Securitization and equilibrium pricing under relative performance concerns

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Outline

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- **Performance Concern/Social Interaction**: Agents optimize their own gains from trading but also pay attention to what other are doing.

▷ How to price the derivative such that demand matches some constant supply?

▷ How to design the derivative s.t. the market "completes"?

▷ How does the social interaction component affects prices and individual risk perceptions?

Horst et al ('10); Espinosa & Touzi('10,'14); Frei & dR ('11)

The underlyings

$(\Omega, \mathcal{F}, \mathbb{P})$; $t \in [0, T]$; with $W = (W^S, W^R)$.

Agents $\mathbb{A} = \{a, b, c, ...\}$ are exposed to tradable and non-tradable risk factors:

- Non-tradable risk: diffusion with additive noise:

$$dR_t = \mu^R(t, R_t)dt + \sigma^R(t, R_t)dW_t^R,$$

- The tradable asset is a GBM type process

$$\begin{aligned} dS_t &= S_t\mu^S(t, R_t, S_t)dt + S_t\sigma^S(t, R_t, S_t)dW_t^S \\ &= S_t\mu^S dt + \langle \sigma^S, dW_t \rangle, \qquad \sigma_t := (S_t\sigma_t^S, 0) \end{aligned}$$

- Zero interest rates

Endowments and pricing measures

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Assumption:

all functions are C_b^b, σ^S elliptic
H^D completes the market

Pricing schemes

- Set of martingale measures $\mathbb{P}^\theta$ equivalent to $\mathbb{P}$

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \mathcal{E}\Big(- \int_0^T \langle \theta, W \rangle \Big),$$

$dW^\theta = dW + \theta dt$ is a $\mathbb{P}^\theta$-Brownian motion

- $\theta = (\theta^S, \theta^R)$ is the *market price of risk*
  - $\theta^S = \mu^S / \sigma^S \in \mathcal{S}^\infty$ is exogenously given
  - θ^R is endogenously given by an equilibrium condition.

H^D's price process

- For a MPR $\theta = (\theta^S, \theta^R)$

$$B_t^\theta = \mathbb{E}^\theta[H^D|\mathcal{F}_t] = \mathbb{E}^\theta[H^I] + \int_0^t \langle \kappa_s^\theta, dW_s^\theta \rangle$$
$$= B_0^\theta + \int_0^t \langle \kappa_s^\theta, \theta_s \rangle ds + \int_0^t \langle \kappa_s^\theta, dW_s \rangle$$

▷ Where volatility of H^D is $\kappa^\theta = (\kappa^{\theta,S}, \kappa^{\theta,R})$;

Assumption (Market completion)

$\kappa^{\theta,R} \neq 0 \qquad \mathbb{P}$-a.s..

The wealth process + final payoff

- The gains or losses from trading according to $\pi^{a,\theta} := (\pi^{a,1}, \pi^{a,2})$ are (recall $\theta = (\theta^S, \theta^R)$, $\sigma = (S\sigma^S, 0)$)

$$\begin{aligned} V_t^{a,\theta}(\pi^a) &:= \int_0^t \pi_s^{a,1} dS_s + \int_0^t \pi_s^{a,2} dB_s^\theta \\ &= \int_0^t \langle \pi^{a,1}\sigma + \pi^{a,2}\kappa^\theta, \theta_s \rangle ds + \int_0^t \langle \pi^{a,1}\sigma + \pi^{a,2}\kappa^\theta, dW_s \rangle \end{aligned}$$

and agent's a payoff at terminal horizon T from trading according to $\pi^{a,\theta}$ is

$$H^a + V_T^{a,\theta}(\pi^{a,\theta})$$

Risk assessment and preferences

Risk Assessment of ξ^a via a risk measure $\rho^a(\xi^a)$

- translation invariance: $\rho(\xi + m) = \rho(\xi) - m$, $m \in \mathbb{R}$,
- monotonicity: $\xi_1 \leq \xi_2$ implies $\rho(\xi_1) \geq \rho(\xi_2)$,
- convexity: $\xi \mapsto \rho(\xi)$ is convex .

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A class of them is given by BSDE: $Y_0^a = Y_0^a(\xi^a)$

$$Y_t^a = \xi^a + \int_t^T g^a(s, Z_s^a)ds - \int_t^T Z_s^a dW_s$$

- Preferences of Agent *a* are encoded in g^a (C^1 + convex))
- Peng (2004), Gianin (2006), Cheridito et al. (2009, Delbaen et al. (2009), etc...

see references in Mastrogiacomo's + Tangpi's talks

Individual optimization

$a \in \mathbb{A}$ with $|\mathbb{A}| < \infty$
Agent $a \in \mathbb{A}$ minimizes her risk by minimizing Y^a via π^a:

$$Y_t^a = -\xi^a + \int_t^T g^a(s, Z_s^a) ds - \int_t^T Z_s^a dW_s$$

where

$$\xi^a = H^a + \qquad V_T^{a,\theta}(\pi^a)$$

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where

$$\xi^a = H^a + (1 - \lambda^a) V_T^{a,\theta}(\pi^a) + \lambda^a \left(V_T^{a,\theta}(\pi^a) - \frac{1}{N-1} \sum_{b \neq a} V_T^{b,\theta}(\pi^b) \right)$$

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$\lambda^a > 0$ is the concern rate

Admissibility and equilibrium

Admissibility sets $\mathcal{A}^a$ depend on g^a, hence: as we go!

Definition (Equilibrium and MPR)

For a given θ we call $\{\pi_*^a\}_{a\in\mathbb{A}}$ an equilibrium if $\pi^a \in \mathcal{A}^a$ and

- $Y_0^a(\pi_*^a, \pi_*^{-a}) \leq Y_0^a(\pi^a, \pi_*^{-a})$ for all admissible π^a, i.e. individual optimality, and
- $\sum_{b\in\mathbb{A}} \pi_*^{b,2} \equiv n$, i.e. market clearing condition for fixed net supply of derivatives.

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- $\sum_{b\in\mathbb{A}} \pi_*^{b,2} \equiv n$, i.e. market clearing condition for fixed net supply of derivatives.

θ^R is an equilibrium market price of external risk (EMPR) if

- $\mathcal{E}(-\int\langle(\theta^S, \theta^R), dw\rangle)$ is a true martingale
- exist optimal $\pi^a \in \mathcal{A}^a$, $a \in \mathbb{A}$, such that

$$\sum_{a\in\mathbb{A}} \pi_t^{a,2} = n \quad \mathbb{P}\otimes\lambda - a.s.$$

Solving the optimization

The individual optimization problem - I

$$Y_t^a = -\xi^a + \int_t^T g^a(s, Z_s^a)ds - \int_t^T Z_s^a dW_s$$

$$\xi^a := H^a + V_T^{a,\theta}(\pi^a) - \frac{\lambda^a}{|\mathbb{A}| - 1} \sum_{b \neq a} V_T^{b,\theta}(\pi^b)$$

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By a change of variables (note $V_T = V_t + (V_T - V_t)$):

$$\widehat{Y}_t := Y_t + V_t^{a,\theta} - \frac{\lambda^a}{|\mathbb{A}|-1}\sum_{b\neq a} V_t^{b,\theta}(\pi^b) \qquad \text{plus one for } Z$$

leads to

The individual optimization problem - II

$$\widehat{Y}_t^a = -H^a + \int_t^T \widehat{G}(s, \pi^a, \widehat{Z}_s^a) ds - \int_t^T \widehat{Z}_s^a dW_s$$

with

$$\widehat{G}^a(\omega, t, \pi_t^a, z) := g^a(t, z - \Upsilon_t^a) - \langle \Upsilon_t^a, \theta_t \rangle$$

and

$$\Upsilon^a := \pi^{a,1}\sigma + \pi^{a,2}\kappa^\theta - \frac{\lambda^a}{|\mathbb{A}| - 1} \sum_{b \in \mathbb{A}\setminus\{a\}} \left(\pi^{b,1}\sigma + \pi^{b,2}\kappa^\theta\right).$$

The individual optimization problem - II

$$\widehat{Y}^a_t = -H^a + \int_t^T \widehat{G}(s, \pi^a, \widehat{Z}^a_s) ds - \int_t^T \widehat{Z}^a_s dW_s$$

with

$$\widehat{G}^a(\omega, t, \pi^a_t, z) := g^a(t, z - \Upsilon^a_t) - \langle \Upsilon^a_t, \theta_t \rangle$$

and

$$\Upsilon^a := \pi^{a,1}\sigma + \pi^{a,2}\kappa^\theta - \frac{\lambda^a}{|\mathbb{A}| - 1} \sum_{b \in \mathbb{A}\setminus\{a\}} \left(\pi^{b,1}\sigma + \pi^{b,2}\kappa^\theta \right).$$

Idea: if comparison holds

$$\min_{\pi^a} Y^a \quad \Leftrightarrow \quad \min_{\pi} \widehat{G}(t, \pi^a, z) \quad \text{pointwise.}$$

FOC and entropic risk measure

If $g^a \in C^1$ then

$$\text{(FOC)} \qquad \min_{\pi^a} \widehat{G} \Leftrightarrow \nabla_{\pi^a} \widehat{G} = 0$$

For the entropic risk measure ($\Leftrightarrow$ Exponential Utility):

$$g^a(z) = \frac{1}{2\gamma_a}|z|^2, \qquad \gamma_a > 0 \ \text{ risk aversion}$$

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$$g^a(z) = \frac{1}{2\gamma_a}|z|^2, \qquad \gamma_a > 0 \text{ risk aversion}$$

then (using fixed net supply of H^D i.e. $\sum_a \pi^{a,2} = n$)

$$\Pi^{a,2} := \left(1 + \frac{\lambda^a}{|\mathbb{A}| - 1}\right)^{-1} \frac{\gamma_a \theta^R + \widehat{Z}^{a,2}}{\kappa^{\theta,R}}.$$

$$\Pi^{a,1} := \frac{\gamma_a[\theta^S \kappa^{\theta,R} - \theta^R \kappa^{\theta,S}] + \widehat{Z}^{a,1}\kappa^{\theta,R} - \widehat{Z}^{a,2}\kappa^{\theta,S}}{\sigma^S S \kappa^{\theta,R}} + \frac{\lambda^a}{|\mathbb{A}| - 1} \sum_{b \neq a} \Pi^{b,1}.$$

Under the optimum

For the entropic risk measure, the BSDE for optimal strategy

$$\widehat{Y}_t^a = -H^a + \int_t^T \widehat{G}(s, \Pi^a, \widehat{Z}_s^a)ds - \int_t^T \widehat{Z}_s^a dW_s$$

with

$$\begin{aligned}\widehat{G}^a(\omega, t, \Pi_t^a, z) &:= g^a(t, z - \Upsilon_t^a(\Pi)) - \langle \Upsilon_t^a(\Pi), \theta_t \rangle \\ &= -\langle z, \theta \rangle - \frac{\gamma_a}{2}|\theta|^2 \quad \to \text{no } \kappa^\theta = (\kappa^S, \kappa^R) \text{ - Great!}\end{aligned}$$

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but θ^R is unknown! Not great!

How to find θ^R?

Aggregation of the risk measures

- *Representative agent*: for which risk minimization is equivalent to the existence of an equilibrium for the whole system, see Negishi ('60)
- the risk measure Y^{ab} of the rep. agent follows from inf-convolution techniques, see El Karoui & Barrieu (2005) and Mastrogiacomo's talk

For $\xi^a := H^a + \frac{n}{2}H^D$, $a \in \mathbb{A}$

$$Y_t^{ab} := \inf\Big\{ Y_t^a\Big(\xi^a - F\Big) + Y^b\Big(\xi^b + F\Big)\Big\}$$

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For $\xi^a := H^a + \frac{n}{2} H^D$, $a \in \mathbb{A}$

$$Y_t^{ab} := \inf \left\{ Y_t^a \Big(\xi^a - F \Big) + Y^b \Big(\xi^b + F \Big) \right\}$$

▷ This does not work if $\lambda^a \neq \lambda^b$!

Weighted weightings of risk measures

$$Y_t^{ab} := \inf_{F\in L^\infty} \left\{ w^a Y_t^a\Big(\frac{\xi^a - F}{w^a}\Big) + w^b Y^b\Big(\frac{\xi^b + F}{w^b}\Big) \right\}.$$

Since the risk measures are given by BSDEs, this leads to a combination of the drivers

$$\widehat{g}^{ab}(z) := \inf_{x\in\mathbb{R}^2} \left\{ w^a g^a\left(\frac{z-x}{w^a}\right) + w^b g^b\left(\frac{x}{w^b}\right) \right\}.$$

with $w^a := 1/(1+\lambda^a)$.

Again entropic

If $g^a(z) = |z|^2/(2\gamma_a)$ we obtain for the simple case $\mathbb{A} = \{a, b\}$:

$$g^{ab}(z) = \frac{1}{2\gamma_R}|z|^2, \qquad \gamma_R := \frac{\gamma_a}{1+\lambda^a} + \frac{\gamma_b}{1+\lambda^b}.$$

and after variable transformation (as in single agent):

$$\widehat{Y}^{ab}_T := -\sum_a \frac{1}{1+\lambda^a}\left(H^a + \frac{n}{2}H^D\right)$$

and FOC for representative agent yields

$$\frac{\widehat{Z}^{ab,1} - \pi^{ab,1}\sigma^S S}{\gamma_R} = -\theta^S, \qquad \frac{\widehat{Z}^{ab,2}}{\gamma_R} = -\theta^R.$$

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and after variable transformation (as in single agent):

$$\widehat{Y}_T^{ab} := -\sum_a \frac{1}{1+\lambda^a}(H^a + \frac{n}{2}H^D)$$

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Theorem

Optimizing for the rep. agent $\Leftrightarrow$ existence of equilibrium

Recipe

- Solve the BSDE for representative agent
  - Get the $\widehat{Z}^{ab}$ and hence $\theta^R = -\widehat{Z}^{ab,2}/\gamma_R$
- Solve the BSDE for the derivative price
  - Get the κ^R and κ^S
  - Get π^a for single agent from FOC
- Solve the BSDE for the individual agents injecting the computed θ^R and κ

Following this we have way to analyze the system!

Entropic risk measure:

$$g^a(z) = \frac{|z|^2}{2\gamma_a}$$

Completion is attainable

Assumption: Everything is C_b^2 and $\partial_{x_2} H^D(x_1, x_2) > 0$

Theorem (Equilibrium MPR exists + market completion)

$(\theta^S, \theta^R) = (\theta^S, -\widehat{Z}^{ab,2}/\gamma_R)$ is the EMPR and $\kappa^R > 0$ $\mathbb{P} - a.s.$

Proof.

- Malliavin calculus on the BSDE for the price of H^D

$$B_t^\theta = H^D - \int_t^T \langle \kappa_s^\theta, \theta_s \rangle ds - \int_t^T \langle \kappa_s^\theta, dW_s \rangle$$

  using the representation $\kappa^R = D^{W^R} B$
- Lots of work because $\theta^R = -\frac{1}{\gamma_R} Z^{ab,2}$

□

Parameter analysis

Theorem

▷ $\gamma_R = \sum \gamma_a (1+\lambda^a)^{-1}$ *is Rep. agent risk aversion*
▷ *n is number of units of H^D in the market*

$$\partial_{\gamma_R} Y_t^w < 0$$
$$\partial_n Y_t^w \leq 0 \qquad \partial_n B_0^\theta < 0, \quad \textit{and} \quad \partial_n \theta_t^R > 0, \quad \forall t,\ \mathbb{P} - a.s.$$

Next?

- more parameter analysis (e.g. $\lambda^a = 0$ vs $\lambda^a > 0$)
    - We expect: if only 1 agent has perf. concerns then all are better off having them!
- other drivers g^a
- Compare with other relative performance concerns
- analysis for $|\mathbb{A}| \to \infty$
- Any other ideas?

Thank you!

Thank you for your time!

Some References

- G.-E. Espinosa: Stochastic control methods for optimal portfolio investment, 2010, PhD Thesis, Ecole Polytechnique
- G.-E. Espinosa and N. Touzi : Optimal Investment under Relative Performance Concerns, 2013 (*to appear in Math. Finance*)
- C. Frei and G. Dos Reis (2011), A financial market with interacting investors: does an equilibrium exist? Mathematics and financial economics, 4, 161-182