Bang-Bang Type Nash Equilibrium Point for Nonzero-sum Stochastic Differential Game

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- Nonzero-sum stochastic differential game
  - A system which is described by a stochastic differential equation.
  - Played by two or multiple players. Each player imposes a control on the system.
  - Payoffs.
  - The problem is to find the equilibrium point.
- What's new?
  - The control is of bang-bang type.
  - The corresponding BSDE is of multiple-dimensional and coupled by the volatility process of each dimension. Besides, the generator is discontinuous on the volatility term.

A 2-player and 1-dimensional case:

- Dynamic: $\forall s \leq T, \ X_s^{t,x} = x + (B_{s\vee t} - B_t)$.
  - $T > 0$ is fixed;
  - For fixed $(t, x) \in [0, T] \times \mathbf{R}$;
  - B is a 1-d Brownian motion.
- The set of admissible controls: $\mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2$.
  $\mathcal{M}_1 = \{u = (u_t)_{t\leq T} | u : [0, T] \times \Omega \to U \text{ is adapted}\}$;
  $\mathcal{M}_2 = \{v = (v_t)_{t\leq T} | v : [0, T] \times \Omega \to V \text{ is adapted}\}$.
- $U = [0, 1], \ V = [-1, 1]$.

Weak formulation:

- Probability transform: $d\mathbf{P}^{u,v}_{t,x} = \zeta_T(\Gamma(.,X^{t,x}_{.},u.,v.))d\mathbf{P}$ with $\zeta_t(\Theta) := 1 + \int_0^t \Theta_s \zeta_s dB_s,\ \ t \le T.$
  - $\Gamma : [0,T] \times \mathbf{R} \times U \times V \to \mathbf{R}$, $\Gamma(t,x,u,v) = f(t,x) + u + v$.
  - $f : (t,x) \in [0,T] \times \mathbf{R} \to \mathbf{R}$, $|f(t,x)| \le C(1+|x|)$.

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  - $\Gamma : [0,T] \times \mathbf{R} \times U \times V \to \mathbf{R}$, $\Gamma(t,x,u,v) = f(t,x) + u + v$.
  - $f : (t,x) \in [0,T] \times \mathbf{R} \to \mathbf{R}$, $|f(t,x)| \le C(1+|x|)$.
- The Process $P^{u,v}_{t,x}$ is a probability since:

Lemma (Haussmann)

For any admissible control $(u,v) \in \mathcal{M}$ and $(t,x) \in [0,T] \times \mathbf{R}$, there exists a constant $p_0 \in (1,2)$ such that $\mathbf{E}[|\zeta_T|^{p_0}] \le C$.

- $B^{u,v} = (B_s - \int_0^s \Gamma(r, X^{t,x}_r, u_r, v_r)dr)_{s \le T}$ is a $(\mathcal{F}_s, \mathbf{P}^{u,v}_{t,x})$-B.M.;

By Girsanov's transformation:

$$\begin{cases} dX_s^{t,x} = \Gamma(s, X_s^{t,x}, u_s, v_s)ds + dB_s^{u,v}, \ \forall s \in [t, T]; \\ X_s^{t,x} = x, \ s \in [0, t] \end{cases}$$

with $\Gamma(t, x, u, v) = f(t, x) + u + v$ which is of linear growth on x.

- Terminal function: $g_i : x \in \mathbf{R} \to \mathbf{R}$ for player $i = 1, 2$.

  $|g_1(x)| + |g_2(x)| \leq C(1 + |x|^\gamma)$, $\forall x \in \mathbf{R}$ and fixed constant $\gamma > 1$.

- Payoffs for the players: For fixed $(0, x)$, and any $(u, v) \in \mathcal{M}$,

$$J_1(u, v) := \mathbf{E}^{u,v}[g_1(X_T^{0,x})];$$
$$J_2(u, v) := \mathbf{E}^{u,v}[g_2(X_T^{0,x})].$$

Objective: to find a couple of controls $(u^*, v^*) \in \mathcal{M}$ s.t. for all $(u, v) \in \mathcal{M}$,

$$J_1(u^*, v^*) \geq J_1(u, v^*) \text{ and } J_2(u^*, v^*) \geq J_2(u^*, v).$$

Such (u^*, v^*) is called the Nash equilibrium point for this game.

Hamiltonian functions: $H_i : [0, T] \times \mathbf{R} \times \mathbf{R} \times U \times V \to \mathbf{R}$

$$\begin{aligned} H_1(t, x, p, u, v) &:= p\Gamma(t, x, u, v) = p(f(t, x) + u + v); \\ H_2(t, x, q, u, v) &:= q\Gamma(t, x, u, v) = q(f(t, x) + u + v). \end{aligned}$$

Isaacs' condition: $\forall (t, x, p, q, u, v) \in [0, T] \times \mathbf{R} \times \mathbf{R} \times U \times V$, there exists a pair of applications (u^*, v^*) such that,

$$\begin{aligned} H_1(t, x, p, u^*(t, x, p, q), v^*(t, x, p, q)) &\geq H_1(t, x, p, u, v^*(t, x, p, q)), \\ H_2(t, x, q, u^*(t, x, p, q), v^*(t, x, p, q)) &\geq H_2(t, x, q, u^*(t, x, p, q), v). \end{aligned}$$

Candidate bang-bang type optimal controls: $\bar{u}$ (resp.$\bar{v}$): $\mathbf{R} \times U \to U$ (resp.$\mathbf{R} \times V \to V$), $\forall p, q \in \mathbf{R}$, $\epsilon \in U$, $\kappa \in V$,

$$\bar{u}(p,\epsilon) = \begin{cases} 1, & p > 0, \\ \epsilon, & p = 0, \\ 0, & p < 0, \end{cases} \quad \text{and} \quad \bar{v}(q,\kappa) = \begin{cases} 1, & q > 0, \\ \kappa, & q = 0, \\ -1, & q < 0. \end{cases}$$

The pair $(\bar{u}, \bar{v})$ satisfies the Isaacs' condition:
$\forall (t,x,p,q,u,v) \in [0,T] \times \mathbf{R} \times \mathbf{R} \times U \times V$ and $(\epsilon,\kappa) \in U \times V$,

$$\begin{aligned} H_1^*(t,x,p,q,\kappa) &:= H_1(t,x,p,\bar{u}(p,\epsilon),\bar{v}(q,\kappa)) \geq H_1(t,x,p,u,\bar{v}(q,\kappa)), \\ H_2^*(t,x,p,q,\epsilon) &:= H_2(t,x,q,\bar{u}(p,\epsilon),\bar{v}(q,\kappa)) \geq H_2(t,x,q,\bar{u}(p,\epsilon),v). \end{aligned}$$

$$\bar{u}(p,\epsilon) = \begin{cases} 1, & p > 0, \\ \epsilon, & p = 0, \\ 0, & p < 0. \end{cases}$$

$$\bar{v}(q,\kappa) = \begin{cases} 1, & q > 0, \\ \kappa, & q = 0, \\ -1, & q < 0. \end{cases}$$

$$H_1^*(t,x,p,q,\kappa) = p(f(t,x) + \bar{u}(p,\epsilon) + \bar{v}(q,\kappa)).$$

Remarks

1. H_1^* (resp. H_2^*) does not depend on ϵ (resp. κ);
2. H_i^* is discontinuous w.r.t. (p,q).

S. Hamadène, J.-P. Lepeltier and S. Peng, *BSDEs with continuous coefficients and stochastic differential games*, Pitman Research Notes in Mathematics Series, (1997), pp. 115-128.

P. Mannucci, *Nonzero-sum stochastic differential games with discontinuous feedback*, SIAM journal on control and optimization, 43.4(2004), pp. 1222-1233.

Proposition (Link between SDG and BSDE)

For all $(u, v) \in \mathcal{M}$ *and player* $i = 1, 2$, *there exists a couple of* $\mathcal{P}$*-measurable processes* $(Y^{i;u,v}, Z^{i;u,v})$, *with values on* $\mathbf{R} \times \mathbf{R}$, *such that:*

(i) *For all constant* $q > 1$,

$$\mathbf{E}^{u,v}\left[\sup_{s\in[0,T]} |Y_s^{i;u,v}|^q + \left(\int_0^T |Z_s^{i;u,v}|^2 ds\right)^{\frac{q}{2}}\right] < \infty.$$

(ii) $\forall s \leq T$,

$$-dY_s^{i;u,v} = H_i(s, X_s^{0,x}, Z_s^{i;u,v}, u_s, v_s)ds - Z_s^{i;u,v} dB_s, \;\; Y_T^{i;u,v} = g_i(X_T^{0,x}).$$

(iii) *The solution is unique, besides,* $Y_0^{i;u,v} = J_i(u, v)$.

Proposition (Existence of Nash equilibrium point)

Let us suppose there exist η^1, η^2, (Y^1, Z^1), (Y^2, Z^2) *and* θ, ϑ *such that:*

(i) η^1 *and* η^2 *are two deterministic measurable functions with polynomial growth from* $[0, T] \times \mathbf{R}$ *to* $\mathbf{R}$;

(ii) (Y^1, Z^1) *and* (Y^2, Z^2) *are two couples of* $\mathcal{P}$*-measurable processes with values on* $\mathbf{R}^{1+1}$;

(iii) θ*(resp.* ϑ*) is a* $\mathcal{P}$*-measurable process valued on* U*(resp.* V*),*

and satisfy:

(a) $\mathbf{P}$*-a.s.,* $\forall s \leq T$, $Y^i_s = \eta^i(s, X^{0,x}_s)$ *and* $Z^i(\omega) := (Z^i_s(\omega))_{s \leq T}$ *is ds-square integrable;*

(b) *For all* $s \leq T$,

$$\begin{cases} -dY^1_s = H^*_1(s, X^{0,x}_s, Z^1_s, Z^2_s, \vartheta_s)ds - Z^1_s dB_s, \; Y^1_T = g_1(X^{0,x}_T); \\ -dY^2_s = H^*_2(s, X^{0,x}_s, Z^1_s, Z^2_s, \theta_s)ds - Z^2_s dB_s, \; Y^2_T = g_2(X^{0,x}_T). \end{cases}$$

Then, the pair of controls $(\bar{u}(Z^1_s, \theta_s), \bar{v}(Z^2_s, \vartheta_s))_{s \leq T}$ *is a bang-bang type Nash equilibrium point of the nonzero-sum stochastic differential game.*

Remarks:

- H_1^*(resp. H_2^*) is discontinuous w.r.t. Z^2 (resp. Z^1) and is of linear growth w.r.t Z^1(resp. Z^2) ω by ω.
- The BSDE system is multiple dimensional. Once this system has solution, it will provide us the NEP for the game problem.
- Hint to the proof: let u be an arbitrary element of $\mathcal{M}_1$. The mind is to show that $Y^1 \geq Y^{1;u,\bar{v}}$, which yields $Y_0^1 = J_1(\bar{u}, \bar{v}) \geq Y_0^{1;u,\bar{v}} = J_1(u, \bar{v})$.

Theorem

There exist η^1, η^2, (Y^1, Z^1), (Y^2, Z^2) *and* θ, ϑ *which satisfy (i)-(iii) and (a),(b) of the former Proposition. Finally, the nonzero-sum stochastic differential game has a bang-bang type Nash equilibrium point.*

We aim at providing the solution for the following BSDE: For all $s \leq T$,

$$\begin{cases} -dY_s^1 = H_1^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \vartheta_s)ds - Z_s^1 dB_s, \ Y_T^1 = g_1(X_T^{0,x}); \\ -dY_s^2 = H_2^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \theta_s)ds - Z_s^2 dB_s, \ Y_T^2 = g_2(X_T^{0,x}). \end{cases}$$

$$H_1^*(s, x, p, q, \vartheta) = p(f(s, x) + \bar{u}(p, \theta) + \bar{v}(q, \vartheta));$$

$$H_2^*(s, x, p, q, \theta) = q(f(s, x) + \bar{u}(p, \theta) + \bar{v}(q, \vartheta)).$$

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1. Approximation;
2. Uniform estimates;
3. Convergence results;
4. Verify procedure.

Step 1: Approximation

The functions $p \in \mathbf{R} \mapsto p\bar{u}(p,\epsilon)$ and $q \in \mathbf{R} \mapsto q\bar{v}(q,\kappa)$ are Lipschitz.
Hereafter $\bar{u}(p,0)$ and $\bar{v}(q,0)$ will be simply denoted by $\bar{u}(p)$ and $\bar{v}(q)$.
Define:

$$\bar{u}^n(p) = \begin{cases} 0 \text{ if } & p \leq -1/n, \\ 1 \text{ if } & p \geq 0, \\ np+1 \text{ if } & p \in (-1/n, 0), \end{cases} \qquad \bar{v}^n(q) = \begin{cases} -1 \text{ if } & q \leq -1/n, \\ 1 \text{ if } & q \geq 1/n, \\ nq \text{ if } & q \in (-1/n, 1/n). \end{cases}$$

Truncation function: Φ_n: $x \in \mathbf{R} \mapsto \Phi_n(x) = (x \wedge n) \vee (-n) \in \mathbf{R}$.

$$\begin{cases} -dY_r^{1,n;t,x} = \{\Phi_n(Z_r^{1,n;t,x})\Phi_n(f(r,X_r^{t,x})) + \Phi_n(Z_r^{1,n;t,x}\bar{u}(Z_r^{1,n;t,x}))+ \\ \qquad\qquad \Phi_n(Z_r^{1,n;t,x})\bar{v}^n(Z_r^{2,n;t,x})\}dr - Z_r^{1,n;t,x}dB_r, \ \ Y_T^{1,n;t,x} = g_1(X_T^{t,x}); \\ -dY_r^{2,n;t,x} = \{\Phi_n(Z_r^{2,n;t,x})\Phi_n(f(r,X_r^{t,x})) + \Phi_n(Z_r^{2,n;t,x}\bar{v}(Z_r^{2,n;t,x}))+ \\ \qquad\qquad \Phi_n(Z_r^{2,n;t,x})\bar{u}^n(Z_r^{1,n;t,x})\}dr - Z_r^{2,n;t,x}dB_r, \ \ Y_T^{2,n;t,x} = g_2(X_T^{t,x}). \end{cases}$$

[Pardoux and Peng 1990]

The solution $(Y^{i,n;t,x}, Z^{i,n;t,x}) \in \mathcal{S}_T^2 \times \mathcal{H}_T^2$ exists.

$\mathcal{S}_T^p = \{Y = (Y_s)_{s\in[0,T]} :\ \mathcal{P}$-measurable stochastic process s.t. $\mathbf{E}[\sup_{s\in[0,T]}|Y_s|^p] < \infty\}$;
$\mathcal{H}_T^p = \{Z = (Z_s)_{s\in[0,T]} :\ \mathcal{P}$-measurable stochastic process s.t. $\mathbf{E}[(\int_0^T |Z_s|^2 ds)^{p/2}] < \infty\}$.

[El-Karoui et al. 1997] There exist measurable deterministic functions $\eta^{i,n}$ and $\varsigma^{i,n}$ of $(s,x) \in [t,T] \times \mathbf{R}$, $i = 1,2$ and $n \geq 1$, such that:

$$Y_s^{i,n;t,x} = \eta^{i,n}(s, X_s^{t,x}) \quad \text{and} \quad Z_s^{i,n;t,x} = \varsigma^{i,n}(s, X_s^{t,x}).$$

For $n \geq 1$ and $i = 1,2$, the functions $\eta^{i,n}$ verify: $\forall (t,x) \in [0,T] \times \mathbf{R}$,

$$\eta^{i,n}(t,x) = \mathbf{E}[g_i(X_T^{t,x})] + \int_t^T H_i^n(r, X_r^{t,x})dr]$$

with, for any $(s,x) \in [0,T] \times \mathbf{R}$,

$$\begin{cases} H_1^n(s,x) = \Phi_n(\varsigma^{1,n}(s,x))\Phi_n(f(s,x)) + \Phi_n\big(\varsigma^{1,n}(s,x)\bar{u}(\varsigma^{1,n}(s,x))\big) + \\ \qquad + \Phi_n(\varsigma^{1,n}(s,x))\bar{v}^n(\varsigma^{2,n}(s,x)); \\ H_2^n(s,x) = \Phi_n(\varsigma^{2,n}(s,x))\Phi_n(f(s,x)) + \Phi_n\big(\varsigma^{2,n}(s,x)\bar{v}(\varsigma^{2,n}(s,x))\big) + \\ \qquad + \Phi_n(\varsigma^{2,n}(s,x))\bar{u}^n(\varsigma^{1,n}(s,x)). \end{cases}$$

Step 2: Uniform integrability

Conclusion: There exists a constant C independent of n s.t., for $(t,x) \in [0,T] \times \mathbf{R}$, $i = 1,2$,

$$\begin{cases} \text{(a) } |\eta^{i,n}(t,x)| \leq C(1+|x|^{\lambda}), \text{ for any } \lambda > 2; \\ \text{(b) For any } \alpha \geq 1, \mathbf{E}[\sup_{s\in[t,T]} |Y_s^{i,n;t,x}|^{\alpha}] \leq C; \\ \text{(c) } \mathbf{E}[\int_0^T |Z_s^{i,n;t,x}|^2 ds] \leq C. \end{cases}$$

Step 3: Convergence of sequence $(Y^{i,n;0,x}, Z^{i,n;0,x})_{n\geq 1}$

$H_i^n(s,y) \in \mathcal{L}^q([0,T]\times \mathbf{R}; \mu(0,x;s,dy)ds)$ for fixed $q\in(1,2)$, $i=1,2$. i.e.
$\mathbf{E}[\int_0^T |H_i^n(s,X_s^{0,x})|^q ds] < \infty$.
Note $\mu(0,x;s,dy)$ is the law of $X_s^{0,x}$, i.e. $\forall A\in\mathcal{B}(\mathbf{R}), \mu(0,x;s,A) = \mathbf{P}(X_s^{0,x}\in A)$ for $s\in[0,T]$. $\mu(0,x;s,dy) = \int_A \frac{1}{\sqrt{2\pi s}} e^{-\frac{(y-x)^2}{2s}} dy$.

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A subsequence $H_i^n \to H_i$ weakly in $\mathcal{L}^q([0,T] \times \mathbf{R}; \mu(0,x;s,dy)ds)$, for fixed $q \in (1,2)$, $i = 1,2$.

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$(\eta^{i,n}(t,x))_{n\geq 1}$ is a Cauchy sequence for each $(t,x) \in [0,T] \times \mathbf{R}$, $i = 1,2$. Therefore, $\lim_{n\to\infty} \eta^{i,n}(t,x) = \eta^i(t,x)$.

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Note $\mu(0,x;s,dy)$ is the law of $X_s^{0,x}$, i.e. $\forall A\in\mathcal{B}(\mathbf{R}), \mu(0,x;s,A)=\mathbf{P}(X_s^{0,x}\in A)$ for $s\in[0,T]$. $\mu(0,x;s,dy)=\int_A \frac{1}{\sqrt{2\pi s}}e^{-\frac{(y-x)^2}{2s}}dy$.

A subsequence $H_i^n \to H_i$ weakly in $\mathcal{L}^q([0,T]\times\mathbf{R};\mu(0,x;s,dy)ds)$, for fixed $q\in(1,2)$, $i=1,2$.

$(\eta^{i,n}(t,x))_{n\geq 1}$ is a Cauchy sequence for each $(t,x)\in[0,T]\times\mathbf{R}$, $i=1,2$. Therefore, $\lim_{n\to\infty}\eta^{i,n}(t,x)=\eta^i(t,x)$.

For any $t\in[0,T]$, $\lim_{n\to\infty} Y_t^{i,n;0,x}(\omega)=\eta^i(t,X_t^{0,x}(\omega)), \mathbf{P}-a.s.$

To summarize this step, we have the following results: for $i = 1, 2$,

$$\begin{cases} \text{(a) } H_i^n(s,y) \in \mathcal{L}^q([0,T]\times \mathbf{R};\ \mu(0,x;s,dy)ds) \text{ uniformly w.r.t. } n; \\ \text{(b) } Y^{i,n;0,x} \to_{n\to\infty} Y^i \text{ in } \mathcal{H}_T^\alpha \text{ for any } \alpha \geq 1, \text{ besides,} \\ \qquad Y^{i,n;0,x} \to_{n\to\infty} Y^i \text{ in } \mathcal{S}_T^2; \\ (c)\ Z^{i,n;0,x} \to_{n\to\infty} Z^i \text{ in } \mathcal{H}_T^2, \text{ additionally, there exists a} \\ \qquad \text{subsequence } \{n\} \text{ s.t } Z^{i,n;0,x} \to_{n\to\infty} Z^i\ dt\otimes d\mathbf{P} - a.e. \text{ and} \\ \qquad \sup_{n\geq 1} |Z^{i,n;0,x}| \in \mathcal{H}_T^2. \end{cases}$$

Step 4: Convergence of $(H_i^n)_{n\geq 1},\ i = 1, 2.$

Problem: Whether $H_1^n(s, X_s)$ converge to $H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta)$ or not?
Recall:

$$H_1^n(s, X_s) = \Phi_n(Z_s^{1,n})\Phi_n(f(s, X_s)) + \Phi_n(Z_s^{1,n}\bar{u}(Z_s^{1,n})) + \Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n}).$$

There exists $\mathcal{P}$-measurable process ϑ valued on V such that, for integer $k \geq 0$,

$$H_1^{n_k}(s, X_s) \rightarrow_{k\rightarrow\infty} H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta) \quad \text{weakly in} \quad \mathcal{H}_T^2.$$

$$\Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n}) = \Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n})1_{\{Z_s^2\neq 0\}} + \Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n})1_{\{Z_s^2=0\}}.$$

Define a $\mathcal{P}$-measurable process $(\vartheta_s)_{s\leq T}$ valued on V as the weak limit in $\mathcal{H}^2_T$ of some subsequence $(\bar{v}^{n_k}(Z^{2,n_k})1_{\{Z^2=0\}})_{k\geq 0}$.

The weak limit exists since $(\bar{v}^{n_k})_{k\geq 0}$ is bounded. Then, for $s \leq T$,

$$\Phi_n(Z_s^{1,n_k})\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}} \rightarrow_{k\rightarrow\infty} Z_s^1\vartheta_s 1_{\{Z_s^2=0\}} \text{ weakly in } \mathcal{H}^2_T.$$

For any stopping time τ,

$$\int_0^\tau H_1^{n_k}(s, X_s)ds \rightarrow_{k\rightarrow\infty} \int_0^\tau H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta_s)ds \text{ weakly in } \mathcal{L}^2(\Omega, d\mathbf{P}).$$

Finally,

$$\mathbf{P}\text{-a.s.},\ \forall t \le T,\ \ Y_t^1 = g_1(X_T) + \int_t^T H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta_s)ds - \int_t^T Z_s^1 dB_s.$$

Similarly, for player 2, there exists a $\mathcal{P}$-measurable process $(\theta_s)_{s\le T}$ valued on U, such that,

$$\mathbf{P}\text{-a.s.}\ ,\forall t \le T,\ \ Y_t^2 = g_2(X_T) + \int_t^T H_2^*(s, X_s, Z_s^1, Z_s^2, \theta_s)ds - \int_t^T Z_s^2 dB_s.$$

Generalizations

1. For the drift term Γ in SDE which reads,

$$\Gamma(t, x, u, v) = f(t, x) + u + v,$$

one can replace u (resp. v) by $h(u)$ (resp. $l(v)$) with continuous function:

$$h(u) : U \to U' (\text{ resp. } l(v) : V \to V').$$

2. Multiple dimensional case.

3. $\sigma(t,x) : [0,T] \times \mathbf{R}^m \to \mathbf{R}^{m\times m}$.
   $X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x}) dB_r, \ \forall s \in [t,T]$ and $X_s^{t,x} = x$ for $s \in [0,t]$.

Assumptions: The function $\sigma(t,x)$ is uniformly Lipschitz w.r.t. x; It is invertible and bounded and its inverse is bounded.

With these assumptions, the following results hold true:

- Hörmander's condition. $\Upsilon.I \leq \sigma(t,x).\sigma^\top(t,x) \leq \Upsilon^{-1}.I$;
- Measure domination property.
  $\mu(t,x;s,dy)ds = \phi_t^\delta(s,x)\mu(0,x;s,dy)ds$ on $[t+\delta, T]\times$ $\mathbf{R}^m$; $\forall k \geq 1$,
  $\phi_t^\delta(s,x) \in \mathcal{L}^q([t+\delta,T] \times [-k,k]^m; \mu(0,x;s,dy)ds)$;
- Haussmann's result.

$$\begin{cases} -dY_s^1 = H_1^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \vartheta_s)ds - Z_s^1 dB_s, \ \ Y_T^1 = g_1(X_T^{0,x}); \\ -dY_s^2 = H_2^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \theta_s)ds - Z_s^2 dB_s, \ \ Y_T^2 = g_2(X_T^{0,x}). \end{cases}$$

$$H_1^*(s, x, p, q, \vartheta) = p(f(s, x) + \bar{u}(p, \theta) + \bar{v}(q, \vartheta));$$

$$H_2^*(s, x, p, q, \theta) = q(f(s, x) + \bar{u}(p, \theta) + \bar{v}(q, \vartheta)).$$

Game over