BSDE Representation for Stochastic Control Problems with Controlled Intensity

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Outline

1. Introduction

2. Randomization of the control

3. BSDE with partially zero diffusive component and nonlinear Feynman-Kac formula
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1 Introduction

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Problem

- Hamilton-Jacobi-Bellman equation with controlled intensity:

\[
\frac{\partial v}{\partial t} + \sup_{a \in A} \left[ b(x, a) \cdot D_x v + \frac{1}{2} \text{tr} (\sigma \sigma^\top (x, a) D^2_x v) + f(x, a) \right] + \int_E (v(t, x + \beta(x, a, e)) - v(t, x) - \beta(x, a, e).D_x v(t, x)) \lambda(a, de) = 0,
\]

on \([0, T) \times \mathbb{R}^d\), with terminal condition

\[ v(T, x) = g(x), \quad x \in \mathbb{R}^d. \]

- **Existence and Uniqueness** of a viscosity solution \( v \).

- **Nonlinear Feynman-Kac formula:**

\[ v(t, x) \overset{?}{=} Y_{t,x}^t. \]
How to solve it: two approaches

- **Second-order BSDE with jumps (2BSDEJs):**
  
  

- **Randomization of the control:**
  

► **Main goals:**

- Try to extend the *randomization of the control method* to get existence and the nonlinear Feynman-Kac formula.

- Comparison theorem: implement the *nonlocal version of Jensen-Ishii’s lemma* of Barles & Imbert.
Motivation

• Model uncertainty:

\[
\frac{\partial v}{\partial t} + \sup_{(b,c,F) \in \Theta} \left[ b.D_x v + \frac{1}{2} \text{tr}(cD_x^2 v) \right]
+ \int_E \left( v(t, x + z) - v(t, x) - D_x v(t, x) \cdot z 1_{\{|z| \leq 1\}} \right) F(dz) = 0,
\]

on \([0, T) \times \mathbb{R}^d\), with terminal condition

\[
v(T, x) = g(x), \quad x \in \mathbb{R}^d.\]

\(\Theta\) denotes a set of Lévy triplets \((b, c, F)\).

▶ The unique viscosity solution \(v\) is represented as follows:

\[
v(t, x) = \mathcal{E}(g(x + \mathcal{X}_t)),
\]

where \(\mathcal{X}\) is a nonlinear Lévy process under the nonlinear expectation \(\mathcal{E}(\cdot)\).


We expect that $v$ has the stochastic control representation

$$v(t, x) = \sup_{\alpha \in A} \mathbb{E}^\alpha \left[ \int_t^T f(X_{s}^{t,x,\alpha}, \alpha_s) \, ds + g(X_T^{t,x,\alpha}) \right],$$

where $X_{t,x,\alpha}$ has the controlled dynamics on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$

$$dX_{s}^{\alpha} = b(X_s^{\alpha}, \alpha_s) \, ds + \sigma(X_s^{\alpha}, \alpha_s) \, dW_s + \int_E \beta(X_{s}^{\alpha}, \alpha_s, e) \tilde{\pi}(ds, de)$$

$$X_{t}^{\alpha} = x$$

with

$$\tilde{\pi}(dt, de) = \pi(dt, de) - \lambda(\alpha_t, de) \, dt$$

the compensated martingale measure of $\pi$.

**Randomization of the control** will be developed having in mind this stochastic control representation.
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Forward process

- **Randomization** by an independent Brownian motion $B$ mapped on $A \subset \mathbb{R}^q$ by means of a $C^2$ surjection $h : \mathbb{R}^q \rightarrow A$:

\[
X_s = x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r
+ \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de),
\]

\[
I_s = h(a + B_s - B_t), \quad t \leq s \leq T,
\]

where $\tilde{\pi}(dr, de) = \pi(dr, de) - \lambda(I_r, de)dr$ is the compensated martingale measure of $\pi$.

- **Main issues:**
  - Why do we randomize with a Brownian motion $B$?
  - Existence and uniqueness of $(X^{t,x,a}, I^{t,a})$?
Brownian randomization

- **Poisson random measure**: in Kharroubi & Pham an independent Poisson random measure $\mu$ on $\mathbb{R}_+ \times A$ is used to randomize the control. No surjection is needed.

▶ **Martingale representation theorem**

- Unlike Kharroubi & Pham, we have *dependence* between $B$ (or $\mu$) and $\pi$ through the compensator of $\pi$.
- However, $B$ is **orthogonal** to $\pi$, since $B$ is a continuous martingale.

$\Rightarrow$ *Martingale representation for* $(W, B, \pi)$, while not clear for $(W, \mu, \pi)$ due to the dependence between $\mu$ and $\pi$. 
Wellposedness of the SDE

- **Nonstandard SDE**: The jump part of the driving factors is not given, but depends on the solution via its intensity.
  

- **Dominated case** $\lambda(a, de) = m(a)\bar{\lambda}(de)$: we can solve the forward SDE bringing it back to a standard SDE, via a change of intensity “à la Girsanov”.

  ▶ **Nondominated case**

  1. We solve first the SDE for the process $I^{t,a}$.
  2. Then, we construct a probability measure $\mathbb{P}^{t,a}$ on $(\Omega, \mathcal{F})$ such that the random measure $\pi(dt, de)$ admits $\lambda(I^{t,a}_s, de)ds$ as compensator.
  3. Finally, we solve by standard methods the SDE for $X^{t,x,a}$.
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BSDE with partially zero diffusive component

- **BSDE**: for any \((t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q,\)

\[
Y_s = g(X_{T,t,x,a}^T) + \int_s^T f(X_{t,x,a}^r, I_{t,a}^r)dr + K_T - K_s - \int_s^T Z_r dW_r

- \int_s^T V_r dB_r - \int_s^T \int_E U_r(e) \tilde{\pi}(dr, de), \quad t \leq s \leq T, \quad \mathbb{P}^{t,a} \text{ a.s.}
\]

and

\[
V_s = 0 \quad ds \otimes d\mathbb{P}^{t,a} \text{ a.e.}
\]

**Main issues:**

- We look for a solution for which the \(B\)-component resulting from the martingale representation theorem is zero.

- Existence is guaranteed if we *add* the increasing process \(K\).

- Uniqueness is guaranteed if we look for the **minimal solution** \((Y, Z, V, U, K)\), i.e., for any solution \((\bar{Y}, \bar{Z}, \bar{V}, \bar{U}, \bar{K})\) we must have \(Y \leq \bar{Y}\).
Minimal solution: existence

- **Penalized BSDE:** for any $n \in \mathbb{N}$, $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$,

$$
Y_s^n = g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a}) dr + K^n_T - K^n_s - \int_s^T Z^n_r dW_r
- \int_s^T V^n_r dB_r - \int_s^T \int_E U^n_r(e) \tilde{\pi}(dr, de), \quad t \leq s \leq T, \quad \mathbb{P}^{t,a} \text{ a.s.}
$$

where

$$
K^n_s = n \int_t^s |V^n_r| dr, \quad t \leq s \leq T.
$$

- **Wellposedness** is based, as usual, on the martingale representation theorem and on a fixed point argument.

- **Monotonicity:** from the comparison theorem for BSDEs with jumps we have

$$
Y^0 \leq \cdots \leq Y^n \nearrow Y \leq \bar{Y}, \quad \forall n, \text{ for any other solution } \bar{Y}.
$$

- **Uniform estimates** for $(Z^n, V^n, U^n, K^n)_n$ allow to obtain weak convergence of these components, so to pass to the limit in the BSDE, and to end up with the minimal solution.
Viscosity property of the penalized BSDE

- **Penalized HJB equation**

\[
\frac{\partial v^h_n}{\partial t} + b(x, h(a)).D_x v^h_n + \frac{1}{2}\text{tr}(\sigma \sigma^\top (x, h(a)) D^2_x v^h_n) + f(x, h(a)) \\
+ \int_E [v^h_n(t, x + \beta(x, h(a), e)) - v^h_n(t, x) \\
- \beta(x, h(a), e).D_x v^h_n(t, x)] \lambda(h(a), de) \\
+ \frac{1}{2}\text{tr}(D^2_a v^h_n(t, x, a)) + n \|D_a v^h_n(t, x, a)\| = 0,
\]

on \([0, T) \times \mathbb{R}^d \times \mathbb{R}^q\), with terminal condition

\[
v^h_n(T, x, a) = g(x), \quad (x, a) \in \mathbb{R}^d \times \mathbb{R}^q.
\]

- **Nonlinear Feynman-Kac formula**: the unique continuous viscosity solution, satisfying a linear growth condition, is given by

\[
v^h_n(t, x, a) := Y^{n,t,x,h(a)}_t, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q,
\]
Viscosity property of the BSDE

\[ Y_t^{n,t,x,h(a)} = v^h_n(t, x, a) \uparrow v(t, x, a) = Y_t^{t,x,a}. \]

- \( v \) does not depend on \( a \), but only on \((t, x)\).

- **Existence:** \( v \) is a (discontinuous) **viscosity solution** to the HJB equation. The result follows from:
  - the viscosity property of \( v^h_n \)
  - the convergence \( v^h_n \uparrow v \) as \( n \to \infty \)
  - stability arguments for viscosity solutions.

- **Uniqueness:** \( v \) is the unique viscosity solution to the HJB equation, satisfying a linear growth condition.
  - The result follows from the *comparison theorem*, which is proved relying on the nonlocal version of Jensen-Ishii’s lemma of Barles & Imbert.
Conclusion

- Study of a class of Hamilton-Jacobi-Bellman equations with:
  - Controlled diffusion coefficient; possibly degenerate.
  - Controlled intensity; possibly nondominated.

- Introduction of a class of BSDEs with partially zero diffusive component which provides:
  - Existence of a viscosity solution.
  - Nonlinear Feynman-Kac formula.

- Comparison theorem and uniqueness.
Thank you!