assertions hold:

- *The processes Y, Z and U are Malliavin differentiable.*

$$Y_t = \xi + \int_t^T f\Big(s, X, Y_s, Z_s, \int_{\mathbb{R}_0} g(U_s(x))g_1(x)\nu(dx)\Big)\,ds - \int_t^T Z_s dW(s) - \int_{(t,T]\times\mathbb{R}_0} U_s(x)\tilde{N}(ds,dx), \quad 0 \le t \le T$$

Theorem

Under assumptions given below the following assertions hold:

- *The processes Y, Z and U are Malliavin differentiable.*
- *A version of $\mathcal{D}Y$ and $\mathcal{D}Z$ satisfies the following BSDE $\mathbb{P}\otimes\mathbb{m}$-a.e.*

$$\mathcal{D}_{r,v}Y_t = \mathcal{D}_{r,v}\xi + \int_t^T \mathcal{D}_{r,v}f\,(s,X,\Phi_s)\,ds - \int_t^T \mathcal{D}_{r,v}Z_s dW_s - \int_{]t,T]\times\mathbb{R}_0} \mathcal{D}_{r,v}U_s(x)\tilde{N}(ds,dx)$$

Theorem

- *The differentiated generator is given by*

$$\mathcal{D}_{r,v} f(t, X, \Phi_s) = \begin{cases} \mathcal{D}_{r,0} f(t, X, \phi)|_{\phi=\Phi_s} + \langle \nabla_\phi f(s, X, \Phi_s), \mathcal{D}_{r,0}\Phi_s \rangle, & v = 0, \\ f(s, X + v\mathbb{1}_{[r,T]}, \Phi_s + \mathcal{D}_{r,v}\Phi_s) - f(s, X, \Phi_s), & v \neq 0 \end{cases}$$

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- *For* $\mathbb{m}$*-almost all* (r, v), $\mathcal{D}_{r,v} Y$ *admits a càdlàg version in* t.
- $\mathcal{D}_{r,v} Y_r := \lim_{t \searrow r} \mathcal{D}_{r,v} Y_t$ *is well defined and it holds*

$$Z \overset{version}{=} {}^p\left((\mathcal{D}_{r,0} Y_r)_{r \in [0,T]} \right),$$

$$U \overset{version}{=} {}^p\left((\mathcal{D}_{r,v} Y_r)_{(r,v) \in [0,T] \times \mathbb{R}_0} \right)$$

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  is $\mathbb{P}$-a.s. bounded and continuous.
- $\forall (t,\phi) \in [0,T] \times \mathbb{R}^3 : f(t,X,\phi) \in \mathbb{D}_{1,2}$
- $\forall G \in (\mathrm{L}_2)^3 : \exists \Gamma \in L_2(\mathbb{P} \otimes \mathrm{m})$, such that for a.e. t it holds

$$|(D_{r,v} f)(t,\cdot,G)| \leq \Gamma_{r,v} \quad \mathbb{P} \otimes \mathrm{m} - a.e.$$

Assumptions on f which admit differentiation:

- $\mathbb{E} \int_0^T |f(s, X, 0, 0, 0)|^2 \, ds < \infty$.
- $\forall t \in [0, T]$:

$$\mathbb{R}^3 \ni \phi \mapsto \partial_{\phi_i} f(t, X, \phi), \quad i = 1, 2, 3$$

  is $\mathbb{P}$-a.s. bounded and continuous.

- $\forall (t, \phi) \in [0, T] \times \mathbb{R}^3 : f(t, X, \phi) \in \mathbb{D}_{1,2}$
- $\forall G \in (\mathrm{L}_2)^3 : \exists \Gamma \in L_2(\mathbb{P} \otimes \mathrm{m})$, such that for a.e. t it holds

$$|(D_{r,v} f)(t, \cdot, G)| \leq \Gamma_{r,v} \quad \mathbb{P} \otimes \mathrm{m} - a.e.$$

- $\forall t \in [0, T], \forall N \in \mathbb{N} \;\; \exists \; K_N^t \in \bigcup_{p>0} \mathrm{L}_p$:
  for $\eta, \tilde{\eta} \in B_N(0)$ and a. a. ω

$$\| (D_{\cdot,0} f)(\cdot, t, \eta)(\omega) - (D_{\cdot,0} f)(\cdot, t, \tilde{\eta})(\omega) \|_{L_2[0,T]} < K_N^t(\omega) |\eta - \tilde{\eta}|.$$

Exceeding the assumptions:

Exceeding the assumptions:

If $\nu(\mathbb{R}) < \infty$, and $\xi \in L_\infty$ then the theorem above remains also true for the BSDE

$$Y_t = \xi + \int_t^T \Big(f_g\,(s, X, Y_s, Z_s, U_s) + \int_{\mathbb{R}_0} \frac{e^{\alpha U_s(x)} - \alpha U_s(x) - 1}{\alpha} \nu(dx) \Big) ds \\ - \int_t^T Z_s dW(s) - \int_{(t,T]\times\mathbb{R}_0} U_s(x) \tilde{N}(ds, dx), \quad 0 \leq t \leq T$$

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