Localization method for non-Lipschitz Stochastic Differential Equations Driven by G-Brownian Motion (GSDEs)

Yiqing LIN

with Xinpeng LI and Xiangyun LIN

Faculty of Mathematics, University of Vienna, Austria

yiqing.lin@univie.ac.at

Bordeaux, July 8th, 2014

Outline

- **Brief introduction to the G-framework**
  - G-expectation and G-Brownian motion
  - G-stochastic integrals
  - Localization methods
  - Generalized G-Itô's formula
- **Non-Lipschitz GSDEs**
  - Lipschitz GSDEs
  - Local Lipschitz GSDEs
  - Stability of GSDEs

G-expectation and G-Brownian motion (d=1)

Ref.: Peng (arXiv, 2010), Denis-Hu-Peng (Potential Anal., 2011), Soner-Touzi-Zhang (EJP, 2013)

Basic settings:

Ω: the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{+\infty} 2^{-N} [(\max_{t\in[0,N]} |\omega_t^1 - \omega_t^2|) \wedge 1];$$

B: the canonical process, $B_t(\omega) = \omega_t$, $t \geq 0$;

$\mathcal{F}_t$: the filtration generated by the canonical process $(B_t)_{t\geq 0}$,

$$\mathcal{F}_t := \sigma\{B_s,\ 0 \leq s \leq t\};$$

$\mathcal{P}_1$: a subset of martingale measures induced by the strong formulation;

$\mathcal{P}_G$: the weak closure of $\mathcal{P}_1$;

$\bar{\mathbb{E}}[\cdot]$: an upper expectation $\bar{\mathbb{E}}[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}_G} E^{\mathbb{P}}[\cdot]$.

G-normal distribution:

A dimensional random vector X is called G-normal distributed with parameters $(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, if for each $\varphi \in C_{b,lip}(\mathbb{R})$,

$$u(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}X)], \; t \geq 0, \; x \in \mathbb{R},$$

is the viscosity solution of the following PDE defined on $[0,\infty) \times \mathbb{R}$:

$$\frac{\partial u}{\partial t} - G(\partial^2_{xx} u) = 0, \; u|_{t=0} = \varphi,$$

where

$$G(a) := \frac{1}{2}(a^+ \overline{\sigma}^2 - a^- \overline{\sigma}^2).$$

G-Brownian motion:

Under $\bar{\mathbb{E}}[\cdot]$, the canonical process B is a G-Brownian Motion, that is,

- $B_0 = 0$;
- For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}(0, [s\underline{\sigma}^2, s\overline{\sigma}^2])$-distributed and for $t_1, \ldots, t_n \in [0,t]$, we have

$$\bar{\mathbb{E}}[\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t+s} - B_t)] = \bar{\mathbb{E}}[\psi(B_{t_1}, \ldots, B_{t_n})],$$

  where $\psi(x_{t_1}, \ldots, x_{t_n}) = \bar{\mathbb{E}}[\varphi(x_{t_1}, \ldots, x_{t_n}, \sqrt{s}B_1)]$.

The corresponding Choquet capacity $\bar{C}(\cdot)$:

$$\bar{C}(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A), \; A \in \mathcal{B}(\Omega).$$

Definition (Quasi-surely)

A set $A \in \mathcal{B}(\Omega)$ is called polar if $\bar{C}(A) = 0$. A property is said to hold quasi-surely (q.s.), if it holds outside a polar set.

G-stochastic integrals

Ref.: Peng (arXiv, 2010) and Li and Peng (SPA, 2011)

Element function spaces:

$$L_{ip}^0(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1,\ 0 \leq t_1 \leq \ldots \leq t_n \leq T,\ \varphi \in C_{b,lip}(\mathbb{R}^n)\},$$

and

$$B_b(\Omega_T) := \{X : X \text{ is a bounded element in } \mathcal{B}(\Omega_T)\}.$$

Completions under $\bar{\mathbb{E}}[\cdot]$: for $1 \leq p < +\infty$,

$L_G^p(\Omega_T)$: the norm completion of $L_{ip}^0(\Omega_T)$;

$L_*^p(\Omega_T)$: the norm completion of $B_b(\Omega_T)$;

$$L_G^p(\Omega_T) \subset L_*^p(\Omega_T) \subset \{X \in \mathcal{B}(\Omega_T) : \bar{\mathbb{E}}[|X|^p] < +\infty\}.$$

Simple process spaces:

$$M_G^0([0,T]) := \left\{ \eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}(t) : \xi_k \in L_{ip}^0(\Omega_{t_k}) \right\},$$

and

$$M_b^0([0,T]) := \left\{ \eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}(t) : \xi_k \in B_b(\Omega_{t_k}) \right\}.$$

Norms and integrand spaces:

M^p norm in the G-framework:

$$\|\eta\|_{M_G^p} = \left(\frac{1}{T} \bar{\mathbb{E}} \left[\int_0^T |\eta_t|^p \, dt \right] \right)^{1/p}.$$

For each $p \geq 1$, $M_G^p([0,T])$ denotes the completion of $M_G^0([0,T])$ under the above norm.

For each $p \geq 1$, $M_*^p([0,T])$ denotes the completion of $M_b^0([0,T])$ under the above norm.

G-stochastic integrals with respect to B:

These stochastic integrals of G-Itô type has been first defined by Riemann sums

$$M_G^0([0,T]) \to L_G^2(\Omega_T),\ (resp.\ M_b^0([0,T]) \to L_*^2(\Omega_T)),$$

then the mappings can be continuously extended to $M_G^2([0,T]) \to L_G^2(\Omega_T)$ (resp. $M_*^2([0,T]) \to L_*^2(\Omega_T)$).

BDG type inequality:

For $\eta \in M_G^p([0,T])$ (resp. $M_b^p([0,T])$), $p \geq 1$, we have

$$\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}\left|\int_0^t \eta_s dB_s\right|^p\right] \leq C_p \overline{\sigma}^p \bar{\mathbb{E}}\left[\left|\int_0^T |\eta_t|^2 dt\right|^{\frac{p}{2}}\right],$$

where C_p is a positive constant independent of η and $\overline{\sigma}$. One can find a t-continuous $\bar{C}$-modification of $\int \eta dB_s$.

Advantage of this extension from $L_G^p(\Omega_T)$ to $L_*^p(\Omega_T)$ (resp. from $M_G^p([0,T])$ to $M_*^p([0,T])$):

- Consider a stopping time τ, $\mathbf{1}_{\tau \leq t} \in B_b(\Omega_t)$;
- The process $\mathbf{1}_{\{[0,\tau]\}}(\cdot) \in M_*^p([0,T])$;
- For each $\eta \in M_*^p([0,T])$, $\eta.\mathbf{1}_{[0,\tau]}(\cdot) \in M_*^p([0,T])$.

Localization method

Ref.: Li and Peng (SPA, 2011)

Definition (The extended space of integrands)

Fixing $p \geq 1$, a stochastic process η is said to be in $M_w^p([0, T])$, if there exists a sequence of increasing stopping times $\{\sigma_m\}_{m \in \mathbb{N}}$ satisfying the following conditions:

- For each $m \in \mathbb{N}$, $\eta \mathbf{1}_{[0,\sigma_m]} \in M_*^p([0, T])$;
- $\Omega^m := \{\omega : \sigma_m(\omega) \wedge T = T\} \uparrow \bar{\Omega} \subset \Omega$, where $\bar{C}(\bar{\Omega}^c) = 0$.

Remark

The following claim can be implied from the conditions in the above definition: $\int_0^T |\eta_t|^p \, dt < +\infty, \;\; q.s.$.

In the classical framework, for a predictable càdlàg process X on $[0, T]$, one can define the localized process $X_{\cdot \wedge \tau_N}$ by $\tau_N := \inf\{t : X_t \geq N\}$, and then define the stochastic integral of X by the limit of the localized one.

Lemma

For a given stopping time τ and $\eta \in M_*^p([0,T])$, $p \geq 2$, we consider the t-continuous $\bar{C}$-modifications of these two processes

$$\left(\int_0^t \eta_s dB_s\right)_{0\leq t\leq T} \text{ and } \left(\int_0^t \eta_s \mathbf{1}_{[0,\tau]}(s) dB_s\right)_{0\leq t\leq T}.$$

Then, we can find a polar set A such that for all $\omega \in A^c$ and $t \in [0,T]$,

$$\int_0^{t\wedge\tau} \eta_s dB_s(\omega) = \int_0^t \eta_s \mathbf{1}_{[0,\tau]}(s) dB_s(\omega).$$

Then, for $\eta \in M_w^2([0, T])$ with $\{\sigma_m\}_{m\in\mathbb{N}}$, we define the G-Itô type stochastic integrales in the following way:

- For each $m \in \mathbb{N}$, define $X_t^m := \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s$, $0 \le t \le T$;
- For each m, $n \in \mathbb{N}$, find a polar set $\hat{A}^{m,n}$ such that:

$$\int_0^{t\wedge\sigma_m} \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s(\omega) = \int_0^{t\wedge\sigma_m} \eta_s \mathbf{1}_{[0,\sigma_n]}(s) dB_s(\omega),$$
$$0 \le t \le T, \text{ for all } \omega \in (\hat{A}^{m,n})^c;$$

- Define on $\hat{A}^c$:

$$\int_0^t \eta_s dB_s(\omega) := \lim_{m\to+\infty} X_t^m(\omega),\ 0 \le t \le T,$$

where

$$\hat{A} := \bigcup_{m=1}^{+\infty} \bigcup_{n=m+1}^{+\infty} \hat{A}^{m,n}.$$

The quadratic variation $\langle B\rangle$ of B:

For $t>0$, let $\pi^N_{[0,t]}=\{t_0,t_1,\dots,t_N\}$ be a sequence of partition of $[0,t]$ and set

$$\mu(\pi^N_{[0,t]}):=\max_{k=0,1,\dots,N-1}|t_{k+1}-t_k|.$$

The quadratic variation $\langle B\rangle$ of B is defined in $M^2_G([0,T])$ by

$$\langle B\rangle_t:=\lim_{\mu(\pi^N_{[0,T]})\to 0}\sum_{k=0}^{N-1}(B_{t_{k+1}}-B_{t_k})^2=B_t^2-2\int_0^t B_s dB_s.$$

For $0\le s\le t\le T$,

$$\underline{\sigma}^2(t-s)\le\langle B\rangle_t-\langle B\rangle_s\le\overline{\sigma}^2(t-s),\ q.s..$$

Remark

From the inequality above, we see that for q.s. ω, $\langle B\rangle_\cdot(\omega)$ induce a measure that absolutely continuous with respect to the Lebesgue measure on $[0,T]$. Thus, G-stochastic integrals with respect to $\langle B\rangle$ can be regarded pathwisely in the sense of Lebesgue.

Generalized G-Itô's formula

Lemma

Assume that $\Phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$, $f,\ h \in M_w^1([0, T])$, $g \in M_w^2([0, T])$. *Consider*

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t h_s d\langle B\rangle_s + \int_0^t g_s dB_s,$$

then

$$\begin{aligned}\Phi(T, X_T) - \Phi(0, X_0) &= \int_0^T \frac{d\Phi}{dt}(X_t)dt + \int_0^T \frac{d\Phi}{dx}(X_t)f_t dt \\ &+ \int_0^T \left(\frac{d\Phi}{dx}(X_t)h_t + \frac{1}{2}\frac{d^2\Phi}{dx^2}(X_t)g_t^2\right)d\langle B\rangle_t \\ &+ \int_0^T \frac{d\Phi}{dx}(X_t)g_t dB_t,\ \ q.s..\end{aligned}$$

Formulation to GSDEs

We consider the GSDE of the following form:

$$X_t = x + \int_0^t f(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle_s + \int_0^t g(s, X_s)dB_s,\ 0 \le t \le T,\ q.s.,$$

where $x \in \mathbb{R}$ is the initial value, B is the a d-dimensional G-Brownian motion, $\langle B, B\rangle = (\langle B^i, B^j\rangle)_{i,j=1,\dots,d}$ is the mutual variation matrix of B.

Lipschitz GSDEs

(H1) For some $p \geq 2$ and each $x \in \mathbb{R}^n$, $f(\cdot, x)$, $h^{ij}(\cdot, x)$, $g^j(\cdot, x) \in M_*^p([0, T]; \mathbb{R}^n)$, $i, j = 1, \ldots, d$;

(H2) The coefficients f, h and g are uniformly Lipschitz in x, i.e., for each $t \in [0, T]$ and $x, x' \in \mathbb{R}^n$,

$$|f(t,x) - f(t,x')| + ||h(t,x) - h(t,x')|| + ||g(t,x) - g(t,x')|| \leq C_L|x - x'|, \ q.s.,$$

where $||\cdot||$ is the Hilbert-Schmidt norm of a matrix.

Theorem

Let (H1) and (H2) hold. Then, the GSDE above admits a unique solution $X \in M_*^p([0, T]; \mathbb{R}^n)$, for any $p \geq 2$. For two initial values x, $y \in \mathbb{R}^n$, then there exists a constant $C > 0$ that depends only on p, T and C_L, such that

$$\bar{E}[\sup_{t \in [0,T]} |X_t^x - X_t^y|^p] \leq C|x - y|^p.$$

Remark

Gao (SPA, 2009) and Peng (arXiv, 2010) have considered this type of equations on a smaller space $M_G^2([0, T]; \mathbb{R}^n)$.

Non-Lipschitz GSDEs

(H2") The coefficients $f(\cdot,\cdot)$, $h^{ij}(\cdot,\cdot)$, $g^j(\cdot,\cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are deterministic functions continuous in t and locally Lipschitz in x, i.e., on each $\{x : |x| \leq R\}$, there exists a positive constant C_R that only depends on R, such that for each $t \in [0, T]$,

$$|f(t,x) - f(t,x')| + ||h(t,x) - h(t,x')|| + ||g(t,x) - g(t,x')|| \leq C_R|x - x'|.$$

(H3") There exists a deterministic Lyapunov function $V \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n)$, which is nonnegative, such that

$$\inf_{|x|\geq R} \inf_{t\in[0,T]} V(t,x) \to +\infty, \text{ as } R \to +\infty,$$

and there exists a constant $C_{LY} \geq 0$, such that for all $(t,x) \in [0, T] \times \mathbb{R}^n$,

$$\mathcal{L}V(t,x) \leq C_{LY}V(t,x),$$

where $\mathcal{L}$ is a differential operator defined by

$$\mathcal{L}V = \partial_t V + \partial_{x^\nu} V f^\nu + G\left(\left(\partial_{x^\nu} V \cdot (h^{\nu ij} + h^{\nu ji}) + \partial^2_{x^\mu x^\nu} V \cdot g^{\mu i} g^{\nu j}\right)_{i,j=1}^{d}\right).$$

Lemma

Assume that $\Phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, $f,\ h \in M_w^1([0,T])$, $g \in M_w^2([0,T])$. *Consider*

$$X_t = X_0 + \int_0^t f_s\,ds + \int_0^t h_s\,d\langle B\rangle_s + \int_0^t g_s\,dB_s,$$

then

$$\begin{aligned}\Phi(T,X_T)-\Phi(0,X_0) &= \int_0^T \frac{d\Phi}{dt}(X_t)\,dt + \int_0^T \frac{d\Phi}{dx}(X_t) f_t\,dt \\ &+ \int_0^T \left(\frac{d\Phi}{dx}(X_t)h_t + \frac{1}{2}\frac{d^2\Phi}{dx^2}(X_t)g_t^2\right) d\langle B\rangle_t \\ &+ \int_0^T \frac{d\Phi}{dx}(X_t) g_t\,dB_t,\ \ q.s..\end{aligned}$$

Theorem

Let (H2") and (H3") hold. Then, the GSDE admits a unique solution $X \in M_w^p([0,T];\mathbb{R}^n)$ that has t-continuous paths on $[0,T]$, q.s., and the following estimate holds:

$$\bar{\mathbb{E}}[V(t, X_t^x)] \leq e^{C_{LY}T} V(0,x).$$

Main idea of Proof: The uniqueness is evident. For the existence, we try to find a sequence of stopping times $\{\tau_N\}_{N\in\mathbb{N}}$, and define X as a solution of some Lipschitz GSDE on $[0,\tau_N]$.

Step 1:

For each $N \in \mathbb{N}$, we first consider the following truncated GSDE:

$$X_t^N = x + \int_0^t f^N(s, X_s^N)ds + \int_0^t h^N(s, X_s^N)d\langle B, B\rangle_s + \int_0^t g^N(s, X_s^N)dB_s,\ 0 \le t \le T,\ q.s.,$$

where f^N, $(h^{ij})^N$ and $(g^j)^N$, i, $j = 1, \ldots, d$, are defined in the following form:

$$\zeta^N(t,x) = \begin{cases} \zeta(t,x) & , \text{ if } |x| \le N; \\ \zeta(t, Nx/|x|), & \text{if } |x| > N. \end{cases}$$

The Lipschitz GSDE above admits a unique solution in $M_G^p([0,T];\mathbb{R}^n)$, for $p \ge 2$, denoted by X^N, whose paths are t-continuous.

Define a sequence of stopping times by

$$\tau_N := \inf\{t : |X_t^N| \ge N\} \wedge T.$$

Step 2:

Aim: We verify that X^N and X^{N+1} are distinguishable (in the q.s. sense) on $[0, \tau_N]$.

$$
\begin{aligned}
X^N_{t\wedge\tau_N} &= x + \int_0^{t\wedge\tau_N} f^N(s, X^N_s)\,ds + \int_0^{t\wedge\tau_N} h^N(s, X^N_s)\,d\langle B, B\rangle_s + \int_0^{t\wedge\tau_N} g^N(s, X^N_s)\,dB_s \\
&= x + \int_0^{t} f^{N+1}(s, X^N_s)\mathbf{1}_{[0,\tau_N]}(s)\,ds \\
&\quad + \int_0^{t\wedge\tau_N} h^{N+1}(s, X^N_s)\,d\langle B, B\rangle_s + \int_0^{t\wedge\tau_N} g^{N+1}(s, X^N_s)\,dB_s,\ 0 \le t \le T,\ q.s..
\end{aligned}
$$

On the other hand, by the definition of X^{N+1}, we have

$$
\begin{aligned}
X^{N+1}_{t\wedge\tau_N} = x + \int_0^{t\wedge\tau_N} f^{N+1}(s, X^{N+1}_s)\,ds &+ \int_0^{t\wedge\tau_N} h^{N+1}(s, X^{N+1}_s)\,d\langle B, B\rangle_s \\
&+ \int_0^{t\wedge\tau_N} g^{N+1}(s, X^{N+1}_s)\,dB_s,\ 0 \le t \le T,\ q.s..
\end{aligned}
$$

By the uniqueness of the solution to the GSDE with coefficients f^{N+1}, h^{N+1} and g^{N+1}, we have the desired result. This implies that the sequence $\{\tau_N\}_{N\in\mathbb{N}}$ are q.s. increasing.

Step 3:

Now we aim to show that

$$\bar{C}\left(\bigcup_{N=1}^{+\infty}\{\omega : \tau_N(\omega) = T\}\right) = 1.$$

First, we have

$$\begin{aligned} X^N_{t\wedge\tau_N} &= x + \int_0^{t\wedge\tau_N} f^N(s, X^N_s)\,ds + \int_0^{t\wedge\tau_N} h^N(s, X^N_s)\,d\langle B, B\rangle_s + \int_0^{t\wedge\tau_N} g^N(s, X^N_s)\,dB_s \\ &= x + \int_0^t f(s, X^N_s)\mathbf{1}_{[0,\tau_N]}(s)\,ds + \int_0^t h(s, X^N_s)\mathbf{1}_{[0,\tau_N]}(s)\,d\langle B, B\rangle_s \\ &+ \int_0^t g(s, X^N_s)\mathbf{1}_{[0,\tau_N]}(s)\,dB_s,\ 0 \le t \le T,\ q.s.. \end{aligned}$$

Then, apply G-Itô's formula to

$$\Phi(t \wedge \tau_N, X^N_{t\wedge\tau_N}) := \exp(-C_{LY}(t \wedge \tau_N))\, V(t \wedge \tau_N, X^N_{t\wedge\tau_N}),$$

and define

$$\eta^{ij}_{\cdot}(\Phi, X) := \partial_{x^\nu}\Phi(\cdot, X_\cdot)(h^{\nu ij}(\cdot, X_\cdot) + h^{\nu ji}(\cdot, X_\cdot)) + \partial^2_{x^\mu x^\nu}\Phi(\cdot, X_\cdot) g^{\mu i}(\cdot, X_\cdot) g^{\nu j}(\cdot, X_\cdot).$$

We have

$$
\begin{aligned}
\Phi(t\wedge\tau_N, X^N_{t\wedge\tau_N}) - \Phi(0,x)
&= \int_0^{t\wedge\tau_N} \left(\partial_t\Phi(s,X^N_s) + \partial_{x^\nu}\Phi(s,X^N_s) f^\nu_s(s,X^N_s)\right) ds \\
&+ \int_0^{t\wedge\tau_N} \partial_{x^\nu}\Phi(s,X^N_s) h^{\nu ij}_s(s,X^N_s) \\
&+ \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(s,X^N_s) g^{\mu i}_s(s,X^N_s) g^{\nu j}_s(s,X^N_s)\, d\langle B^i, B^j\rangle_s \\
&+ \int_0^{t\wedge\tau_N} \partial_{x^\nu}\Phi(s,X^N_s) g^{\nu j}_s(s,X^N_s)\, dB^j_s \\
&= \int_0^t \mathcal{L}\Phi(s,X^N_s)\mathbf{1}_{[0,\tau_N]}(s)\, ds \\
&+ \int_0^t \eta^{ij}_s(\Phi, X^N)\mathbf{1}_{[0,\tau_N]}(s)\, d\langle B^i, B^j\rangle_s - \int_0^t G(\eta_s(\Phi, X^N)\mathbf{1}_{[0,\tau_N]}(s))\, ds \\
&+ \int_0^t \partial_{x^\nu}\Phi(s,X^N_s) g^{\nu j}_s(s,X^N_s)\mathbf{1}_{[0,\tau_N]}(s)\, dB^j_s, \ \ q.s..
\end{aligned}
$$

Since $\mathcal{L}V \leq C_{LY}V$, $\mathcal{L}\Phi \leq 0$. From the fact that $\partial_x V(t,x)$ is uniformly continuous in t and uniformly Lipschitz in x on $[0,T] \times B(0,N)$, it is readily observed that

$$\partial_\nu \Phi(\cdot, X^N_\cdot) g^{\nu j}(X^N_\cdot) \mathbf{1}_{[0,\tau_N]}(\cdot) \in M^p_*([0,T]),$$

for any $p \geq 2$. Then, we obtain

$$\bar{\mathbb{E}}\left[\int_0^t \partial_{x^\nu}\Phi(s, X^N_s) g^{\nu j}_s(s, X^N_s) \mathbf{1}_{[0,\tau_N]}(s)\, dB^j_s\right] = 0.$$

On the other hand,

$$\bar{\mathbb{E}}\left[\int_0^t \eta^{ij}_s(\Phi, X^N)\mathbf{1}_{[0,\tau_N]}(s)\, d\langle B^i, B^j\rangle_s - \int_0^t G\Big(\eta_s(\Phi, X^N)\mathbf{1}_{[0,\tau_N]}(s)\Big)\, ds\right] \leq 0,$$

Therefore,

$$\bar{\mathbb{E}}[\Phi(T\wedge\tau_N, X^N_{T\wedge\tau_N})] - \Phi(0,x) = \bar{\mathbb{E}}\left[\int_0^{T\wedge\tau_N} \mathcal{L}\Phi(t, X^N_t)\, dt\right] \leq 0,$$

and consequently,

$$\bar{\mathbb{E}}[V(T\wedge\tau_N, X^N_{T\wedge\tau_N})] \leq V(0,x)\exp(C_{LY}T).$$

Note: the estimate in the theorem can be obtained by applying Fatou's lemma to the above inequality.

In particular, we have

$$\bar{\mathbb{E}}[\mathbf{1}_{\{\tau_N<T\}}V(T\wedge\tau_N, X^N_{T\wedge\tau_N})] \le V(0,x)\exp(C_{LY}T).$$

Since X^N has t-continuous paths, $\tau_N < T$ implies $|X^N_{T\wedge\tau_N}| = N$, q.s., from which we deduce

$$\bar{C}(\{\omega : \tau_N(\omega) < T\}) \inf_{|x|\ge N} \inf_{t\in[0,T]} V(t,x) \le V(0,x)\exp(C_{LY}T).$$

As $N \to +\infty$, by (H3"), we obtain

$$1 \ge \lim_{N\to+\infty} \bar{C}(\{\omega : \tau_N(\omega) = T\}) \ge 1 - \lim_{N\to+\infty} \bar{C}(\{\omega : \tau_N(\omega) < T\}) = 1.$$

Since $\{\omega : \tau_N(\omega) = T\}$ is increasing, the upwards convergence theorem yields the desired result.

Step 4:

Therefore, there exists a polar set A, such that for all $\omega \in A^c$, the following assertion holds: one can find an $N_0(\omega)$ that depends on ω, such that for all $N \geq N_0(\omega)$, $N \in \mathbb{N}$, $\tau_N(\omega) = T$. Then, we define

$$X_t(\omega) = \begin{cases} X_t^{N_0(\omega)}(\omega),\ 0 \leq t \leq T\ ,\ \omega \in A^c; \\ \hfill 0\ ,\ \omega \in A. \end{cases}$$

From the argument above, we have for each τ_N,

$$X\mathbf{1}_{[0,\tau_N]} = X^N\mathbf{1}_{[0,\tau_N]} \in M_*^p([0,T];\mathbb{R}^n)$$

and thus, $X \in M_w^p([0,T];\mathbb{R}^n)$, for any $p \geq 2$. Moreover,

$$\begin{aligned} X_{t\wedge\tau_N} = X^N_{t\wedge\tau_N} &= x + \int_0^{t\wedge\tau_N} f^N(s, X^N_s)\,ds \\ &\quad + \int_0^{t\wedge\tau_N} h^N(s, X^N_s)\,d\langle B, B\rangle_s + \int_0^{t\wedge\tau_N} g^N(s, X^N_s)\,dB_s \\ &= x + \int_0^{t\wedge\tau_N} f(s, X_s)\,ds + \int_0^{t\wedge\tau_N} h(s, X_s)\,d\langle B, B\rangle_s + \int_0^{t\wedge\tau_N} g(s, X_s)\,dB_s, \end{aligned}$$

which implies that X satisfies the GSDE.

Stability of GSDEs

We still consider the GSDE of the following form:

$$X_t = x + \int_0^t f(s, X_s)\,ds + \int_0^t h(s, X_s)\,d\langle B, B\rangle_s + \int_0^t g(s, X_s)\,dB_s,\ 0 \le t \le T,\ q.s..$$

However, we restrict ourself to study conditions for stability of the trivial solution $X \equiv 0$. Accordingly, we assume that

(H4")

$$f(t, 0) = h(t, 0) = g(t, 0) \equiv 0.$$

Definition

Denote by $X^{s,x}$ the solution of the GSDE starting with $X_s = x$, $x \in \mathbb{R}^n$. The solution $X \equiv 0$ of this G-stochastic system in $\mathbb{R}^n$ is said to be

(i) p-stable, for some $p > 0$, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\sup_{|x|\leq\delta} \sup_{t\geq s} \bar{\mathbb{E}}[|X_t^{s,x}|^p] < \varepsilon;$$

(ii) asymptotically p-stable, if it is p-stable and moreover

$$\bar{\mathbb{E}}[|X_t^{s,x}|^p] \to 0, \text{as } t \to +\infty;$$

(iii) exponentially p-stable, if for some positive constants A and α

$$\bar{\mathbb{E}}[|X_t^{s,x}|^p] \leq A|x|^p e^{-\alpha(t-s)}.$$

In particular, when $p = 2$, we call this system is (asymptotically, exponentially) stable in mean square.

We now give the sufficient condition for the stability of GSDEs in terms of Lyapunov functions. Without loss of generality, we consider only the case $s = 0$.

Theorem

Consider the GSDE with (H2") and (H4"). Suppose that there exists a function $V(t, x) \in \mathcal{C}^{1,2}([0, +\infty) \times \mathbb{R}^n)$ *such that for all* $t \geq 0$ *and some positive constant* c_1 *and* c_2*,*

$$c_1|x|^p \leq V(t, x) \leq c_2|x|^p.$$

Then,

(a) *the trivial solution is p-stable, if*

$$\mathcal{L}V \leq 0,$$

(b) *the trivial solution is exponentially p-stable, if there exists a* $\lambda > 0$ *such that for all* $(t, x) \in ([0, +\infty) \times \mathbb{R}^n)$*,*

$$\mathcal{L}V(t, x) \leq -\lambda V(t, x).$$

Since $\mathcal{L}V \leq C_{LY}V$, $\mathcal{L}\Phi \leq 0$. From the fact that $\partial_x V(t,x)$ is uniformly continuous in t and uniformly Lipschitz in x on $[0, T] \times B(0, N)$, it is readily observed that

$$\partial_\nu \Phi(\cdot, X_\cdot^N) g^{\nu j}(X_\cdot^N) \mathbf{1}_{[0,\tau_N]}(\cdot) \in M_*^p([0, T]),$$

for any $p \geq 2$. Then, we obtain

$$\bar{\mathbb{E}}\left[\int_0^t \partial_{x^\nu}\Phi(s, X_s^N) g_s^{\nu j}(s, X_s^N)\mathbf{1}_{[0,\tau_N]}(s)\,dB_s^j\right] = 0.$$

On the other hand,

$$\bar{\mathbb{E}}\left[\int_0^t \eta_s^{ij}(\Phi, X^N)\mathbf{1}_{[0,\tau_N]}(s)\,d\langle B^i, B^j\rangle_s - \int_0^t G\Big(\eta_s(\Phi, X^N)\mathbf{1}_{[0,\tau_N]}(s)\Big)\,ds\right] \leq 0,$$

Therefore,

$$\bar{\mathbb{E}}[\Phi(T \wedge \tau_N, X_{T\wedge\tau_N}^N)] - \Phi(0, x) = \bar{\mathbb{E}}\left[\int_0^{T\wedge\tau_N} \mathcal{L}\Phi(t, X_t^N)\,dt\right] \leq 0,$$

and consequently,

$$\bar{\mathbb{E}}[V(T \wedge \tau_N, X_{T\wedge\tau_N}^N)] \leq V(0, x)\exp(C_{LY}T).$$

Note: the estimate in the theorem can be obtained by applying Fatou's lemma to the above inequality.

Theorem

Consider the GSDE with (H2”), (H4”) and the following conditions: for some $K > 0$

$$x^{\mathrm{Tr}} f(t,x) \vee ||x^{\mathrm{Tr}} h(t,x)|| \vee ||g^{\mathrm{Tr}}(t,x) g(t,x)|| \leq K|x|^2, \tag{1}$$

for all $(t,x) \in [0,+\infty) \times \mathbb{R}^n$*. If the trivial solution* $X \equiv 0$ *is exponential* p*-stable, for some* $p \geq 2$*, then it is q.s. exponentially stable, i.e. for all* $x \in \mathbb{R}^n$*,*

$$\limsup_{t \to +\infty} \frac{1}{t} \log(|X_t^x|) < 0, \ \ q.s..$$

Thank you for your attention!