Numerical approximation of monotone BSDEs

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Summary of the talk

We consider the Euler schemes for BSDEs with a driver f that is monotone with polynomial growth in Y.

Implicit scheme : good (converges).

Explicit : bad (explodes).

[Based on joint work with Gonçalo dos Reis and Lukasz Szpruch.]

BSDEs

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t \\ Y_T = \xi. \end{cases}$$

Several connections, applications, points of view.

- stochastic control
- mathematical finance
- a target/inverse problem for a controlled SDE
- nonlinear expectations
- a probabilistic representation for PDEs —backward stochastic ODE.

Connection between FBSDEs and PDEs.

Forward and backward SDE : for $s \leq t \leq T$,

$$\begin{cases} X_s = x \\ dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t \\ Y_T = \xi = g(X_T). \end{cases}$$

PDE :

$$\begin{cases} v_t + \frac{1}{2}\sigma^2 v_{xx} + bv_x + f(t, x, v, v_x\sigma) = 0 \\ v(T, x) = g(x). \end{cases}$$

Connection : $v(t, X_t) = Y_t$ and $(v_x\sigma)(t, X_t) = Z_T$.
Nonlinear Feynman–Kac formula $v(s, x) = Y_s$.
Generalization of the method of characteristics to 2nd order, parabolic PDEs.

BSDE : backward stochastic ODE.

The BSDE can be re-writen

$$Y_t = E\left(\xi + \int_t^T f(Y_u, Z_u)du \middle| \mathcal{F}_t\right)$$

where
$\int ZdW$ = martingale part of the semimartingale Y.

Two things taking place continuously :

- ODE dynamics : $\xi + \int_t^T f(Y_u)du$ (backward in time)
- conditional expectation.

Numerical approximation of BSDEs

Discretize the time interval $[0, T]$: let $h = T/N$ and

$$0 = t_0 < t_1 = h < t_2 = 2h < \ldots < t_N = Nh = T.$$

Over an interval $[t_i, t_{i+1}]$,

$$Y_{t_i} = E\left(Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_u, Z_u) du \middle| \mathcal{F}_{t_i} \right)$$

and $\int Z_t dW_t$ is the martingale part of Y.

Numerical approximation of BSDEs

Over an interval $[t_i, t_{i+1}]$,

$$Y_{t_i} = E\left(Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_u, Z_u)du \middle| \mathcal{F}_t \right)$$

and $\int Z_t dW_t$ is the martingale part of Y.

Approximating the integral (several possibilities : choose an integration rule) leads to time-discretization schemes : produces a sequence $(Y_i, Z_i)_i$ approximating $(Y_{t_i}, Z_{t_i})_i$.

Initialization : define (Y_N, Z_N).
Backward computations : from $i = N - 1$ to $i = 0$ compute (Y_i, Z_i), knowing (Y_{i+1}, Z_{i+1}).

Numerical approximation of BSDEs

→ Euler schemes (rectangle rule).

$$Y_{t_i} = E\left(Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_u, Z_u)du \middle| \mathcal{F}_t \right)$$

becomes either

$$Y_i = E\left(Y_{i+1} + f(Y_{i+1}, Z_{i+1})h \middle| \mathcal{F}_i \right)$$

(explicit scheme, right-end point rule)
or

$$Y_i = E\left(Y_{i+1} \middle| \mathcal{F}_i \right) + f(Y_i, Z_i)h$$

(implicit scheme, left-end point rule).

Numerical approximation of BSDEs

What about Z_i ? Since $\int ZdW$ is the martingale part of Y,

$$\langle W, Y\rangle_{t_i}^{t_{i+1}} = \int_{t_i}^{t_{i+1}} d\langle W, Y\rangle_t = \int_{t_i}^{t_{i+1}} Z_t dt$$

so

$$Z_{t_i} \approx E\left(\frac{1}{h}\int_{t_i}^{t_{i+1}} Z_t dt \middle| \mathcal{F}_i\right) \approx E\left(\frac{1}{h}(W_{t_{i+1}} - W_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \middle| \mathcal{F}_i\right)$$
$$= E\left(\frac{\Delta W_{t_{i+1}}}{h} Y_{t_{i+1}} \middle| \mathcal{F}_i\right),$$

which leads to setting

$$Z_i = E\left(\frac{\Delta W_{t_{i+1}}}{h} Y_{i+1} \middle| \mathcal{F}_i\right)$$

Euler schemes : explicit and implicit

Explicit scheme :

$$Y_i = E\left(Y_{i+1} + f(Y_{i+1}, Z_{i+1})h \middle| \mathcal{F}_i\right)$$

$$Z_i = E\left(\frac{\Delta W_{i+1}}{h}\left\{Y_{i+1} + f(Y_{i+1}, Z_{i+1})h\right\} \middle| \mathcal{F}_i\right)$$

Implicit scheme :

$$Y_i = E\left(Y_{i+1} \middle| \mathcal{F}_i\right) + f(Y_i, Z_i)h$$

$$Z_i = E\left(\frac{\Delta W_{i+1}}{h}\left\{Y_{i+1} + 0\right\} \middle| \mathcal{F}_i\right)$$

Analytic framework

Euler scheme(s) : well understood (to some extent ...) and well-behaved in the case of Lipschitz drivers f (moment estimates, regularity, etc).

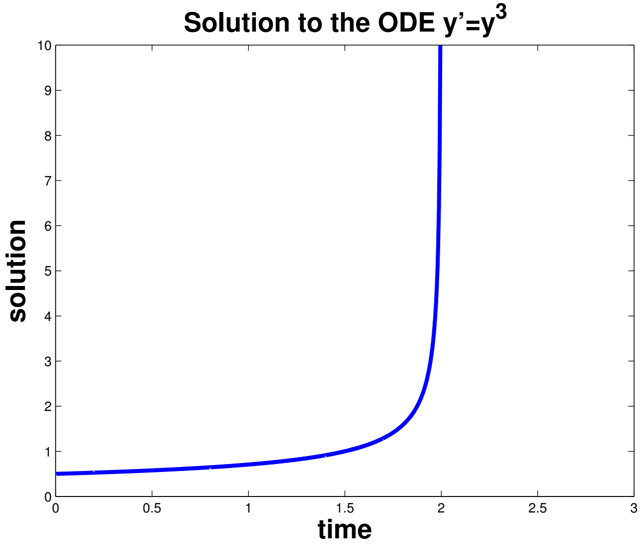
We study the Euler schemes for BSDEs when the driver f is monotone and has polynomial growth in the Y variable.

Motivation : reaction-diffusion PDEs,

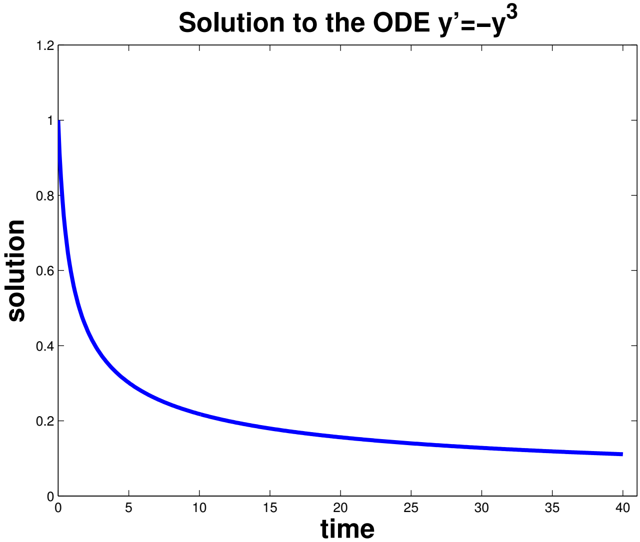
- FitzHugh–Nagumo equation,
- Allen–Cahn equation, ...

Typically the driver in these equations is a polynomial in y.

The monotonicity condition



The monotonicity condition



The monotonicity condition

Linear growth $\rightsquigarrow$ Gronwall $\rightsquigarrow$ bounds. The solution cannot explode.

As soon as there is some superlinear growth (in y, in z, ...), no guarantee anymore.
Need to add a condition on the *structure* of the driver.

"Monotonicity condition" : monotone decreasing.

Precisely, there exists $\mu \in \mathbb{R}$:

$$\left\langle f(y', z) - f(y, z) \middle| y' - y \right\rangle \leq \mu |y' - y|^2$$

(can think of $\mu \leq 0$).

Findings. Main message of this talk.

Implicit scheme : converges as usual.

Explicit scheme : diverges in general.

$\rightarrow$ The time-discretization "breaks the structure".

For a better understanding, we studied the θ-schemes.

Euler schemes : theta, implicit and explicit

θ-scheme :

$$Y_i = E\left(Y_{i+1} + (1-\theta) f(Y_{i+1}, Z_{i+1}) h \middle| \mathcal{F}_i \right) + \theta f(Y_i, Z_i) h$$
$$Z_i = E\left(\frac{\Delta W_{i+1}}{h} \left\{ Y_{i+1} + (1-\theta) f(Y_{i+1}, Z_{i+1}) h \right\} \middle| \mathcal{F}_i \right)$$

θ = degree of implicitness.
$\theta = 1$: implicit scheme,
$\theta = 0$: explicit scheme.

Findings. Main message of this talk.

Explicit scheme : diverges in general.

Implicit scheme : converges as usual.

$\rightarrow$ the time-discretization "breaks the structure".

For a better understanding, we studied the θ-schemes.

$\theta \geq \frac{1}{2}$ (mostly-implicit) : scheme is *somehow stable*, and converges.

$\theta = 1$ (pure implicit) : scheme *is* stable (and so converges).

Why does the explicit scheme explode ?

Comparison with ODEs.

The case of ODEs.

The case of SDEs.

The case of BSDEs.

Explicit Euler scheme for a superlinear ODE

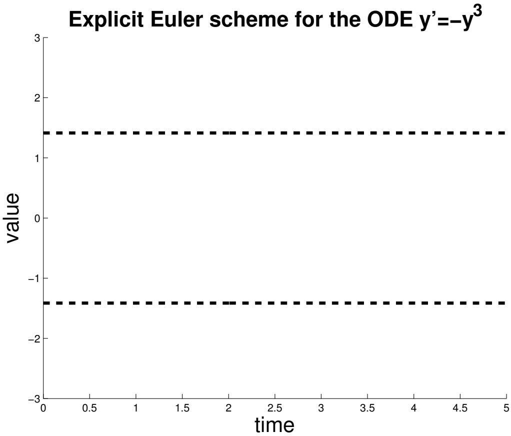
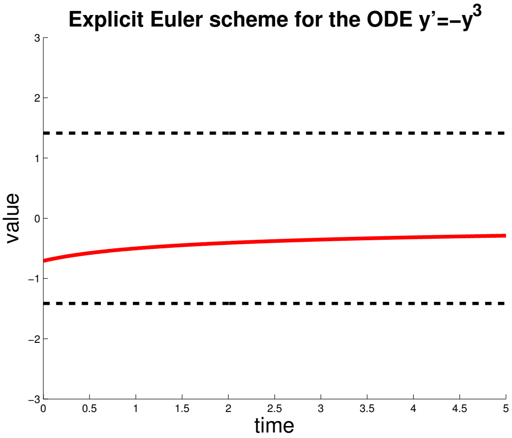


FIGURE: Time step h=1.

Explicit Euler scheme for a superlinear ODE



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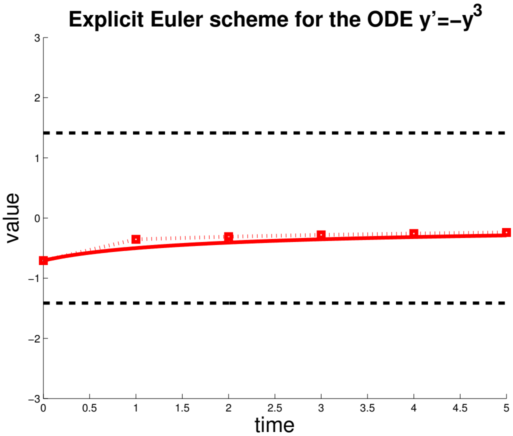

FIGURE: Time step h=1.

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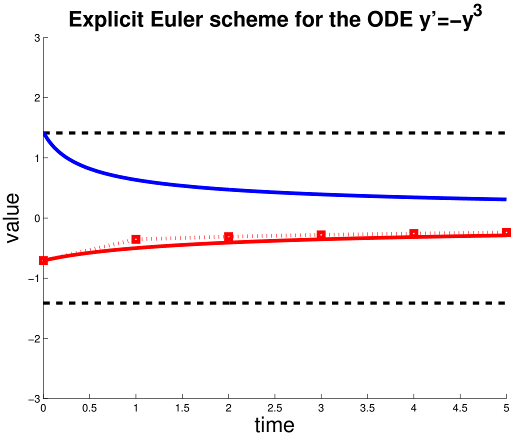

FIGURE: Time step h=1.

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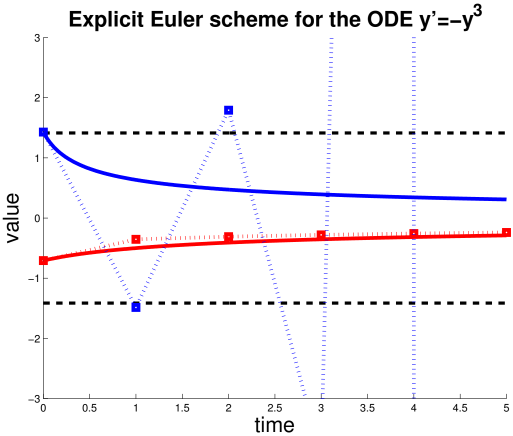

FIGURE: Time step h=1.

Explicit Euler scheme for a superlinear ODE

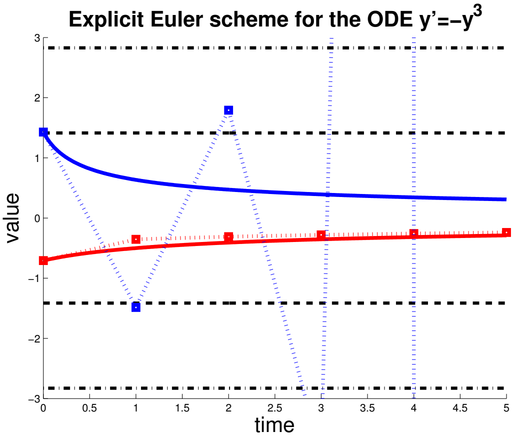

FIGURE: Time step h=0.25.

Explicit Euler scheme for a superlinear ODE

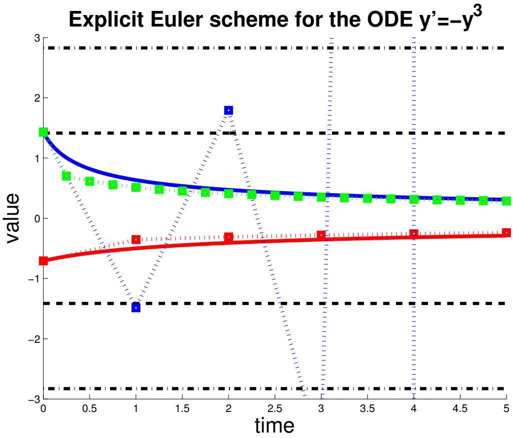

FIGURE: Time step h=0.25.

Why does the explicit scheme explode ?

Comparison with ODEs.

The case of ODEs.

- Thresholds.
- Initial condition is a point : bounded.

The case of SDEs.

The case of BSDEs.

Why does the explicit scheme explode ?

Comparison with ODEs.

The case of ODEs.

The case of SDEs.

- Initial condition is a point : bounded.
- Noise ΔW_t can throw the system out of the safe zone.

The case of BSDEs.

Why does the explicit scheme explode ?

Comparison with ODEs.

The case of ODEs.

The case of SDEs.

The case of BSDEs.

- Terminal condition is a distribution : unbounded.
- ODE component of the BSDE $\rightsquigarrow$ explosion (conditional expectations don't help).

Convergence for the mostly implicit schemes

Recall the θ-scheme :

$$Y_i = E\Big(Y_{i+1} + (1-\theta) f(Y_{i+1}, Z_{i+1}) h \Big| \mathcal{F}_i\Big) + \theta f(Y_i, Z_i) h$$
$$Z_i = E\Big(\frac{\Delta W_{i+1}}{h}\Big\{Y_{i+1} + (1-\theta) f(Y_{i+1}, Z_{i+1}) h\Big\}\Big| \mathcal{F}_i\Big)$$

θ = degree of implicitness.
$\theta = 1$: implicit scheme,
$\theta = 0$: explicit scheme.

Write $S_i = (Y_i, Z_i)$.
In all generality, a scheme does $S_i = \Phi(S_{i+1})$.

Convergence for the mostly implicit schemes

Write $S_{t_i} = (Y_{t_i}, \overline{Z}_{t_i})$ for the BSDE solution on the grid.
BSDE dynamics : $S_{t_i} = \Psi(S_{t_{i+1}})$.

Global error :

$$\mathrm{ERR}\Big((Y_i, Z_i)_i \ , (Y_{t_i}, \overline{Z}_{t_i})_i\Big)^2 = \max_{i=0\ldots N} E\Big[|Y_i - Y_{t_i}|^2\Big] + E\left[\sum_{i=0}^{N-1} |Z_i - \overline{Z}_{t_i}|^2 h\right] .$$

Error at time t_i :
ϵ_i = distance between $S_i = (Y_i, Z_i)$ and $S_{t_i} = (Y_{t_i}, \overline{Z}_{t_i})$, i.e. between output of the scheme for time t_i and BSDE solution at t_i.

Convergence for the mostly implicit schemes

$$
\begin{aligned}
S_i - S_{t_i} &= (S_i - \widehat{S}_i) + (\widehat{S}_i - S_{t_i}) \\
&= \underbrace{\Phi(S_{i+1}) - \Phi(S_{t_{i+1}})}_{\text{error propagation}} + \underbrace{\Phi(S_{t_{i+1}}) - \Psi(S_{t_{i+1}})}_{\text{one-step error}} \ .
\end{aligned}
$$

Distance between $S_i = \Phi(S_{i+1})$ and $\widehat{S}_i = \Phi(S_{t_{i+1}})$: propagation at time t_i of the error ϵ_{i+1} that existed already at time t_{i+1}, denoted by ρ_i.

Distance between $\widehat{S}_i = \Phi(S_{t_{i+1}})$ and $S_{t_i} = \Psi(S_{t_{i+1}})$: one-step time-discretization error $=: \tau_i$.

So we have

$$
\mathrm{dist}\Big(S_i, S_{t_i}\Big) = \epsilon_i \leq \rho_i + \tau_i \ .
$$

Convergence for the mostly implicit schemes

Stability estimate for the scheme : essentially says that

$$\begin{aligned} \rho_i &= \mathrm{dist}\Big(\Phi(S_{i+1}), \Phi(S_{t_{i+1}})\Big) \\ &\leq e^{c\,h}\,\mathrm{dist}\Big(S_{i+1}, S_{t_{i+1}}\Big) + R_i^\theta \\ &= e^{c\,h}\epsilon_{i+1} + R_i^\theta \quad , \end{aligned}$$

where R_i^θ is a remainder term. $R_i^\theta = 0$ corresponds to the usual stability estimate.

Convergence for the mostly implicit schemes

This gives

$$\epsilon_i \leq e^{c\,h}\epsilon_{i+1} + R_i^\theta + \tau_i \ .$$

Gronwall type estimation leads to (Fundamental Lemma)

$$\mathrm{ERR}\Big((S_i)_i, (S_{t_i})_i\Big) \lesssim \mathsf{Err}(\mathsf{Term.Cond}) + \sum_{i=0}^{N-1} \tau_i + \mathcal{R}^{\mathcal{S}} \ ,$$

where $\mathcal{R}^{\mathcal{S}} = \sum_i R_i^\theta$ is the stability remainder.

Convergence for the mostly implicit schemes

$$\mathrm{ERR}\Big((S_i)_i, (S_{t_i})_i\Big) \lesssim \mathsf{Err}(\mathsf{Term.Cond}) + \sum_{i=0}^{N-1} \tau_i + \mathcal{R}^{\mathcal{S}} \ .$$

$\mathsf{Err}(\mathsf{Term.Cond}) + \sum_{i=0}^{N-1} \tau_i$: similar to Lipschitz drivers.
$\lesssim \mathrm{ERR}(X) + \mathrm{REG}^f(Y, Z)$.

Stability remainder $\mathcal{R}^{\mathcal{S}} = \sum_i R_i^\theta$: due to superlinear growth, and scheme-specific.

Convergence for the mostly implicit schemes

Our analysis shows that :

- For $\theta = 1$ (implicit scheme) : $R_i^\theta = 0$. Hence the scheme is stable in the usual sense, and we have convergence.

- For $\theta \in [\frac{1}{2}, 1[$: $R_i^\theta \neq 0$ but further estimation (using $\theta \geq \frac{1}{2}$) allows to show that $\mathcal{R}^{\mathcal{S}} \to 0$ as $h \to 0$.

- For $\theta = \frac{1}{2}$, and under further assumptions on f, we find higher-order estimates for $\mathcal{R}^{\mathcal{S}}$ as well as $\sum_{i=0}^{N-1} \tau_i$, hence we have a higher-order scheme.

This analysis also allows us to study the convergence of a simple *tamed* explicit scheme (cf Lukasz's talk).

Some numerical simulations

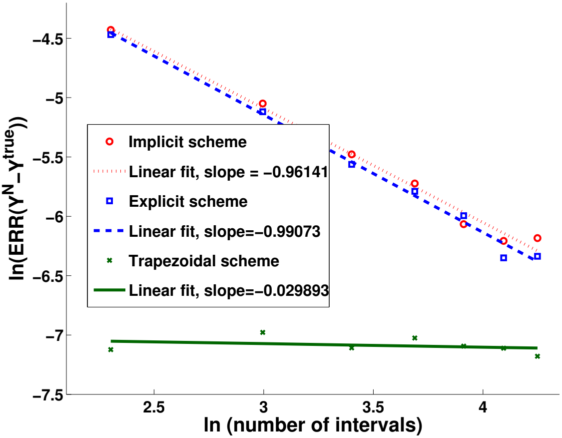

FIGURE: Driver $f(y) = y - y^3$, term.cond. $g(x) = 1/(1 + e^x)$, X=BM. Time grid with $N = 10 \ldots 70$. Expectations computed by regression on a basis of polynomials up to degree $K = 7$, with $M = 200k$ simulated paths.

Some numerical simulations

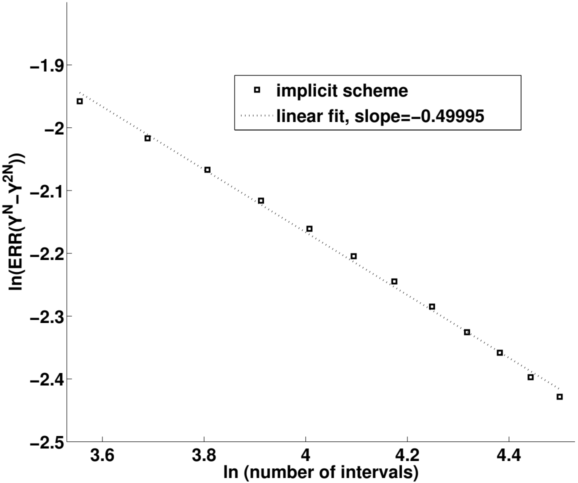

FIGURE: Driver $f(y) = -y^3$, term.cond. $g(x) = x$, X=gBM ($\mu = \sigma = \frac{1}{2}$) Time grid with $N = 30 \dots 90$. Expectations computed by regression on a basis of polynomials up to degree $K = 4$, with $M = 100k$ simulated paths.

Summary

We study the time-discretization for BSDEs with monotone drivers.

Lipschitz/linear growth $\rightsquigarrow$ Gronwall $\rightsquigarrow$ bounds. Explosion cannot happen. BSDE and schemes well behaved.
Superlinear growth : no guarantee. The discretization "breaks the structure".
Explicit Euler scheme : unstable, explodes.
Implicit Euler scheme : stable, converges.

General and abstract analysis of the global error : allows to easily study the convergence of time-discretization schemes.
Study of θ-schemes. $\theta \geq \frac{1}{2} \Rightarrow$ convergence (but $\theta = 1$ more stable).

That's all folks ...

THANK YOU

FOR YOUR

ATTENTION