Markov Chain BSDEs and risk averse networks

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Outline

Markov chains

- Consider a time homogenous finite/countable-state continuous-time Markov chain X.
- X is a process in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the natural filtration of X.
- X takes values in $\mathcal{X}$, the standard basis of $\mathbb{R}^N$ where $N =$ number of states $\leq \infty$.
- X jumps from state X_{t-} to state $e_i \in \mathcal{X}$ at rate $e_i^* A X_{t-}$, so

$$X_t = X_0 + \int_{]0,t]} A X_{u-} du + M_t$$

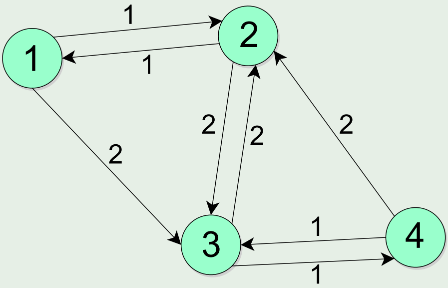
  for some $\mathbb{R}^N$-valued martingale M.
- A is a $\mathbb{R}^{N\times N}$ matrix, with $[A]_{ij} \geq 0$ for $i \neq j$, and $\mathbf{1}^* A e_i = \sum_j [A]_{ij} \equiv 0$.

Example

For $N = 4$, we can consider the Markov chain with matrix

$$A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 2 & 2 \\ 2 & 2 & -3 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Which can be drawn as a weighted directed graph



BSDEs

A Backward Stochastic Differential Equation (BSDE) is an equation of the form

$$Y_t = \xi + \int_{]t,T]} f(\omega, s, Y_{s-}, Z_s)ds - \int_{]t,T]} Z_s^* dM_s$$

or equivalently,

$$dY_t = -f(\omega, t, Y_{t-}, Z_t)dt + Z_t^* dM_t, \qquad Y_T = \xi.$$

- These are useful for various problems, particularly in stochastic control.
- Classically, they are closely linked to semilinear PDEs.

Simple examples:

- $f \equiv 0$ (Martingale representation)
- $f(z) = z^*(B - A)X_{s-}$ (Measure change)

BSDE Existence and Uniqueness

Theorem

Let f be a predictable function with

$$|f(\omega,t,y,z)-f(\omega,t,y',z')|^2 \leq c(|y-y'|^2+\|z-z'\|^2_{M_t})$$

for some $c>0$, and $E[\int_{]0,T]}|f(\omega,t,0,0)|^2dt]<\infty$.
Then for any $\xi\in L^2(\mathcal{F}_T)$ the BSDE

$$Y_t=\xi+\int_{]t,T]}f(\omega,t,Y_{u-},Z_u)du-\int_{]t,T]}Z_u^*dM_u$$

has a unique adapted solution (Y,Z) with appropriate integrability.

Here $\|z\|^2_{M_t}:=z^*\frac{d\langle M\rangle}{dt}z$ is a natural (semi)norm for our problem.

γ-balanced drivers

To get a comparison theorem, we need the following definition.

Definition

f is γ-*balanced* if there exists a random field λ, with $\lambda(\cdot,\cdot,z,z')$ predictable and $\lambda(\omega,t,\cdot,\cdot)$ Borel measurable, such that

- $f(\omega,t,y,z) - f(\omega,t,y,z') = (z-z')^*(\lambda(\omega,t,z,z') - AX_{t-})$,
- for each $e_i \in \mathcal{X}$,
$$\frac{e_i^*\lambda(\omega,t,z,z')}{e_i^*AX_{t-}} \in [\gamma,\gamma^{-1}]$$
for some $\gamma > 0$, where $0/0 := 1$,
- $\mathbf{1}^*\lambda(\omega,t,z,z') \equiv 0$ and
- $\lambda(\omega,t,z+\alpha\mathbf{1},z') = \lambda(\omega,t,z,z')$ for all $\alpha \in \mathbb{R}$.

Comparison theorem

Definition

Summary: f is γ balanced if, for each terminal condition, the BSDE solution can be written in terms of *some* expectation, under a measure with similar jump rates as under $\mathbb{P}$.

The purpose of this definition is to obtain:

Theorem (Comparison theorem)

Let f be γ-balanced for some $\gamma > 0$. Then if

$$\xi \geq \xi' \text{ and } f(\omega, t, y, z) \geq f'(\omega, t, y, z),$$

the associated BSDE solutions satisfy $Y_t \geq Y'_t$ for all t, and $Y_t = Y'_t$ on $A \in \mathcal{F}_t$ iff $Y_s = Y'_s$ on A for all $s \geq t$.

Comparison theorem

Lemma

If $f(u;\cdots)$ is γ-balanced for each u, then $g(\cdots) := \operatorname{ess\,sup}_u\{f(u;\cdots)\}$ is γ-balanced (given it is always finite).

As mentioned before, if B is another rate matrix, we write E^B for the expectation under the corresponding measure.

Lemma

If $B \sim_\gamma A$ (to be defined) then $f(\omega, t, y, z) = z^*(B - A)X_{t-}$ is γ-balanced with $\lambda(\cdots) = BX_{t-}$, and $Y_t = E^B[\xi|\mathcal{F}_t]$.

Definition

For $\gamma > 0$, we say 'A γ-controls B' (and write $A \preceq_\gamma B$) if $B - \gamma A$ is also a rate matrix.

If $A \preceq_\gamma B$ and $B \preceq_\gamma A$ we write $A \sim_\gamma B$.

Example

$$A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 2 & 2 \\ 2 & 2 & -3 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 2 \\ 2 & 2 & -3 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Then for $\gamma \leq 1/2$, $A \sim_\gamma B$.

Markovian Solutions

We can also obtain a Feynman–Kac type result.

Theorem

Let $f : \mathcal{X} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$*, and consider the BSDE with*

$$Y_t = \phi(X_T) + \int_{]t,T]} f(X_{t-}, t, Y_{u-}, Z_u)du - \int_{]t,T]} Z_u^* dM_u$$

for some function $\phi : \mathcal{X} \to \mathbb{R}$*. Then the solution satisfies*

$$Y_t = u(t, X_t) = X_t^* \mathbf{u}_t, \qquad Z_t = \mathbf{u}_t$$

for $\mathbf{u}$ *solving*

$$d\mathbf{u}_t = -(\mathbf{f}(t, \mathbf{u}_t) + A^* \mathbf{u}_t)dt; \qquad e_i^* \mathbf{u}_T = \phi(e_i)$$

where $e_i^* \mathbf{f}(t, \mathbf{u}_t) := f(e_i, t, e_i^* \mathbf{u}_t, \mathbf{u}_t)$*.*

Application to Control

Consider the problem

$$\min_{\mathfrak{u}} E^{\mathfrak{u}}\left[e^{-rT}\phi(X_T) + \int_{]0,T]} e^{-rt}L(\mathfrak{u}_t; X_{t-}, t)dt\right]$$

where $\mathfrak{u}$ is a predictable process with values in U, and $E^{\mathfrak{u}}$ is the expectation under which X has compensator $\lambda_t(\mathfrak{u}_t) \in \mathbb{R}^N$.

- Suppose $\lambda_t(\cdot)$ satisfies $\frac{e_i^* \lambda_t(\cdot)}{e_i^* AX_{t-}} \in [\gamma, \gamma^{-1}]$ for some $\gamma > 0$.
- Define the Hamiltonian

$$f(\omega, t, y, z) = -rY_t + \inf_{u\in U}\{L(u; X_{t-}, t) + z^*(\lambda_t(u) - AX_{t-}(\omega))\}$$

- The dynamic value function Y_t satisfies the BSDE with driver f, terminal value $\phi(X_t)$.
- By the comparison theorem, we have existence and uniqueness of an optimal feedback control.

Ergodic BSDEs

- For the study of graphs, we are often interested in ’steady state’ behaviour of a random walk/Markov chain living on the graph.
- Understanding the steady state of controlled systems naturally leads to ‘Ergodic’ BSDEs.
- These equations are less intuitive than BSDEs, but give a useful framework to study these systems.

Uniform ergodicity

- If $X_0 \sim \mu$, for some probability measure μ on $\mathcal{X}$, we write $X_t \sim P_t\mu$.
- Writing μ as a vector, $P_t\mu \equiv e^{At}\mu$.
- We say X is uniformly ergodic if

$$\|P_t\mu - \pi\|_{\text{TV}} \leq Re^{-\rho t} \qquad \text{for all } \mu$$

  for some $R, \rho > 0$, some probability measure π on $\mathcal{X}$.
- $\|\cdot\|_{\text{TV}}$ is the total variation norm (and writing signed measures as vectors, $\|\cdot\|_{\text{TV}} = \|\cdot\|_{\ell_1}$).
- The measure π is the *ergodic* measure of the Markov chain and is unique.
- It also satisfies $P_t\pi = \pi$, so it is *stationary*.
- We call (R, ρ) the parameters of ergodicity.
- Irreducible finite state chains are always uniformly ergodic.

Stability of Uniform Ergodicity

- We can tie together the matrix A and the measure $\mathbb{P}$.
- Hence a change of measure corresponds to using a different (stochastic, time varying?) matrix B.
- We say A is uniformly ergodic if X is uniformly ergodic under $\mathbb{P}^A$

Question

Suppose the chain is uniformly ergodic under the measure $\mathbb{P}^A$, and has associated parameters (R_A, ρ_A).

If A and B are 'similar', can we say the chain is uniformly ergodic under $\mathbb{P}^B$, and can we say anything about (R_B, ρ_B)?

- We wish to find a relationship between A and B under which we can get a uniform estimate of ρ_B.

Our key result is:

Theorem

Suppose A is uniformly ergodic and $\gamma \in]0,1[$. Then there exist constants $\bar{R}, \bar{\rho}$ dependent on γ and A such that B is uniformly ergodic with parameters $(\bar{R}, \bar{\rho})$ whenever $A \preceq_\gamma B$.

The key is that these constants are uniform in B, so these measures are 'uniformly uniformly-ergodic'.

Using these estimates, we can prove the following...

EBSDE Existence/Uniqueness

Theorem

Suppose X is uniformly ergodic. Then the ergodic BSDE

$$dY_t = -(f(X_t, Z_t) - \lambda)dt + Z_t^* dM_t$$

admits a bounded solution (Y, Z, λ) with $Y_t = u(X_t)$ and $u(\hat{x}) = 0$, whenever f is γ-balanced with $|f(x, 0)| < C$.

Any other bounded solution has the same λ, any other bounded Markovian solution has $Y'_t = Y_t + c$.

Our Feynman–Kac type result gives, with $Y = u(X_t) = X_t^* \mathbf{u}$, $Z = \mathbf{u}$,

$$\mathbf{f}(\mathbf{u}) - \lambda \mathbf{1} = -A^* \mathbf{u}$$

which is readily calculable for small dimensions.

Properties

The comparison theorem does not hold for Y for EBSDEs. However, it does hold for λ.

Theorem

Let f, f' be γ-balanced and $f \geq f'$. Then we have $\lambda \geq \lambda'$ in the EBSDE solutions.

Theorem

Let π^A denote the ergodic measure when X has matrix A. Then for some $B^{\mathbf{u}} \sim_\gamma A$,

$$\lambda = \int_{\mathcal{X}} f(x, \mathbf{u}) d\pi^A(x) = \int_{\mathcal{X}} f(x, 0) d\pi^{B^{\mathbf{u}}}(x)$$

$$Y_t + c = \int_{\mathcal{X}} f(x, \mathbf{u}) \mu^A_{X_0}(x) = \int_{\mathcal{X}} f(x, 0) \mu^{B^{\mathbf{u}}}_{X_0}(x)$$

where $\mu^A_x = \int_{\mathbb{R}+} (P^A_u \delta_x - \pi^A) du$.

Applications

- Application to Ergodic control similar to the finite-horizon case, with value

$$\lambda = \min_u \left\{ \limsup_{T\to\infty} E^u \left[\frac{1}{T} \int_{]0,T]} L(u_t; X_{t-}) dt \right] \right\}$$

- Write f as the Hamiltonian, comparison theorem gives optimal feedback control, etc...

- Instead, we will look at creating spatially stable nonlinear probabilities on graphs.
- These attempt to generalize the ergodic distribution of a Markov chain to a nonlinear setting.

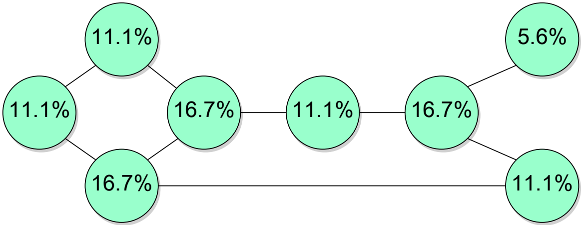
Applications

- Consider the driver $f(x, z) = I_{\{x \in \Xi\}} + g(x, z)$. Then if $g \equiv 0$, the EBSDE solutions will be $\lambda = \pi(\Xi)$.
- If g is convex, we will have a 'convex ergodic probability' or 'ergodic capacity'.

- A common problem when we study graphs is to find the most 'central' nodes.
- One method (connected to 'random walk centrality') is to look at the ergodic distribution of a Markov chain on the graph.
- What happens if we instead dynamically take the minimum over a range of rate matrices?

Nonlinear Graph Centrality

Consider the undirected graph below with a Markov chain moving at a constant exit rate. The ergodic probabilities under this basic model can be simply calculated.



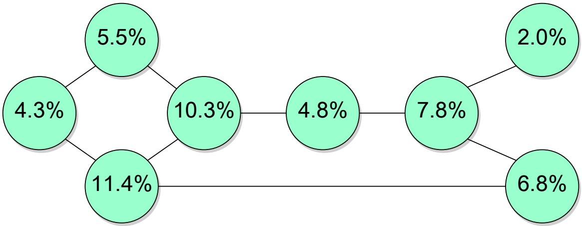
We have three states which are equally highest ranked, even though they have different positions in the graph.

Nonlinear Graph Centrality

Now suppose a bias can be introduced dynamically, so that the relative rate of jumping to the right/left can be doubled. Then the EBSDE can be solved with drivers

$$f(x, z) = I_{\{x\in\Xi\}} + \min_B\{z^*(B - A)x\}$$

for each $\Xi \subseteq \mathcal{X}$. The corresponding values λ^{Ξ} (for Ξ singletons) are

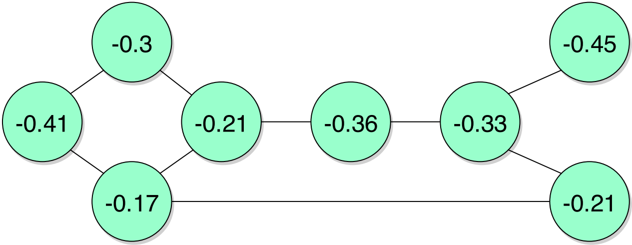


Nonlinear Graph Centrality

- To assess centrality (as opposed to ergodic importance), it is more interesting to look at the change in ergodic probability when the system is controlled.
- The 'controllability centrality'

$$CC = \log\left(\frac{\text{controlled probability}}{\text{uncontrolled probability}}\right)$$

  gives interesting results.
- Nodes near the edge change more (so have $CC \downarrow -\infty$)



Bank networks

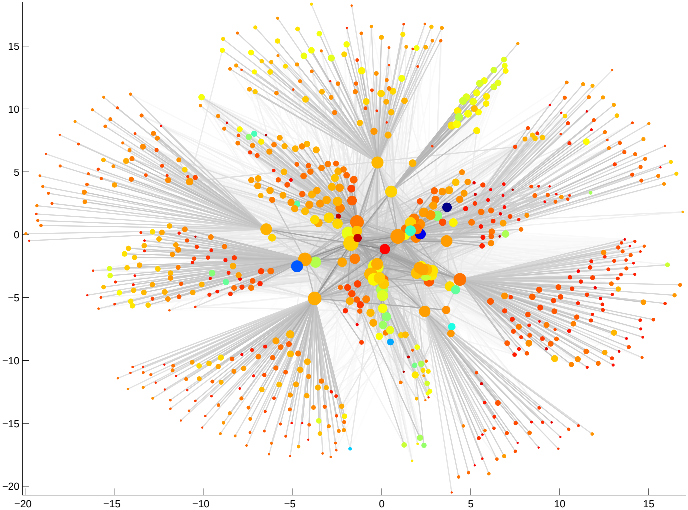
- Consider an interbank liability network.
- Taking liabilities as links, we obtain a weighted directed graph.
- We can consider a continuous time Markov chain with these values as rates of change.
- Data from Austrian banking system, Dec 2008. (Thanks to Martin Summer of the Österreichische Nationalbank)

In the following:

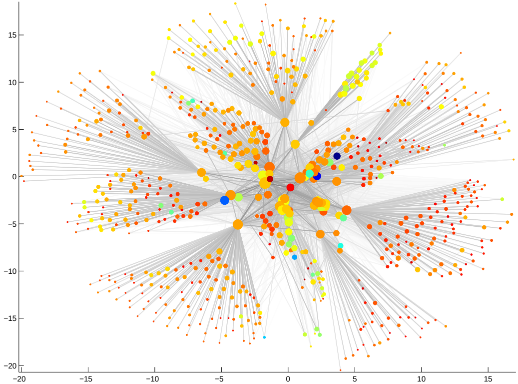
- Size related to total liability
- Colour based on $\log(\text{Ergodic probability})$, Blue= low $\to$ Red=high.

Thanks to Lucas Jeub for visualization help.

Ergodic distribution



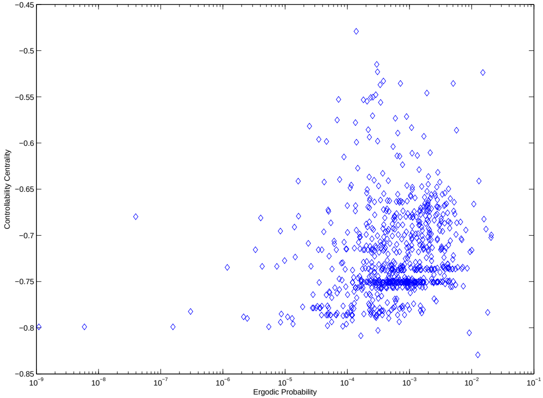
Controlled ergodic distribution



Determined using a 10% change in any/all connection strength, corresponding to BSDE driver

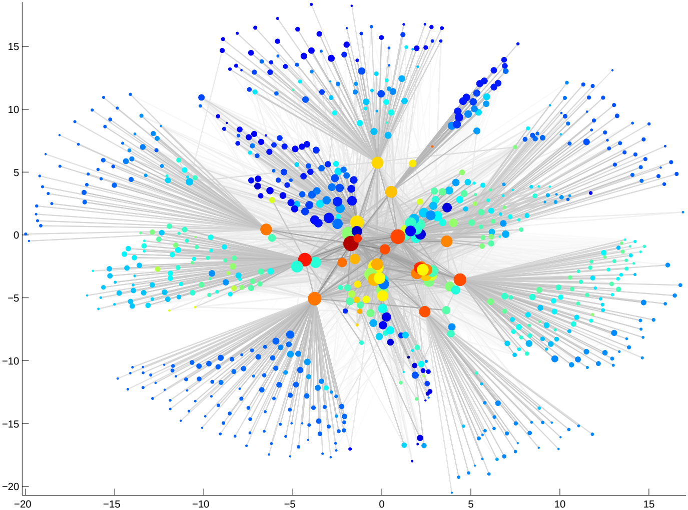
$$f(x,z) = 0.1 \sum_{e_j \neq x} A_{ij} |(e_j - x)^* z|.$$

Controllability Centrality



CC is somewhat related to ergodic probability.

Controllability Centrality



Bank networks

- This gives a natural notion of 'robustness' of the network.
- If a node has a high CC, then only a small change in the network is needed to noticeably change its significance.
- Conversely, low CC means large structural changes would be needed to alter the node's importance.

- Reweighting the graph to have a constant exit rate significantly changes the ergodic distribution.
- The CC is qualitatively similar (Clear linear relationship between the values of CC calculated in each setting, $r^2 = 0.87$).

Conclusion

- BSDEs represent control problems nicely.
- Ergodic BSDEs with Markov chains can be applied to control of graphs.
- This yields a ‘controllability centrality’ measure.
- Future work: Connect these approaches to specific models of interbank lending, compare with other notions of systemic risk.