Indifference fee rate [1]

for variable annuities

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Outline

1. Variable Annuities
2. Model
3. Indifference fees
4. Numerical Results

Outline

What is a Variable Annuity?

Variable annuity is a contract between a policyholder and an insurance company.

- **The policyholder gives an initial amount** of money to the insurer.
- **It is invested in a reference portfolio** until a preset date, until the policyholder withdraws from the contract or dies.
- At the end of the contract, **the insurance pays an amount of money** depending on the performance of the reference portfolio.

Risks

- Actuarial risks:
  - mortality,
  - longevity,
  - ..
- Financial risks:
  - volatility,
  - interest rate,
  - ..

Literature

- Bauer (2008) presents a general framework to define Variable Annuities (VA).
- Boyle and Schwartz (1977), extend the Black-Scholes framework to insurance issues.
- Milvesky and Posner (2001) apply risk neutral option pricing theory to value Guaranteed Minimum Death Benefits (GMDB) in VAs.
- Dai *et al.* (2008) HJB equation is derived for a singular control problem related to VA.
- Belanger *et al.* (2009) describes the GMDB pricing problem as an impulse control problem.

Outline

Model (main points)

- **No restrictive assumptions** on the reference portfolio and the interest rate dynamics (Markovianity of processes is not assumed):
  - Incomplete market, not a unique risk-neutral measure.
  - We introduce a **methodology with BSDEs** with a jump.
- **Indifference pricing** with continuous fees.

Financial Market and Wealth Process

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space, with $\mathbb{F}$ the Brownian filtration.

Financial market:

$$\begin{aligned} dS_t^0 &= r_t S_t^0 dt \ , \quad \forall t \in [0, T] \ , \quad S_0^0 = 1 \ , \\ dS_t &= S_t(\mu_t dt + \sigma_t dB_t) \ , \quad \forall t \in [0, T] \ , \quad S_0 = s > 0 \end{aligned}$$

where μ, σ and r are $\mathbb{F}$-adapted bounded processes and σ is lower bounded by a positive constant.

Discounted wealth process:

$$X_t^{x,\pi} = x + \int_0^t \pi_s(\mu_s - r_s)ds + \int_0^t \pi_s \sigma_s dB_s \ ,$$

with strategy π and initial capital x.

Exit time of a Variable Annuity Policy

Let τ be the exit time which is the minimum time between:

- The time of death of the insured.
- The time of total withdrawal.

The random time τ is not assumed to be an $\mathbb{F}$-stopping time.

We consider $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ with

$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\mathbb{1}_{\tau\leq u}\,, u \in [0,t])$ for all $t \geq 0$.

Hypothesis

- Immersion of $\mathbb{F}$ in $\mathbb{G}$: every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.
- The process $N_. := \mathbb{1}_{\tau\leq .}$ admits an $\mathbb{F}$-compensator $\int_0^{.\wedge\tau} \lambda_t dt$.

Dynamics

Discounted Account Value A^p:

$$dA_t^p = A_t^p\Big[(\mu_t - r_t - \xi_t - p)dt + \sigma_t dB_t\Big] , \quad \forall t \in [0, T] ,$$

with initial value A_0, fee-rate p and withdrawal $(\xi_t)_{0\leq t\leq T}$.

Pay-off

The discounted pay-off including the withdrawals at time $T \wedge \tau$ to the insured is:

$$F(p) := F^L(T, A^p)\mathbb{1}_{\{T<\tau\}} + F^D(\tau, A^p)\mathbb{1}_{\{\tau \leq T\}} + \int_0^{T\wedge\tau} \xi_s A_s^p \, ds \; .$$

Notice that $F(p)$ is $\mathcal{G}_{T\wedge\tau}$-measurable.

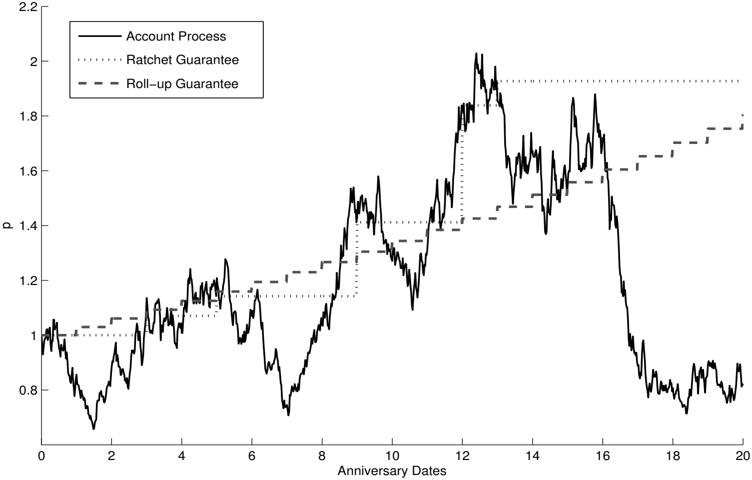
Guarantees without withdrawals

The common guarantees are:

- **Constant guarantee**: $G_t^Q(p) = A_0$.
- **Roll-up guarantee**: Let be $\eta > 0$, then $G_t^Q(p) = A_0(1+\eta)^t$.
- **Ratchet guarantee**: $G_t^Q(p) = \max\left(A_{t_1}^p, A_{t_2}^p, \ldots, A_t^p\right)$.

For t an anniversary date.

Usual Guarantees



Outline

Finding the Indifference Fees

The objective is to find a fee p^* such that

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\left[U\left(X_T^{x,\pi}\right)\right] = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\left[U\left(A_0 + X_T^{x,\pi} - F(p^*)\right)\right] ,$$

where $\mathcal{A}^{\mathbb{F}}[0,T]$ (resp. $\mathcal{A}^{\mathbb{G}}[0,T]$) is the set of admissible strategies between the interval of time $[0,T]$ in $\mathbb{F}$ (resp. in $\mathbb{G}$).

Utility function

$$U(y) = -e^{-\gamma y} , \quad \forall y \in \mathbb{R} ,$$

where $\gamma > 0$.

The classical problem:

$$V_0 = \sup_{\pi \in \mathcal{A}^{\mathbb{F}}[0,T]} \left\{ \mathbb{E}\left[U\left(X_T^{\pi} \right) \right] \right\}$$

Hu, Imkeller and Muller (2004), Rouge and El Karoui (2000)

Theorem

The optimal value is $\mathbf{V_0 = -\exp(\gamma y_0)}$, using the optimal strategy

$$\pi_t^* := \frac{\mu_t - r_t}{\gamma \sigma_t^2} + \frac{z_t}{\sigma_t} \ ,$$

where y_0 and z are given by the BSDE

$$-dy_t = \left(-\frac{\nu_t^2}{2\gamma} - z_t \nu_t \right) dt - z_t dB_t \ , \quad y_T = 0 \ .$$

Utility Maximization with VA (Step 1)

$$V_{\mathbb{G}}(p) := \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\Big[U\Big(X_T^{A_0,\pi} - F(p)\Big)\Big]$$

Proposition

The value function is

$$V_{\mathbb{G}}(p) = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X_{T\wedge\tau}^{A_0,\pi} - \widehat{F}(p)\Big)\Big)\Big],$$

where $\widehat{F}(p) :=$

$$F(p) + \frac{1}{\gamma}\log\left\{\operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\left[e^{-\gamma\left(X_T^{A_0,\pi} - X_{T\wedge\tau}^{A_0,\pi}\right)}\middle|\mathcal{G}_{T\wedge\tau}\right]\right\}.$$

Utility Maximization with VA (Step 2)

Finding $\widehat{F}(p)$

Proposition

There exists a process $Y^{(\tau)}$ such that

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\Big[\exp\big(-(\gamma X_T^{A_0,\pi} - X_{T\wedge\tau}^{A_0,\pi})\big)|\mathcal{G}_{T\wedge\tau}\Big] = \exp(\gamma Y_{T\wedge\tau}^{(\tau)}) ,$$

where $(Y^{(\tau)}, Z^{(\tau)})$ is solution of the BSDE

$$\begin{cases} dY_t^{(\tau)} &= \left[\frac{\nu_t^2}{\gamma} + \nu_t Z_t^{(\tau)}\right]dt + Z_t^{(\tau)} dB_t , \\ Y_T^{(\tau)} &= 0 . \end{cases}$$

Utility Maximization with VA (Step 3)

$$V_{\mathbb{G}}(p) := \sup_{\pi \in \mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\left[U\left(X_T^{A_0,\pi} - F(p)\right)\right]$$

Theorem

The value function is given by

$$V_{\mathbb{G}}(p) \quad = \quad -\exp\Big(-\gamma\big(A_0 - Y_0(p)\big)\Big),$$

where $(Y(p), Z(p), U(p))$ is a solution of

$$\begin{aligned} Y_t(p) \quad = \quad & \widehat{F}(p) + \int_{t\wedge\tau}^{T\wedge\tau} \Big(\lambda_s \frac{e^{\gamma U_s(p)} - 1}{\gamma} - \frac{\nu_s^2}{2\gamma} - \nu_s Z_s(p)\Big) ds \\ & - \int_{t\wedge\tau}^{T\wedge\tau} Z_s(p) dB_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s(p) dH_s \,, \quad t \in [0, T] \,. \end{aligned}$$

The Optimal Strategy

The Strategy

$$\pi_t^* := \begin{cases} \frac{\nu_t}{\gamma\sigma_t} + \frac{Z_t(p)}{\sigma_t}, & t \in [0, T \wedge \tau), \\ \frac{\nu_t}{\gamma\sigma_t} + \frac{Z_t^{(\tau)}}{\sigma_t}, & t \in [T \wedge \tau, T]. \end{cases}$$

Methodology

Recapitulation

Indifference Fees

$$\sup_{\pi\in\mathcal{A}^{\mathcal{F}}[0,T]}\{\mathbb{E}[U(X_T^{\pi})]\} = \sup_{\pi\in\mathcal{A}^{\mathcal{G}}[0,T]}\left\{\mathbb{E}\left[U\left(X_T^{\pi,A_0}-F(p)\right)\right]\right\}.$$

- $\surd$ Utility Maximization:
  - $\surd$ Classical Utility Maximization Problem. $V_0 = -\exp(\gamma y_0)$.
  - $\surd$ Not the Classical Problem.

    $V_{\mathbb{G}}(p) = -\exp\Big(-\gamma\big(A_0 - Y_0(p)\big)\Big).$
- Existence of the Indifference Fees. $Y_0(p^*) - A_0 = y_0$.
- Simulations.

Existence of the Indifference Fees

Consider $\psi(p) := Y_0(p) - y_0 - A_0 \,, \quad \forall p \in \mathbb{R}\,.$

Proposition

The function ψ is continuous and non-increasing on $\mathbb{R}$.

(i) For any $p \in \mathbb{R}$, we have $\psi(p) > 0$ i.e., for any fee p, we have $V_{\mathbb{G}}(p) < V_{\mathbb{F}}$.

(ii) For any $p \in \mathbb{R}$, we have $\psi(p) < 0$ i.e., for any fee p, we have $V_{\mathbb{G}}(p) > V_{\mathbb{F}}$.

(iii) There exist p_1 and p_2 such that $\psi(p_1)\psi(p_2) < 0$. Then, there exists an indifference fee p^*.

Outline

Numerical results.

We assume that r and μ are Markov chains taking values in the states spaces $S^r = \{0,\ 0.01, \dots, 0.25\}$ and $S^\mu = \{0,\ 0.01,\ 0.02, \dots, 0.3\}$.
We give the following numerical values to parameters:

$$\gamma = 1.3, \quad \lambda = 0.05, \quad \xi = 0, \quad A_0 = 1,$$

and, for the financial market parameters:

$$r_0 = 0.02, \quad \mu_0 = 0.15.$$

Market Risk

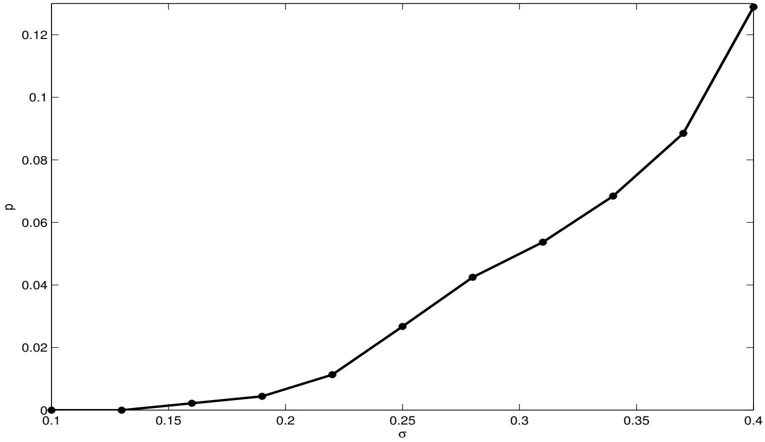


Figure : Ratchet option ($T = 20$).

Actuarial Risk

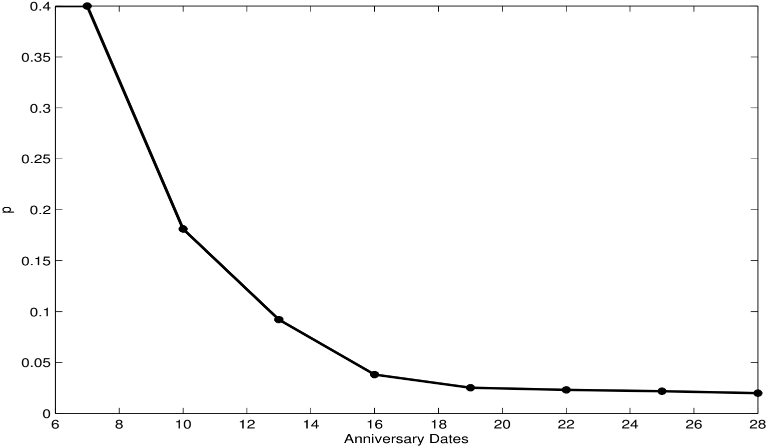


Figure : Ratchet option ($\sigma = 0.3$).

Roll up Guarantee Risk

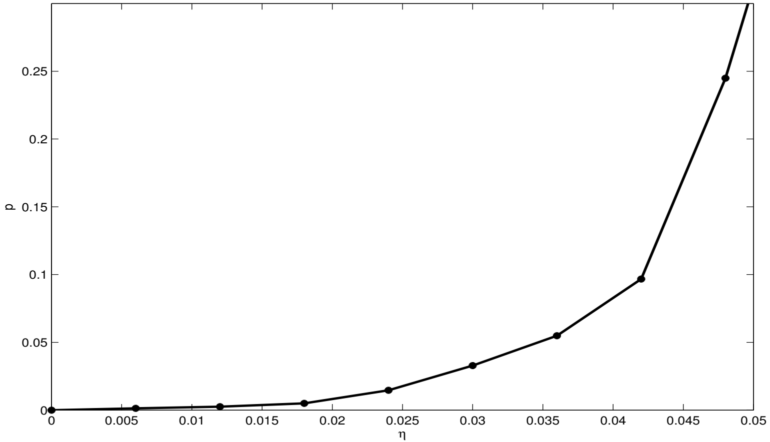


Figure : Roll up option ($T = 20$, $\sigma = 0.3$).

Thank you!

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Questions?

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Their respective transitional matrix are:

$$q^r_{i,j} = \begin{cases} \frac{1}{2} & \text{if} \quad i = j, \\ \frac{1}{2} & \text{if} \quad i = 1 \text{ and } j = 2, \\ \frac{1}{2} & \text{if} \quad i = 27 \text{ and } j = 26, \\ \frac{1}{4} & \text{if} \quad i = j + 1 \text{ and } i \leq 26, \\ \frac{1}{4} & \text{if} \quad i = j - 1 \text{ and } i \geq 2, \\ 0 & \text{else}, \end{cases} \qquad \text{and}$$

$$q^\mu_{i,j} = \begin{cases} \frac{1}{2} & \text{if} \quad i = j, \\ \frac{1}{2} & \text{if} \quad i = 1 \text{ and } j = 2, \\ \frac{1}{2} & \text{if} \quad i = 32 \text{ and } j = 31, \\ \frac{1}{4} & \text{if} \quad i = j + 1 \text{ and } i \leq 31, \\ \frac{1}{4} & \text{if} \quad i = j - 1 \text{ and } i \geq 2, \\ 0 & \text{else}, \end{cases}$$