BSDEs with singular terminal condition and applications to optimal trade execution

Thomas Kruse

based on joint work with Stefan Ankirchner and Monique Jeanblanc

Young Researchers Meeting on BSDEs, Numerics and Finance
July 7, 2014
Bordeaux

Financial support from the French Banking Federation through the Chaire Markets in Transition is gratefully acknowledged.

Optimal position closure

Case study: Sell x shares of Adidas within T minutes using market orders.

Optimal position closure

Case study: Sell x shares of Adidas within T minutes using market orders.

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG							83,680	133	12:33:29	CO	83,140
	Bid/Ask Orders												
			2	505	83,650	83,680	162	2					
			5	586	83,640	83,690	275	2					
			9	925	83,630	83,700	670	7					
			7	869	83,620	83,710	1.125	10					
			5	566	83,610	83,720	1.062	8					
			6	676	83,600	83,730	1.085	8					
			7	583	83,590	83,740	405	4					
			5	790	83,580	83,750	952	9					
			7	776	83,570	83,760	246	4					
			2	117	83,560	83,770	888	6					

Optimal position closure

Case study: Sell x shares of Adidas within T minutes using market orders.

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG							83,680	133	12:33:29	CO	83,140
	Bid/Ask Orders												
			2	505	83,650	83,680	162	2					
			5	586	83,640	83,690	275	2					
			9	925	83,630	83,700	670	7					
			7	869	83,620	83,710	1.125	10					
			5	566	83,610	83,720	1.062	8					
			6	676	83,600	83,730	1.085	8					
			7	583	83,590	83,740	405	4					
			5	790	83,580	83,750	952	9					
			7	776	83,570	83,760	246	4					
			2	117	83,560	83,770	888	6					

Assumption (Almgren&Chriss):

$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta z$$

z: amount sold at time t
η: price impact factor

Stochastic Liquidity

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG	1	397	84,840	84,880	312	2	84,890	89	12:38:40	CO	85,920
		Bid/Ask Orders											
			1	876	84,870	84,900	281	2					
			3	455	84,860	84,910	392	3					
			5	494	84,850	84,920	275	2					
			9	1.187	84,840	84,930	1.040	9					
			9	1.408	84,830	84,940	889	5					
			7	602	84,820	84,950	994	7					
			7	760	84,810	84,960	358	4					
			3	400	84,800	84,970	631	6					
			5	929	84,790	84,980	922	6					
			3	639	84,780	84,990	974	7					

Bid/Ask Orders						
	4	276	84,850	84,900	484	5
	2	275	84,840	84,910	631	5
	7	843	84,830	84,920	808	8
	9	829	84,820	84,930	976	9
	9	1.696	84,810	84,940	937	6
	4	522	84,800	84,950	1.171	7
	6	921	84,790	84,960	358	4
	4	717	84,780	84,970	471	5
	2	134	84,770	84,980	438	3
	4	274	84,760	84,990	723	3

Optimal position closure

Case study: Sell x shares of Adidas within T seconds using market orders.

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG							83,680	133	12:33:29	CO	83,140
	Bid/Ask Orders												
			2	505	83,650	83,680	162	2					
			5	586	83,640	83,690	275	2					
			9	925	83,630	83,700	670	7					
			7	869	83,620	83,710	1.125	10					
			5	566	83,610	83,720	1.062	8					
			6	676	83,600	83,730	1.085	8					
			7	583	83,590	83,740	405	4					
			5	790	83,580	83,750	952	9					
			7	776	83,570	83,760	246	4					
			2	117	83,560	83,770	888	6					

Assumption (Almgren&Chriss):

$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta_t z$$

z: amount sold at time t

(η_t): price impact process

Optimal position closure

Case study: Sell x shares of Adidas within T seconds using market orders.

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG							83,680	133	12:33:29	CO	83,140
	Bid/Ask Orders												
			2	505	83,650	83,680	162	2					
			5	586	83,640	83,690	275	2					
			9	925	83,630	83,700	670	7					
			7	869	83,620	83,710	1.125	10					
			5	566	83,610	83,720	1.062	8					
			6	676	83,600	83,730	1.085	8					
			7	583	83,590	83,740	405	4					
			5	790	83,580	83,750	952	9					
			7	776	83,570	83,760	246	4					
			2	117	83,560	83,770	888	6					

Assumption (Almgren&Chriss):

$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta_t z^{p-1}$$

z: amount sold at time t

(η_t): price impact process, $p > 1$: shape parameter

The model: Trading rates determine remaining position

- $T < \infty$: time horizon
- $x \in \mathbb{R}$: initial position
- X_t: position size at time $t \in [0, T]$
- $\dot{X}_t$: trading rate ($\dot{X} \geq 0$: buying, $\dot{X} \leq 0$: selling)

$$X_t = x + \int_0^t \dot{X}_s ds$$

- **Constraint:** $X_T = 0$

The Model: Stochastic liquidity

- Brownian basis: $(\Omega, \mathcal{F}, P, (\mathcal{F}_t), (W_t))$
- $S = S^{\text{mid}}$: uninfluenced mid-market price (a martingale)
- $(\eta_t)_{t \in [0,T]}$: (positive) price impact process
- $p > 1$: shape parameter of the order book (q its Hölder conjugate)
- If $\dot{X}_t \geq 0$, the realized price is given by

$$S_t^{\text{real}} = S_t + \eta_t \dot{X}_t^{p-1}$$

The Model: Stochastic liquidity

- Brownian basis: $(\Omega, \mathcal{F}, P, (\mathcal{F}_t), (W_t))$
- $S = S^{\text{mid}}$: uninfluenced mid-market price (a martingale)
- $(\eta_t)_{t \in [0,T]}$: (positive) price impact process
- $p > 1$: shape parameter of the order book (q its Hölder conjugate)
- If $\dot{X}_t \geq 0$, the realized price is given by

$$S_t^{\text{real}} = S_t + \eta_t \dot{X}_t^{p-1}$$

- In general: $S_t^{\text{real}} = S_t + \eta_t \operatorname{sgn}(\dot{X}_t) |\dot{X}_t|^{p-1}$

Optimal position closure

- Expected costs:

$$E\left[\int_0^T S_t^{\text{real}}\dot{X}_t dt\right] = -S_0^{\text{mid}}x + E\left[\int_0^T \eta_t|\dot{X}_t|^p dt\right]$$

Optimal position closure

- Expected costs:

$$E\left[\int_0^T S_t^{\text{real}} \dot{X}_t dt\right] = -S_0^{\text{mid}} x + E\left[\int_0^T \eta_t |\dot{X}_t|^p dt\right]$$

- Additive risk functional:

$$E\left[\int_0^T \gamma_t |X_t|^p dt\right], \text{ with e.g. } \gamma_t = const \text{ or } \gamma_t = \lambda(S_t^{\text{mid}})$$

Optimal position closure

- Expected costs:

$$E\left[\int_0^T S_t^{\mathrm{real}} \dot{X}_t \, dt\right] = -S_0^{\mathrm{mid}} x + E\left[\int_0^T \eta_t |\dot{X}_t|^p \, dt\right]$$

- Additive risk functional:

$$E\left[\int_0^T \gamma_t |X_t|^p \, dt\right], \text{ with e.g. } \gamma_t = const \text{ or } \gamma_t = \lambda(S_t^{\mathrm{mid}})$$

- Optimal liquidation problem:

$$E\left[\int_0^T \left(\eta_t |\dot{X}_t|^p + \gamma_t |X_t|^p\right) dt\right] \longrightarrow \min_{X_0 = x, X_T = 0}$$

Our aim & related literature

- We aim at providing a purely probabilistic solution of the control problem
- Characterize the optimal control by means of a BSDE with singular terminal condition

Our aim & related literature

- We aim at providing a purely probabilistic solution of the control problem
- Characterize the optimal control by means of a BSDE with singular terminal condition
- Schied 2013: Solves a variant of this problem in a Markovian framework using superprocesses
- Graewe, Horst, Séré 2013: Allow for jumps in the state process X and show smoothness of the value function in a Markovian framework
- Graewe, Horst, Qiu 2013: Analyze both Markovian and non-Markovian dependence of the coefficients by means of BSPDEs

A maximum principle

$$v(t,x) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \Big| \mathcal{F}_t\right] \qquad (1)$$

Proposition (Maximum Principle)

Let $X \in \mathcal{A}_0(t,x)$ such that

$$M_s = \eta_s|\dot{X}_s|^{p-1} + \int_t^s \gamma_r|X_r|^{p-1} dr$$

is a martingale. Then X is optimal in (1).

A maximum principle

$$v(t,x) = \inf_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \middle| \mathcal{F}_t\right] \qquad (2)$$

Proposition (Maximum Principle)

Let $X \in \mathcal{A}_0(t,x)$, i.e.

$$dX_s = \dot{X}_s ds, \quad X_t = x \quad \& \quad X_T = 0$$

such that

$$M_s = \eta_s|\dot{X}_s|^{p-1} + \int_t^s \gamma_r |X_r|^{p-1} dr$$

is a martingale. Then X is optimal in (2).

Derivation of the BSDE

$$v(t,x) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \middle| \mathcal{F}_t\right]$$

- The value function is explicit in the x variable:

$$v(t,x) = Y_t|x|^p$$

for some coefficient process Y.

Derivation of the BSDE

$$v(t,x) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \middle| \mathcal{F}_t\right]$$

- The value function is explicit in the x variable:

$$v(t,x) = Y_t|x|^p$$

  for some coefficient process Y.
- The maximum principle implies:

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t\right) dt + Z_t dW_t$$

Derivation of the BSDE

$$v(t,x) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s |\dot{X}_s|^p + \gamma_s |X_s|^p\right) ds \middle| \mathcal{F}_t\right]$$

- The value function is explicit in the x variable:

$$v(t,x) = Y_t |x|^p$$

  for some coefficient process Y.

- The maximum principle implies:

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t\right) dt + Z_t dW_t$$

- Terminal constraint leads to singular terminal condition: $Y_T = \infty$

BSDEs with singular terminal condition

So far only considered by Popier 2006, 2007

BSDEs with singular terminal condition

So far only considered by Popier 2006, 2007

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t \right) dt + Z_t dW_t \tag{3}$$

Definition

(Y, Z) is a solution of the BSDE (3) with singular terminal condition $Y_T = \infty$ if it satisfies

(i) *for all $0 \le s \le t < T$:*
$Y_s = Y_t - \int_s^t \left((p-1)\frac{Y_r^q}{\eta_r^{q-1}} - \gamma_r \right) dr - \int_s^t Z_r dW_r$;

(ii) $\liminf_{t \nearrow T} Y_t = \infty$, *a.s.*

(iii) *for all $0 \le t < T$:* $E\left[\sup_{0 \le s \le t} |Y_s|^2 + \int_0^t |Z_r|^2 dr \right] < \infty$;

Integrability Assumptions and Approximation

- For the remainder of the talk we assume that η satisfies

$$E\int_0^T \frac{1}{\eta_t^{q-1}}dt < \infty, \quad E\int_0^T \eta_t^2 dt < \infty$$

and that γ satisfies

$$E\int_0^T \gamma_t^2 dt < \infty$$

Integrability Assumptions and Approximation

- For the remainder of the talk we assume that η satisfies

$$E\int_0^T \frac{1}{\eta_t^{q-1}}dt < \infty, \quad E\int_0^T \eta_t^2 dt < \infty$$

and that γ satisfies

$$E\int_0^T \gamma_t^2 dt < \infty$$

- Approximation

$$\begin{aligned} dY_t^L &= \left((p-1)\frac{(Y_t^L)^q}{\eta_t^{q-1}} - \gamma_t\right)dt + Z_t^L dW_t \\ Y_T^L &= L \end{aligned}$$

Existence and Minimality

Proposition

There exists a solution (Y^L, Z^L). Y^L is bounded from above

$$Y_t^L \leq \frac{1}{(T-t)^p} E\left[\int_t^T (\eta_s + (T-s)^p \gamma_s) ds \middle| \mathcal{F}_t \right].$$

Existence also follows from Briand, Delyon, Hu, Pardoux, Stoica 2003

Existence and Minimality

Proposition

There exists a solution (Y^L, Z^L). Y^L is bounded from above

$$Y_t^L \leq \frac{1}{(T-t)^p} E\left[\int_t^T (\eta_s + (T-s)^p \gamma_s) ds \middle| \mathcal{F}_t\right].$$

Existence also follows from Briand, Delyon, Hu, Pardoux, Stoica 2003

Theorem

There exists a process (Y, Z) such that for every $t < T$ and as $L \nearrow \infty$

- $Y_t^L \nearrow Y_t$ *a.s.*
- $Z^L \to Z$ *in* $L^2(\Omega \times [0, t])$.

The pair (Y, Z) is the minimal solution to (3) with singular terminal condition $Y_T = \infty$.

Optimal Controls - Penalization

Consider the unconstrained minimization problem

$$v^L(0,x) = \inf_{X \in \mathcal{A}(0,x)} E\left[\int_0^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds + L|X_T|^p\right] \qquad (4)$$

Proposition

The control

$$X_t^L = x e^{-\int_0^t \left(\frac{Y_s^L}{\eta_s}\right)^{q-1} ds}$$

is optimal in (4) *and* $v^L(0,x) = Y_0^L|x|^p$.

Optimal Controls

Theorem

The control

$$X_t = xe^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$$

belongs to $\mathcal{A}_0(0, x)$ *and is optimal in* (1). *Moreover,* $v(t, x) = Y_t|x|^p$.

Optimal Controls

Theorem

The control

$$X_t = xe^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$$

belongs to $\mathcal{A}_0(0,x)$ *and is optimal in* (1)*. Moreover,* $v(t,x) = Y_t|x|^p$*.*

Proof

Define $M_t = p\eta_t|\dot{X}_t|^{p-1} + \int_0^t p\gamma_s|X_s|^{p-1}ds$*. Then* $dM_t = X_t^{p-1}Z_t dW_t$*. Hence* M *is a nonnegative, local martingale on* $[0,T)$*. In particular* M *converges almost surely for* $t \nearrow T$*. This implies*

$$0 \leq X_t = \left(\frac{M_t - p\int_0^t \gamma_s X_s^{p-1} ds}{pY_t}\right)^{q-1} \leq \left(\frac{M_t}{pY_t}\right)^{q-1} \to 0$$

a.s. for $t \nearrow T$

Processes with uncorrelated multiplicative increments

Definition

η has uncorrelated multiplicative increments (umi) if

$$E\left[\frac{\eta_t}{\eta_s}\Big|\mathcal{F}_s\right] = E\left[\frac{\eta_t}{\eta_s}\right]$$

for all $s \leq t < T$.

Processes with uncorrelated multiplicative increments

Definition

η has uncorrelated multiplicative increments (umi) if

$$E\left[\frac{\eta_t}{\eta_s}\middle|\mathcal{F}_s\right] = E\left[\frac{\eta_t}{\eta_s}\right]$$

for all $s \leq t < T$.

Examples

- η is deterministic
- η is a martingale
- $d\eta_t = \mu(t)\eta_t dt + \sigma(t, \eta_t) dW_t$
- $\eta_t = e^{Z_t}$ where Z is a Lévy process

umi processes $\leftrightarrow$ deterministic controls

Assume $\gamma = 0$.

Proposition

Suppose that η has umi, then

$$Y_t = \frac{1}{\left(\int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds\right)^{p-1}}$$

is the minimal solution to (3) with singular terminal condition. The deterministic control

$$X_t = x \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}} ds} \int_t^T \frac{1}{E[\eta_s]^{q-1}} ds$$

is optimal in (1). In particular, if $p = 2$, then $\dot{X}_t = -c \frac{1}{E[\eta_t]}$.

umi processes $\leftrightarrow$ deterministic controls

Assume $\gamma = 0$.

Proposition

Suppose that η has umi, then

$$Y_t = \frac{1}{\left(\int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds\right)^{p-1}}$$

is the minimal solution to (3) with singular terminal condition. The deterministic control

$$X_t = x \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}} ds} \int_t^T \frac{1}{E[\eta_s]^{q-1}} ds$$

is optimal in (1). In particular, if $p = 2$, then $\dot{X}_t = -c \frac{1}{E[\eta_t]}$.

Vice versa, assume that the optimal control $X_t = x e^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$ is deterministic. Then η has umi.

Relaxing the liquidation constraint

Consider

$$\inf_{X \in \mathcal{A}(0,x)} E\left[\int_0^T \left(\eta_s |\dot{X}_s|^p + \gamma_s |X_s|^p\right) ds + \xi |X_T|^p\right] \tag{5}$$

where ξ is nonnegative and $\mathcal{F}_T$-measurable with $P[\xi = \infty] > 0$.

Relaxing the liquidation constraint

Consider

$$\inf_{X \in \mathcal{A}(0,x)} E\left[\int_0^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds + \xi|X_T|^p\right] \qquad (5)$$

where ξ is nonnegative and $\mathcal{F}_T$-measurable with $P[\xi = \infty] > 0$.

Examples

- binding liquidation: $\xi = \infty$
- excepted scenarios: $\xi = \infty 1_A$ (e.g. $A = \{\int_0^T \eta_t dt \leq k\}$)

Relaxing the liquidation constraint

Associated BSDE

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t \right) dt + Z_t dW_t, \quad Y_T = \xi \tag{6}$$

Theorem

There exists a minimal supersolution Y ($\liminf_{t \nearrow T} Y_t \geq \xi$) to (6). The strategy

$$X_t = x e^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$$

is optimal in the relaxed liquidation problem (5).

Position targeting & directional views

Consider

$$v(x) = \inf_{X \in \tilde{\mathcal{A}}_0(0,x)} E\left[\int_0^T \left(\left(S_u + \eta_u \dot{X}_u\right)\dot{X}_u + \gamma_u |X_u|^2\right) du\right], \qquad (7)$$

where S is a semimartingale.

Position targeting & directional views

Consider

$$v(x) = \inf_{X \in \tilde{\mathcal{A}}_0(0,x)} E\left[\int_0^T \left(\left(S_u + \eta_u \dot{X}_u\right)\dot{X}_u + \gamma_u |X_u|^2\right) du\right], \qquad (7)$$

where S is a semimartingale. Let Y be the minimal solution to

$$dY_t = \left(\frac{Y_t^2}{\eta_t} - \gamma_t\right) dt + Z_t dW_t, \quad Y_T = \infty$$

and define

$$H_t = \exp\left(-\int_0^t \frac{Y_s}{\eta_s} ds\right), \quad U_t = -\frac{1}{2} E\left[\int_t^T \frac{H_u}{H_t} dS_u \middle| \mathcal{F}_t\right]$$

Position targeting & directional views

Proposition

The strategy $X \in \tilde{\mathcal{A}}_0(0,x)$ *solving the ODE*

$$\dot{X}_t = -\frac{1}{\eta_t}\left(U_t + Y_t X_t\right)$$

is optimal in (7). *The value function is given by*

$$v(x) = Y_0 x^2 + (2U_0 - S_0)x - E\left[\int_0^T \frac{U_s^2}{\eta_s} ds\right].$$

Position targeting & directional views

Consider

$$v(x) = \inf_{X \in \mathcal{A}_\lambda(0,x)} E\left[\int_0^T \left(\left(S_u + \eta \dot{X}_u\right) \dot{X}_u\right) ds\right], \qquad (8)$$

where $X_T = \int_0^T \lambda_s ds$ and S is a martingale.

Position targeting & directional views

Consider

$$v(x) = \inf_{X \in \mathcal{A}_\lambda(0,x)} E\left[\int_0^T \left(\left(S_u + \eta \dot{X}_u\right) \dot{X}_u\right) ds\right], \qquad (8)$$

where $X_T = \int_0^T \lambda_s ds$ and S is a martingale.

Corollary

The strategy solving the ODE

$$\dot{X}_t = -\frac{1}{T-t}\left(X_t - E\left[\int_0^T \lambda_s ds \middle| \mathcal{F}_t\right]\right)$$

is optimal in (8)*.*

Literature

The talk is based on

- S. Ankirchner, M. Jeanblanc and T. Kruse. BSDEs with singular terminal condition and control problems with constraints. SIAM Journal on Control and Optimization, 52(2):893913, 2014.
- S. Ankirchner, T. Kruse. Optimal position targeting with stochastic linear-quadratic costs. To appear in the AMaMeF volume of Banach Center Publications, 2014.

Further references

- P. Briand, B. Delyon, Y. Hu, Ying, E. Pardoux and L. Stoica. L^p solutions of backward stochastic differential equations. Stochastic Processes and Their Applications, 2003
- P. Graewe, U. Horst, J. Qiu. A Non-Markovian Liquidation Problem and Backward SPDEs with Singular Terminal Conditions, 2013
- P. Graewe, U. Horst, E. Séré. Smooth solutions to portfolio liquidation problems under price-sensitive market impact, 2013
- A. Popier. Backward stochastic differential equations with singular terminal condition. Stochastic Processes and Their Applications, 2006.
- A. Popier. Backward stochastic differential equations with random stopping time and singular final condition. Annals of Probability, 2007.
- A. Schied. A control problem with fuel constraint and Dawson-Watanabe superprocesses, 2013. To appear in Annals of Applied Probability.

Thank you!