BSDE Representation for Stochastic Control Problems with Controlled Intensity

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Outline

1. Introduction
2. Randomization of the control
3. BSDE with partially zero diffusive component and nonlinear Feynman-Kac formula

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Problem

- **Hamilton-Jacobi-Bellman equation with controlled intensity:**

$$\frac{\partial v}{\partial t} + \sup_{a \in A} \Big[b(x,a).D_x v + \frac{1}{2}\mathrm{tr}\big(\sigma\sigma^{\intercal}(x,a)D_x^2 v\big) + f(x,a)$$

$$+ \int_E \big(v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_x v(t,x)\big)\lambda(a,de)\Big] = 0,$$

on $[0,T) \times \mathbb{R}^d$, with terminal condition

$$v(T,x) \;=\; g(x), \qquad x \in \mathbb{R}^d.$$

- **Existence and Uniqueness** of a viscosity solution v.
- **Nonlinear Feynman-Kac formula:**

$$v(t,x) \;\stackrel{?}{=}\; Y_t^{t,x}.$$

How to solve it: two approaches

- **Second-order BSDE with jumps (2BSDEJs)**:
(i) M. N. Kazi-Tani, D. Possamaï, C. Zhou (2014) *Second Order BSDEs with Jumps: Formulation and Uniqueness*, preprint arXiv.
(ii) M. N. Kazi-Tani, D. Possamaï, C. Zhou (2014) *Second Order BSDEs with Jumps: Existence and probabilistic representation for fully-nonlinear PIDEs*, preprint arXiv.

- **Randomization of the control**:
I. Kharroubi and H. Pham (2012) *Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDEs*, to appear on Annals of Probability.

▶ **Main goals:**
  - Try to extend the *randomization of the control method* to get existence and the nonlinear Feynman-Kac formula.
  - Comparison theorem: implement the *nonlocal version of Jensen-Ishii's lemma* of Barles & Imbert.

Motivation

- **Model uncertainty:**

$$\frac{\partial v}{\partial t} + \sup_{(b,c,F)\in\Theta} \Big[b.D_x v + \frac{1}{2}\mathrm{tr}\big(c D_x^2 v\big)$$

$$+ \int_E \big(v(t,x+z) - v(t,x) - D_x v(t,x).z1_{\{|z|\le 1\}}\big) F(dz) \Big] = 0,$$

on $[0,T)\times\mathbb{R}^d$, with terminal condition

$$v(T,x) \; = \; g(x), \qquad x\in\mathbb{R}^d.$$

Θ denotes a set of Lévy triplets (b,c,F).

- ▶ The unique viscosity solution v is represented as follows:

$$v(t,x) \; = \; \mathcal{E}(g(x+\mathcal{X}_t)),$$

where $\mathcal{X}$ is a nonlinear Lévy process under the nonlinear expectation $\mathcal{E}(\cdot)$.

Literature review

- G. Barles and C. Imbert (2008) *Second-order elliptic integro-differential equations: viscosity solutions' theory revisited*, Annales de l'Institut Henri Poincaré.
- M. Hu and S. Peng (2009) *G-Lévy Processes under Sublinear Expectations*, preprint arXiv.
- M. N. Kazi-Tani, D. Possamaï, C. Zhou (2014) *Second Order BSDEs with Jumps: Formulation and Uniqueness*, preprint arXiv.
- M. N. Kazi-Tani, D. Possamaï, C. Zhou (2014) *Second Order BSDEs with Jumps: Existence and probabilistic representation for fully-nonlinear PIDEs*, preprint arXiv.
- I. Kharroubi and H. Pham (2012) *Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDEs*, to appear on Annals of Probability.
- A. Neufeld and M. Nutz (2014) *Nonlinear Lévy Processes and their Characteristics*, preprint arXiv.

Stochastic control representation: intuition

We expect that v has the stochastic control representation

$$v(t,x) \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\alpha}\Big[\int_t^T f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_T^{t,x,\alpha})\Big],$$

where $X^{t,x,\alpha}$ has the controlled dynamics on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}^{\alpha})$

$$\begin{aligned} dX_s^{\alpha} &= b(X_s^{\alpha},\alpha_s)ds + \sigma(X_s^{\alpha},\alpha_s)dW_s + \int_E \beta(X_{s^-}^{\alpha},\alpha_s,e)\tilde{\pi}(ds,de) \\ X_t^{\alpha} &= x \end{aligned}$$

with

$$\tilde{\pi}(dt,de) \;=\; \pi(dt,de) - \lambda(\alpha_t,de)dt$$

the compensated martingale measure of π.

- **Randomization of the control** will be developed having in mind this stochastic control representation.

Outline

Forward process

- **Randomization** by an independent Brownian motion B mapped on $A \subset \mathbb{R}^q$ by means of a C^2 surjection $h\colon \mathbb{R}^q \to A$:

$$\begin{aligned} X_s &= x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r \\ &\quad + \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de), \\ I_s &= h(a + B_s - B_t), \qquad\qquad t \le s \le T, \end{aligned}$$

where $\tilde{\pi}(dr, de) = \pi(dr, de) - \lambda(I_r, de)dr$ is the compensated martingale measure of π.

- **Main issues:**
  - Why do we randomize with a Brownian motion B?
  - Existence and uniqueness of $(X^{t,x,a}, I^{t,a})$?

Brownian randomization

- **Poisson random measure:** in Kharroubi & Pham an independent Poisson random measure μ on $\mathbb{R}_+ \times A$ is used to randomize the control. No surjection is needed.

- **Martingale representation theorem**
  - Unlike Kharroubi & Pham, we have *dependence* between B (or μ) and π through the compensator of π.
  - However, B is **orthogonal** to π, since B is a continuous martingale.

$\Rightarrow$ *Martingale representation* for (W, B, π), while not clear for (W, μ, π) due to the dependence between μ and π.

Wellposedness of the SDE

- **Nonstandard SDE:** The jump part of the driving factors is not given, but depends on the solution via its intensity.
  - J. Jacod and P. Protter (1982) *Quelques remarques sur un nouveau type d'équations différentielles stochastiques*, Séminaire de Probabilités XVI.
- **Dominated case** $\lambda(a, de) = m(a)\bar{\lambda}(de)$: we can solve the forward SDE bringing it back to a standard SDE, via a change of intensity "à la Girsanov".

▶ **Nondominated case**

(1) We solve first the SDE for the process $I^{t,a}$.

(2) Then, we construct a probability measure $\mathbb{P}^{t,a}$ on $(\Omega, \mathcal{F})$ such that the random measure $\pi(dt, de)$ admits $\lambda(I_s^{t,a}, de)ds$ as compensator.

(3) Finally, we solve by standard methods the SDE for $X^{t,x,a}$.

Outline

BSDE with partially zero diffusive component

- **BSDE:** for any $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$,

$$\begin{aligned} Y_s &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T - K_s - \int_s^T Z_r dW_r \\ &\quad - \int_s^T V_r dB_r - \int_s^T \int_E U_r(e)\tilde{\pi}(dr,de), \qquad t \leq s \leq T,\ \mathbb{P}^{t,a}\ a.s. \end{aligned}$$

and

$$V_s = 0 \qquad ds \otimes d\mathbb{P}^{t,a}\ a.e.$$

- **Main issues:**
  - We look for a solution for which the B-component resulting from the martingale representation theorem is zero.
  - Existence is guaranteed if we *add* the increasing process K.
  - Uniqueness is guaranteed if we look for the **minimal solution** (Y,Z,V,U,K), i.e., for any solution $(\bar{Y},\bar{Z},\bar{V},\bar{U},\bar{K})$ we must have $Y \leq \bar{Y}$.

Minimal solution: existence

- **Penalized BSDE:** for any $n \in \mathbb{N}$, $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$,

$$
\begin{aligned}
Y_s^n &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^n - K_s^n - \int_s^T Z_r^n dW_r \\
&\quad - \int_s^T V_r^n dB_r - \int_s^T \int_E U_r^n(e)\tilde{\pi}(dr,de), \quad t \leq s \leq T,\ \mathbb{P}^{t,a}\ a.s.
\end{aligned}
$$

where

$$
K_s^n = n \int_t^s |V_r^n| dr, \qquad t \leq s \leq T.
$$

- **Wellposedness** is based, as usual, on the *martingale representation theorem* and on a *fixed point argument.*
- **Monotonicity:** from the *comparison theorem* for BSDEs with jumps we have

$$
Y^0 \leq \cdots \leq Y^n \nearrow Y \leq \bar{Y}, \qquad \forall\, n, \text{ for any other solution } \bar{Y}.
$$

- **Uniform estimates** for $(Z^n, V^n, U^n, K^n)_n$ allow to obtain weak convergence of these components, so to pass to the limit in the BSDE, and to end up with the *minimal solution.*

Viscosity property of the penalized BSDE

- **Penalized HJB equation**

$$
\begin{aligned}
\frac{\partial v_n^h}{\partial t} + b(x,h(a)).D_x v_n^h + \frac{1}{2}\mathrm{tr}\big(\sigma\sigma^{\intercal}(x,h(a))D_x^2 v_n^h\big) + f(x,h(a)) \\
+ \int_E \big[v_n^h(t,x+\beta(x,h(a),e)) - v_n^h(t,x) \\
-\beta(x,h(a),e).D_x v_n^h(t,x)\big]\lambda(h(a),de) \\
+\frac{1}{2}\mathrm{tr}\big(D_a^2 v_n^h(t,x,a)\big) + n\big|D_a v_n^h(t,x,a)\big| = 0,
\end{aligned}
$$

on $[0,T) \times \mathbb{R}^d \times \mathbb{R}^q$, with terminal condition

$$
v_n^h(T,x,a) \;=\; g(x), \qquad (x,a) \in \mathbb{R}^d \times \mathbb{R}^q.
$$

▶ **Nonlinear Feynman-Kac formula:** the unique continuous viscosity solution, satisfying a linear growth condition, is given by

$$
v_n^h(t,x,a) \;:=\; Y_t^{n,t,x,h(a)}, \qquad (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q,
$$

Viscosity property of the BSDE

- $Y_t^{n,t,x,h(a)} = v_n^h(t,x,a) \nearrow v(t,x,a) = Y_t^{t,x,a}$.
  - v does not depend on a, but only on (t,x).

- **Existence:** v is a (discontinuous) *viscosity solution* to the HJB equation. The result follows from:
  - the viscosity property of v_n^h
  - the convergence $v_n^h \nearrow v$ as $n \to \infty$
  - stability arguments for viscosity solutions.

- **Uniqueness:** v is the unique viscosity solution to the HJB equation, satisfying a linear growth condition.
  - The result follows from the *comparison theorem*, which is proved relying on the nonlocal version of Jensen-Ishii's lemma of Barles & Imbert.

Conclusion

- Study of a class of Hamilton-Jacobi-Bellman equations with:
  - Controlled diffusion coefficient; possibly degenerate.
  - Controlled intensity; possibly nondominated.

- Introduction of a class of BSDEs with partially zero diffusive component which provides:
  - Existence of a viscosity solution.
  - Nonlinear Feynman-Kac formula.

- Comparison theorem and uniqueness.

Thank you!