Random solitons and soliton gasses

Manuela Girotti





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JOYEUSE
ACTION DE GRÂCES
Canada

The mighty "wave of translation"

KdV equation
Scott-Russell, 1834; Korteweg, de Vries, 1895

$$q_t - 6qq_x + q_{xxx} = 0$$

Shallow-water waves, internal waves in the ocean, ion acoustic waves in plasma, acoustic waves on a crystal lattice, ...



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The Spirited Horse, the Engineer, and the Mathematician: Water Waves in Nineteenth-Century Hydrodynamics

OLIVIER DARRIGOL

Communicated by J.Z. BUCHWALD

Once upon a time, a spirited horse was dragging a boat along a peaceful canal of Northern Scotland. Suddenly, a ferocious dog charged from the bushes. The horse took fright and ran off, drawing the boat after him. To the horse's surprise, the boat offered almost no resistance at this high speed. A clever naval engineer, who chanced to witness the scene, pursued the matter through numerous experiments and confirmed the horse's discovery. Not knowing his limits, the engineer ventured to propose a mechanical explanation of this paradox based on the existence of a wonderful, solitary wave that carried the boat with it. A renowned mathematician, who excelled in the learned calculus of water motion, mocked this amateurish attempt. There was nothing in the engineer's equations to justify his reasoning, and much to condemn it. Yet for a few years Scottish canal travelers enjoyed the commercial exploitation of the paradox. Half a century later, the aged mathematician resumed his calculations. Thanks to his long experience, he now saw new meanings in his old symbols. From this enriched analysis the solitary wave and the possibility of vanishing ship resistance emerged as if by magic. At last, the wise man rejoiced, mathematics could do as well as a galloping horse.

This fable is an imaginary simplification of a real story of which the engineer John Scott Russell, and the mathematicians (in a broad sense) George Biddell Airy, George Gabriel Stokes, Joseph Boussinesq, and Lord Kelvin were the main actors (besides the spirited horse). It is intended to indicate a major nineteenth-century transformation of the mathematical physicist's tool kit through which practically important solutions of long-known equations became much more easily accessible. The main purpose of the present article is to analyze the nature and the water-wave circumstances of this transformation.

Waves on the surface of water were an obvious field of application of the new hydrodynamics of Jean le Rond d'Alembert, Leonhard Euler, and Joseph Louis Lagrange.

Classical solutions

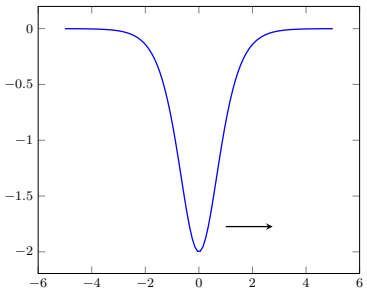
Travelling wave ansatz:

$$q(x,t) = \varphi\,(x - vt)$$

1. rapidly decreasing, localized travelling wave (*soliton*):

$$q_{\mathrm{sol}}(x,t) = -2\kappa^2 \operatorname{sech}^2\Big(2\kappa(x - 4\kappa^2 t - x_0)\Big)$$

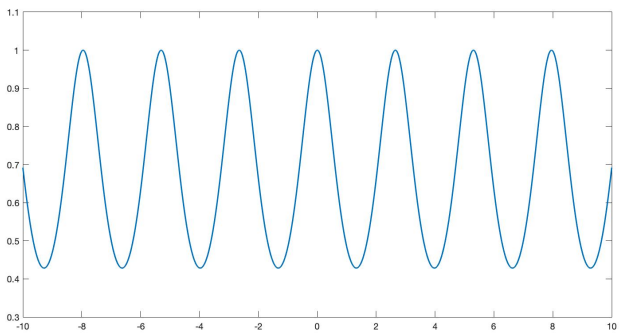
with $\kappa > 0$.



2. *periodic travelling wave* solutions:

$$q(x,t) = \beta_1 - \beta_2 - \beta_3 - 2(\beta_1 - \beta_3)\,\mathrm{dn}^2\Big(\sqrt{\beta_1 - \beta_3}(x - 2(\beta_1 + \beta_2 + \beta_3)t) + x_0 \,|\, m\Big)$$

where $\mathrm{dn}\,(z \mid m)$ is Jacobi elliptic function of modulus $m \in (0,1)$.



General solutions

The KdV equation is a *nonlinear integrable* PDE, arising as the compatibility condition of a Lax pair of linear differential operators [Lax, '68]

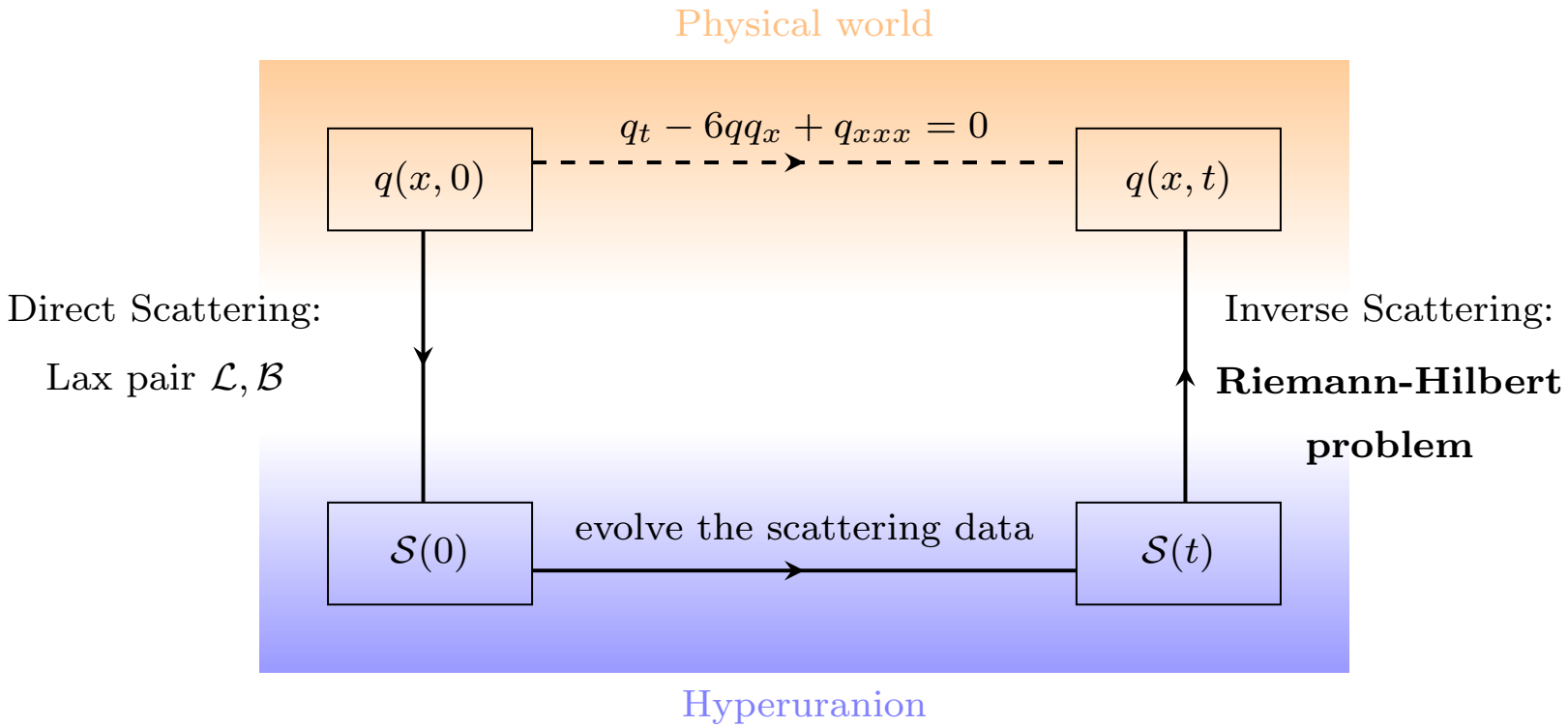
$$\dot{\mathcal{L}} = [\mathcal{B}, \mathcal{L}]$$

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q, \qquad \mathcal{B} = -4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 6q\frac{\mathrm{d}}{\mathrm{d}x} + 3q_x \ ,$$

The compatibility condition can be presented as the existence of simultaneous solutions to

$$\mathcal{L}\psi = \lambda\psi, \qquad \mathrm{i}\psi_t = \mathcal{B}\psi \ .$$

Recipe [Gardner, Greene, Kruskal, Miura, '67]:



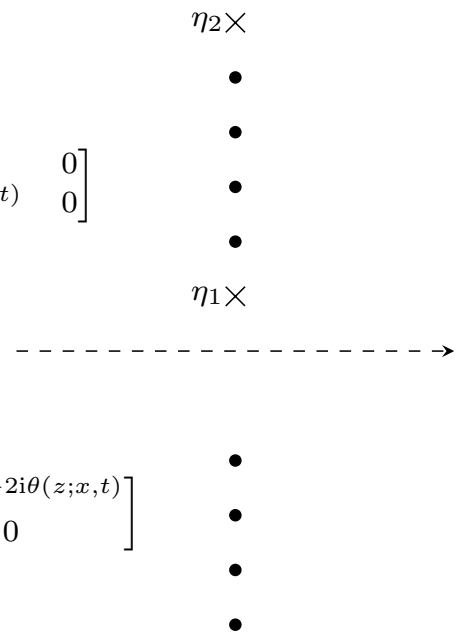
The scattering data are the L^2-eigenvalues $\{\mathrm{i}\kappa_j\}$, the norming constants $\{\mathrm{i}c_j\}$, and the reflection coefficient $\rho(z)$.
Their evolution is *linear*!

A Riemann-Hilbert problem

1. $\boldsymbol{m}(k;x,t)$ is meromorphic in $\mathbb{C}$, with simple poles at $\{\mathrm{i}\kappa_j\}_{j=1}^N \subset \mathrm{i}\mathbb{R}_+$ and at the corresponding conjugate points $\{-\mathrm{i}\kappa_j\}_{j=1}^N \subset \mathrm{i}\mathbb{R}_-$;
2. $\boldsymbol{m}$ satisfies the residue conditions

$$\operatorname*{res}_{z=\mathrm{i}\kappa_j} \boldsymbol{m} = \lim_{z\to\mathrm{i}\kappa_j} \boldsymbol{m}(z) \begin{bmatrix} 0 & 0 \\ \mathrm{i}c_j\,\mathrm{e}^{2\mathrm{i}\theta(z;x,t)} & 0 \end{bmatrix}$$

$$\operatorname*{res}_{z=-\mathrm{i}\kappa_j} \boldsymbol{m} = \lim_{z\to-\mathrm{i}\kappa_j} \boldsymbol{m}(z) \begin{bmatrix} 0 & -\mathrm{i}c_j\,\mathrm{e}^{-2\mathrm{i}\theta(z;x,t)} \\ 0 & 0 \end{bmatrix}$$



where $\theta(z;x,t) = zx + 4tz^3$ and the norming constants $c_j \in \mathbb{R}_+$;

3. $\boldsymbol{m}(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\dfrac{1}{z}\right)$ as $z \to \infty$,

Ansatz:

$$\boldsymbol{m}(z;x,t) = \Big[1 + \sum_{j=1}^{N} \frac{\mathrm{i}\alpha_j(x,t)}{z-\mathrm{i}\kappa_j}\,, \quad 1 - \sum_{j=1}^{N} \frac{\mathrm{i}\alpha_j(x,t)}{z+\mathrm{i}\kappa_j}\Big]$$

Then,

$$\begin{aligned} q_N(x,t) &= 2\mathrm{i}\,\partial_x\Big(\lim_{z\to\infty} z\,(\boldsymbol{m}_1(z;x,t)-1)\Big) \\ &= -2\partial_x^2\Big(\log\det(\boldsymbol{I}+\boldsymbol{A})\Big) \qquad \text{[Kay, Moses, '56]} \end{aligned}$$

with $\boldsymbol{A} \in \mathbb{R}^{N\times N}$,

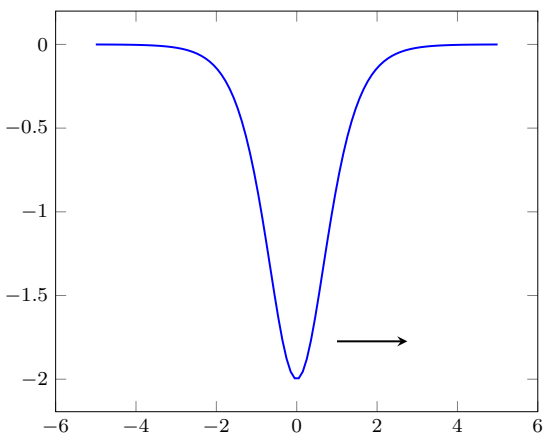
$$\boldsymbol{A}_{j\ell} = \frac{\sqrt{c_j}\sqrt{c_\ell}\,e^{-\mathrm{i}\left[\theta(\mathrm{i}\kappa_j;x,t)+\theta(\mathrm{i}\kappa_\ell;x,t)\right]}}{\kappa_j+\kappa_\ell}\,.$$

Example: $N = 1$

$$q_{\text{sol}}(x,t) = -2\kappa^2 \operatorname{sech}^2\left(2\kappa(x - 4\kappa^2 t) + x_0\right)$$

with initial position of the peak

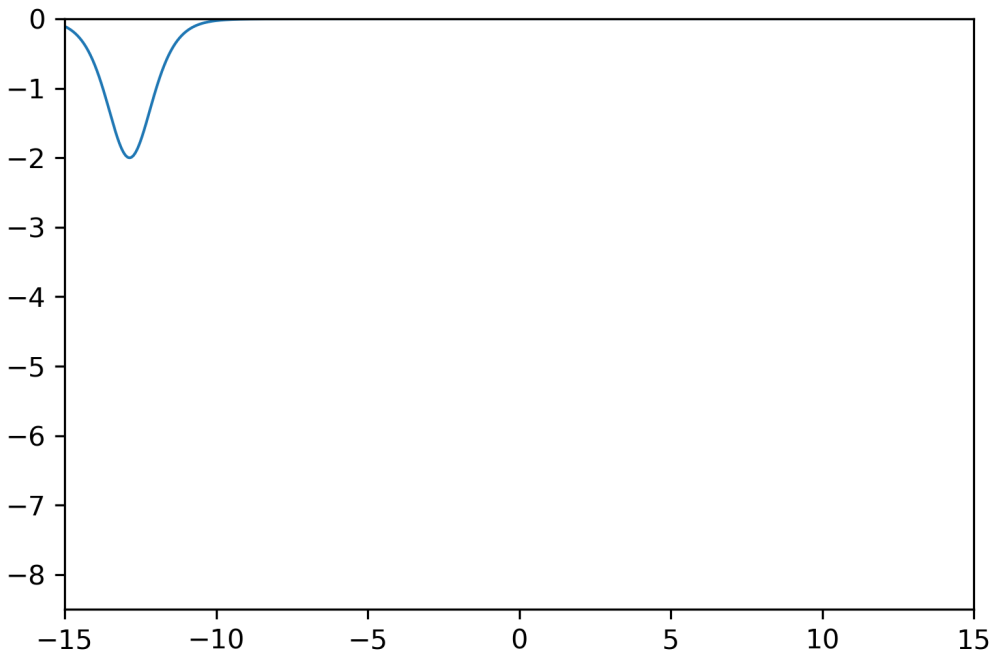
$$x_0 = \frac{1}{2\kappa} \log \frac{2\kappa}{c} \ .$$



Note: the wave solution can move only to the right; also, larger solitons move faster than smaller ones.

Example: $N = 2$, $\kappa_1 < \kappa_2$

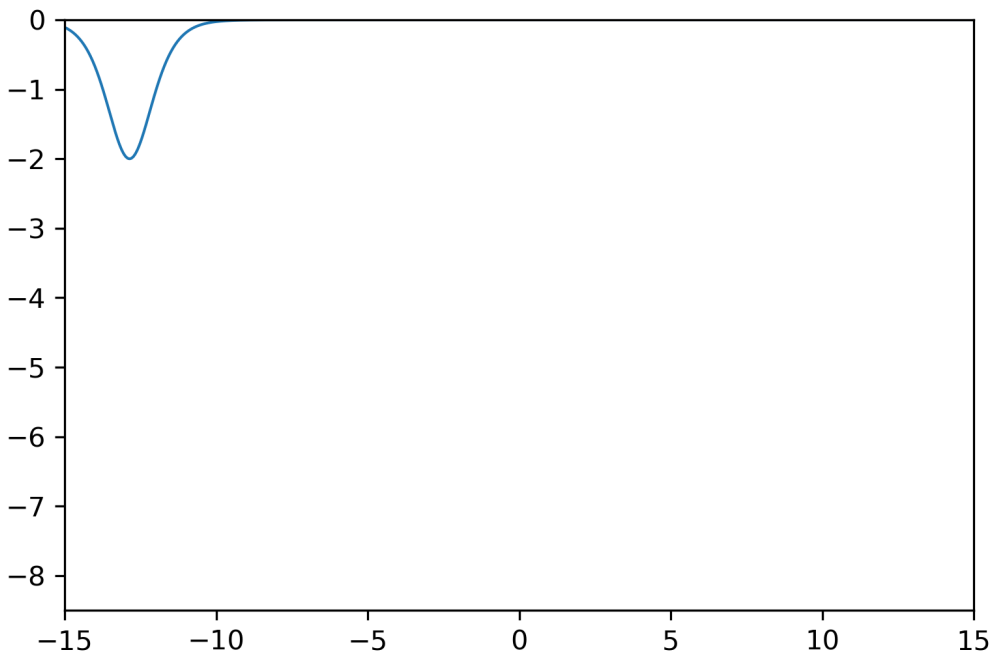
$$q_2(x,t) = \frac{\left(4\kappa_1^2 - 4\kappa_2^2\right)\left[2\kappa_1 \cosh\left(2\kappa_2(x - 4\kappa_2^2 t)\right) + 2\kappa_2 \cosh\left(2\kappa_1(x - 4\kappa_1^2 t)\right)\right]}{16\kappa_1\kappa_2 + (2\kappa_1 - 2\kappa_2)^2 \cosh\left(2\kappa_1(x - 4\kappa_1^2 t) + 2\kappa_2(x - 4\kappa_2^2 t)\right) + (2\kappa_1 + 2\kappa_2)^2 \cosh\left(2\kappa_1(x - 4\kappa_1^2 t) - 2\kappa_2(x - 4\kappa_2^2 t)\right)}$$



Example: $N = 2$, $\kappa_1 < \kappa_2$

$$q_2(x,t) \approx -12\kappa_1^2 \operatorname{sech}^2\left(\kappa_1(x-4\kappa_1^2 t)\right) - 12\kappa_2^2 \operatorname{sech}^2\left(\kappa_2(x-4\kappa_2^2 t)\right)\,, \qquad t \to -\infty\,,$$

$$q_2(x,t) \approx -12\kappa_1^2 \operatorname{sech}^2\left(\kappa_1(x-4\kappa_1^2 t) + \frac{1}{\kappa_1}\ln\left|\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right|\right)$$
$$-12\kappa_2^2 \operatorname{sech}^2\left(\kappa_2(x-4\kappa_2^2 t) - \frac{1}{\kappa_2}\ln\left|\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right|\right)\,, \quad t \to +\infty\,.$$



What's so special about solitons?

- Solitons = localized travelling wave solutions, fundamental solutions to the KdV equation.

- Solitons = discrete eigenvalues of the operator $\mathcal{L}$:

$$\begin{aligned} \{\text{velocity, amplitude}\} &\quad\Leftrightarrow\quad \kappa_j \\ \{\text{position}\} &\quad\Leftrightarrow\quad c_j \end{aligned}$$

They arise in the long-time behaviour!
The reflection coefficient $\rho(z)$ corresponds to a radiative part that propagates to the left and decays at rate t^{-1}.

- Soliton interaction is elastic ($\Leftrightarrow$ particles)

solitons "survive" collisions [Zabusky-Kruskal, '65]

$$q(x,t) = \sum_{j=1}^{N} q_{\text{sol,j}}(x + \delta_j^{\pm}, t) + o(1) \qquad \text{as } t \to \pm\infty$$

$$\delta_j^{\pm} \sim \sum_{\ell} \frac{\sigma_{j\ell}}{\kappa_j} \log \left| \frac{\kappa_j + \kappa_\ell}{\kappa_j - \kappa_\ell} \right|$$

Goals

1. *Study the behaviour of an N-soliton solution $q_N(x,t)$ in the limit as $N \to \infty$. We call such solution $q(x,t)$ a* soliton gas.

   → *Full description of a regular, dense soliton gas in the large time regime, over the whole spatial domain.*
   [G., Grava, Jenkins, McLaughlin, '21]
   [G., Grava, Jenkins, McLaughlin, Minakov, '23]

2. *Consider random configurations of N solitons and study the behaviour of the random variable*

$$q_N(x,t) - q(x,t)$$

   as $N \to \infty$.

   → *The gas is deterministic, but it is the universal limit of a large class of random N-soliton potentials over non-compact sets of (x,t).*
   [G., Grava, McLaughlin, Najnudel, '26+]

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What is a soliton gas? [Zakharov, '71]

Let

$$f_\lambda(x,t)\,\mathrm{d}\lambda\,\mathrm{d}x = \left\{ \begin{array}{c} \text{number of solitons with the spectral parameter } (\lambda, \lambda+\mathrm{d}\lambda) \\ \text{located in the spatial interval } (x, x+\mathrm{d}x) \text{ at time } t \end{array} \right\}$$

Definition

A **soliton gas**/soliton ensemble is an infinite collection of solitons (randomly) distributed on $\mathbb{R}$ with non-zero (physical) density

$$\varrho(x,t) = \int_\Sigma f_\lambda(x,t)\,\mathrm{d}\lambda.$$

The nonlinear wave field $q(x,t)$ is called **integrable soliton turbulence**.

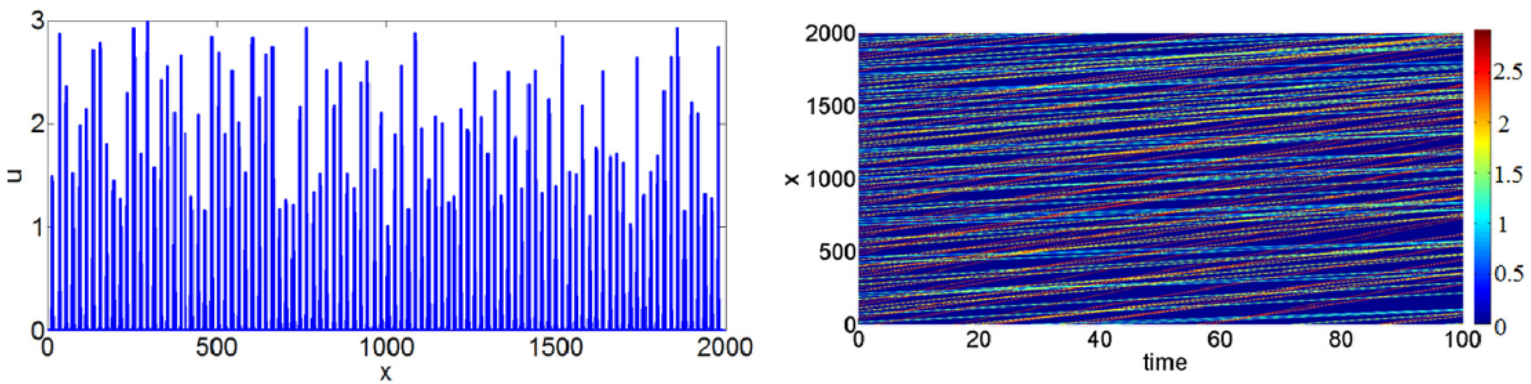


Figure: Initial distribution of the soliton gas (left). (t,x)-diagram of soliton field (right). From [Shurgalina, Pelinovski, '16].

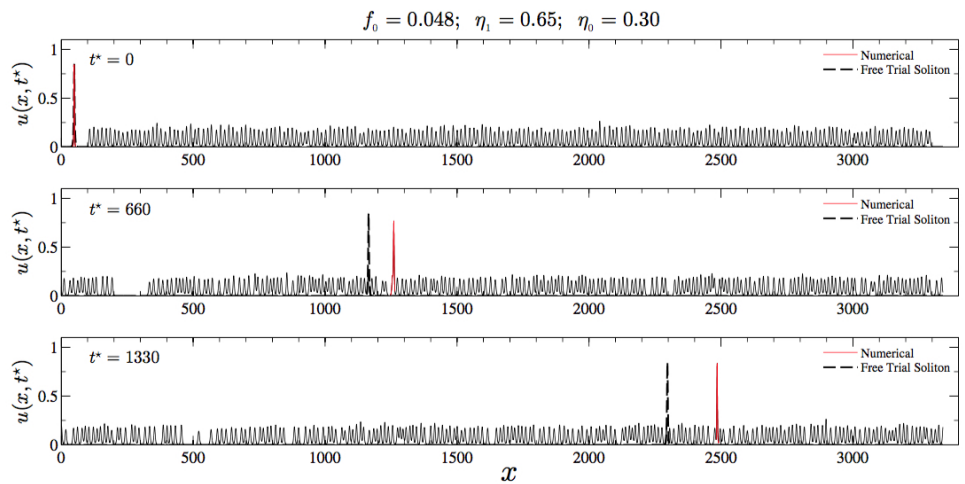
Formulation of a soliton gas

1. **Continuum limit of solitons:** gas solutions belong to the closure of the set of multisoliton potentials
[Marchenko '88–'91], [Zaitsev, Whitham, '83], [Boyd, '84], [Gesztesy, Karwowski, Zhao, '92]
2. **Kinetic theory:** soliton gas as a special large genus (thermodynamic) limit of a finite gap (N-phase nonlinear wave) solution to the PDE

and a trial soliton velocity satisfies kinetic+continuity equations

$$v(\kappa) = 4\kappa^2 + \frac{1}{\kappa}\int_0^\infty \ln\left|\frac{s+\kappa}{s-\kappa}\right| (v(\kappa) - v(s))\, f(s;x,t)\, \mathrm{d}s\ , \qquad f_t + (vf)_x = 0$$

[Zakharov, '71], [El, '03], [El-Tovbis, '20]



From [Carbone, Dutyk, El, '16].

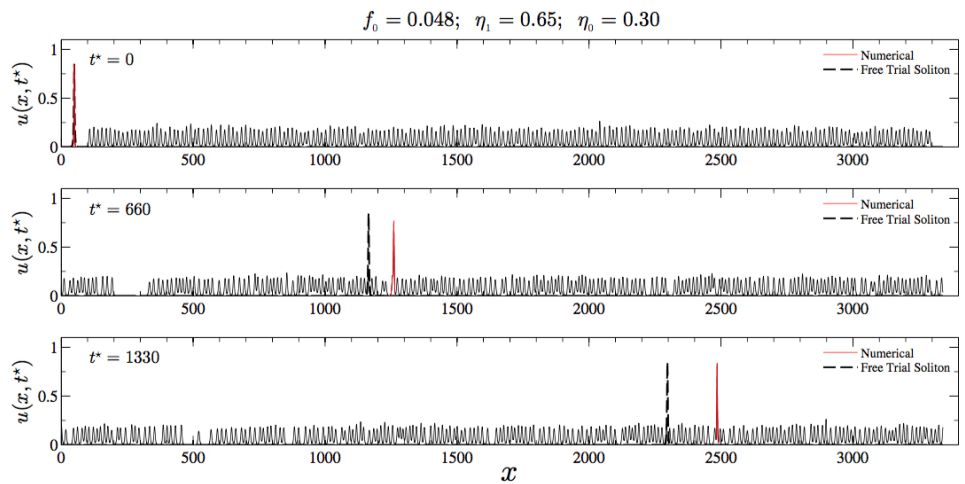
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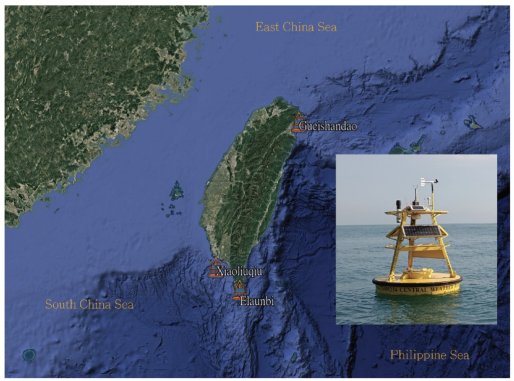
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Motivations

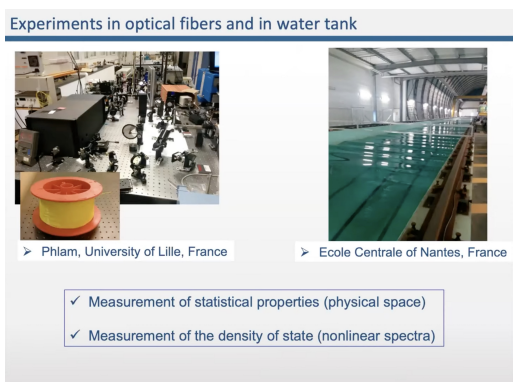
- *real-life observation* in lagoons (Currituck Sound, NC [Costa *et al.*, '14] and Laguna Càhuil, Chile [Cienfuegos *et al.*, '25]) and deep ocean (Taiwan waters [Lee, Wahls, '25])

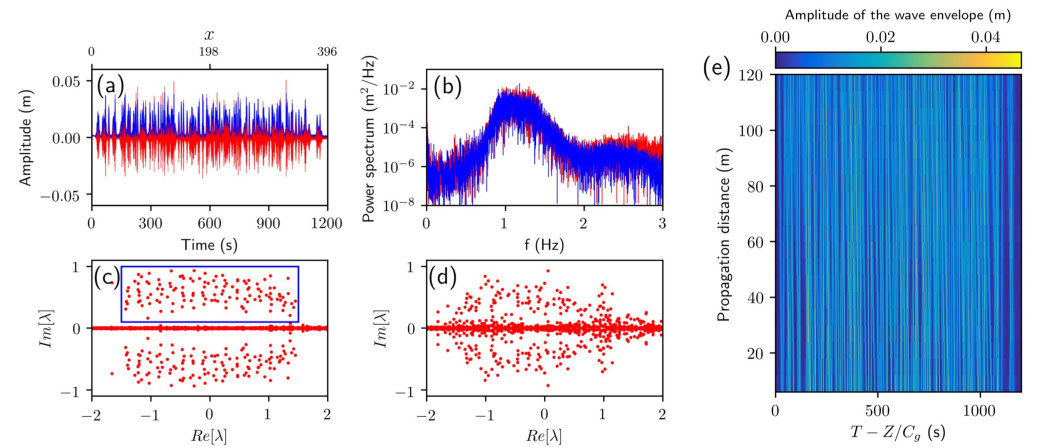






- *experimental observation* in water waves [Costa *et al.*, '14; Redor *et al.*, '19; Suret *et al.*, '20; ...] and in optics [Marcucci *et al.*, '19; ...]





- *numerical studies* [Gelash, Agafontsev, '18]. [Bonnemain, Congy, El, Roberti *et al.*, '18–'26], [Pelinovsky, Didenkulova *et al.*, '16–'24], etc.

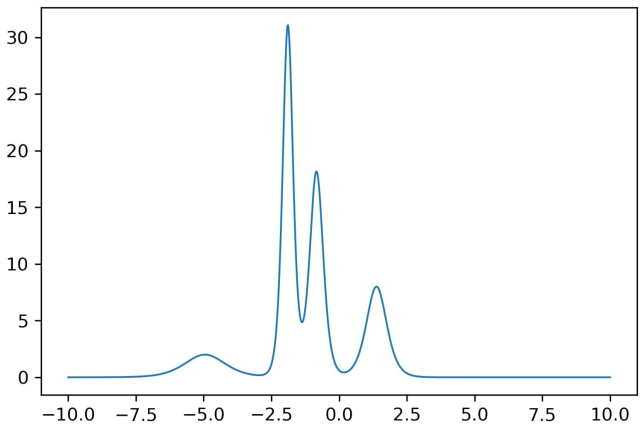
Soliton gas as continuum limit of multi-soliton potentials

Given the scattering data for $q_t - 6qq_x + q_{xxx} = 0$

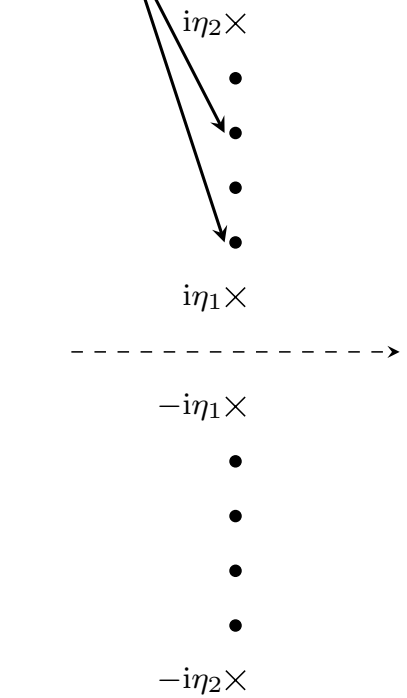
$$\mathcal{S} = \left\{ \{\mathrm{i}\kappa_j\}_{j=1}^N \, , \{\mathrm{i}c_j\}_{j=1}^N \right\}$$

the solution is

$$q_N(x,t) = 2\mathrm{i}\partial_x \Big(\lim_{z\to\infty} z \left(\boldsymbol{m}_{N;1}(z;x,t) - 1 \right) \Big)$$



$$\operatorname*{res}_{z=\mathrm{i}\kappa_j} \boldsymbol{m}_N = \lim_{z\to\mathrm{i}\kappa_j} \boldsymbol{m}_N(z) \begin{bmatrix} 0 & 0 \\ \mathrm{i}c_j \, \mathrm{e}^{2\mathrm{i}\theta(z;x,t)} & 0 \end{bmatrix}$$



$$\theta = xz + 4tz^3$$

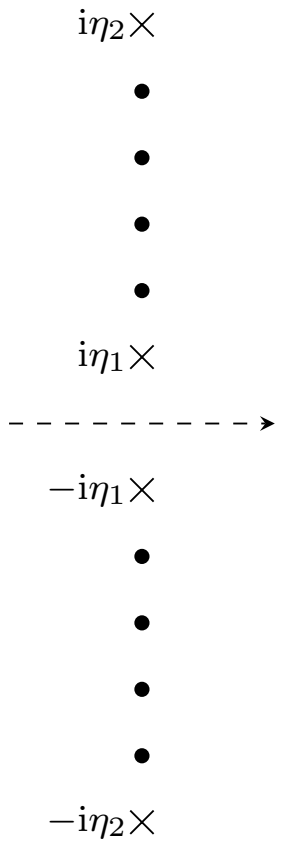
Take the limit $N \to +\infty$ (for (x,t) bounded):

- Assume the eigenvalues are filling a bounded interval $[\mathrm{i}\eta_1, \mathrm{i}\eta_2]$
- Rescale and sample:

$$c_j \mapsto \frac{c_j}{N} = \frac{r(\kappa_j)}{N}$$

for some smooth function $r(z)$ (sufficiently regular).

This N-soliton solution has *very broad* support.
For $N \gg 1$, it decays (exponentially) only for $|x| > CN$ for some $C \in \mathbb{R}_+$.

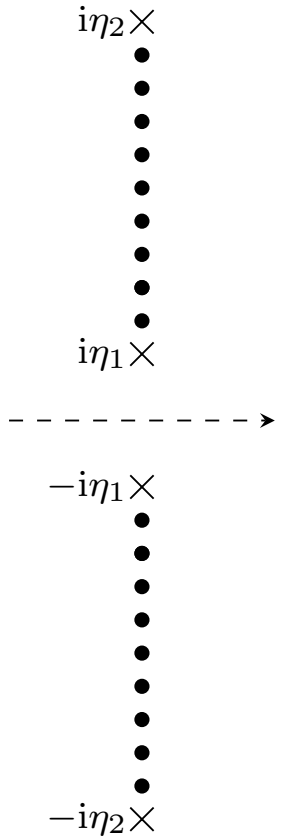


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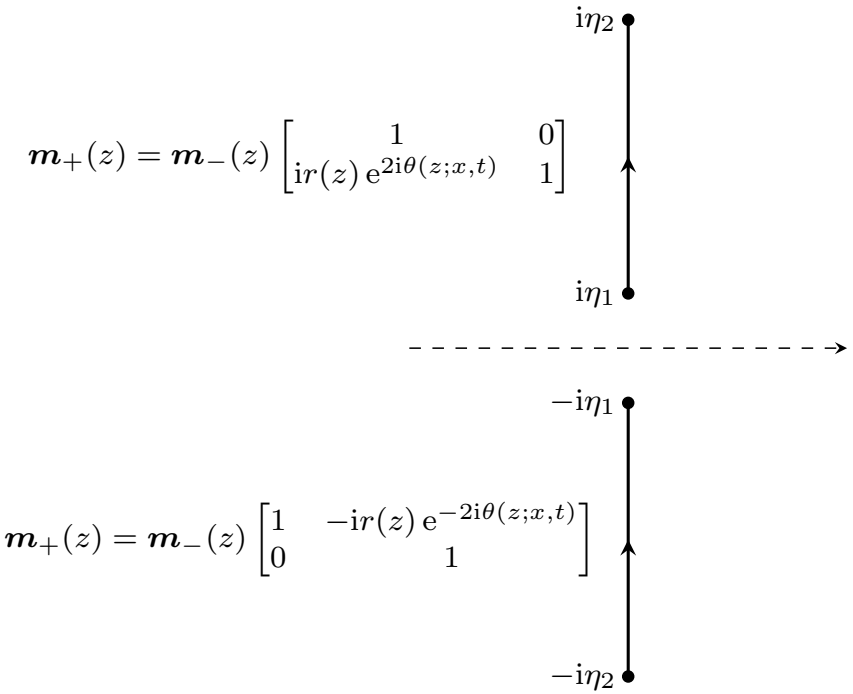
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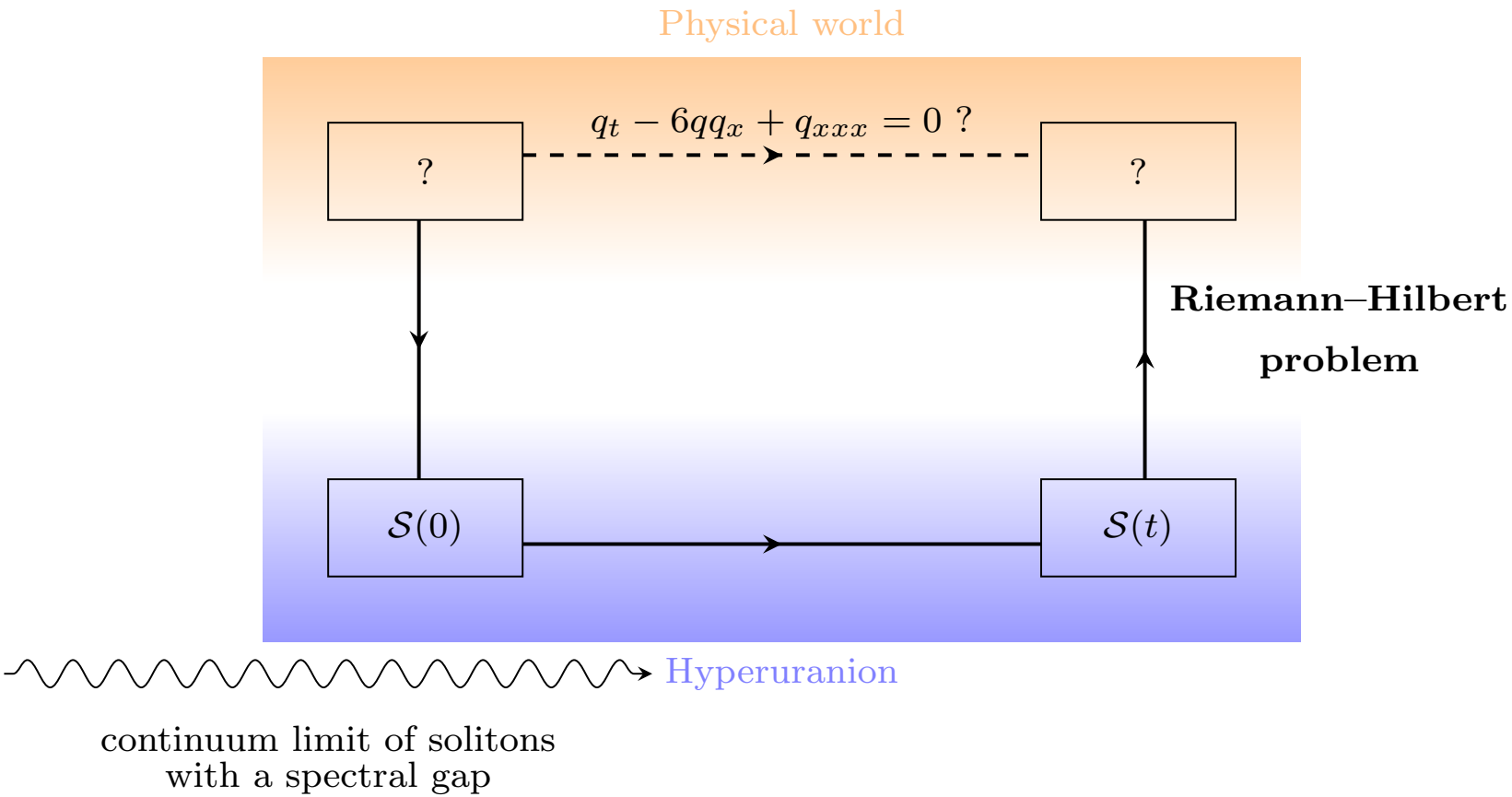


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For $N \gg 1$, it decays (exponentially) only for $|x| > CN$ for some $C \in \mathbb{R}_+$.

The limiting RH problem: $\boldsymbol{m}(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \dfrac{\boldsymbol{m}^{(1)}}{z} + \mathcal{O}(z^{-1}) \qquad k \to \infty$



$$\boxed{\theta = xz + 4tz^3}$$



$$q(x,t) = 2\mathrm{i}\,\partial_x \lim_{z\to\infty} z\,(\boldsymbol{m}_1(z;x,t) - 1) \quad ?$$

The free gas

1. Existence of a KdV solution [G., Grava, Jenkins, McLaughlin, '21; G., Grava, Jenkins, McLaughlin, Minakov, '23]:

$$\begin{aligned} q(x,t) &= 2\mathrm{i}\,\partial_x \lim_{z\to\infty} z\,(\boldsymbol{m}_1(z;x,t)-1) \\ &= -2\partial_x^2 \ln\det\left(\mathrm{Id}_{L^2([\mathrm{i}\eta_1,\mathrm{i}\eta_2])}+\mathcal{K}\right) \end{aligned}$$

where the integral operator $\mathcal{K}$ has kernel

$$K(k,z)=\frac{\sqrt{r(k)}\sqrt{r(z)}\,e^{\mathrm{i}\,(\theta(k;x,t)+\theta(z;x,t))}}{2\pi\mathrm{i}(k+z)}, \qquad k,z\in[\mathrm{i}\eta_1,\mathrm{i}\eta_2]\ .$$

$$q(x,t) = -2\partial_x^2 \ln\det\left(\mathrm{Id}_{L^2([\mathrm{i}\eta_1,\mathrm{i}\eta_2])} + \mathcal{K}\right)$$

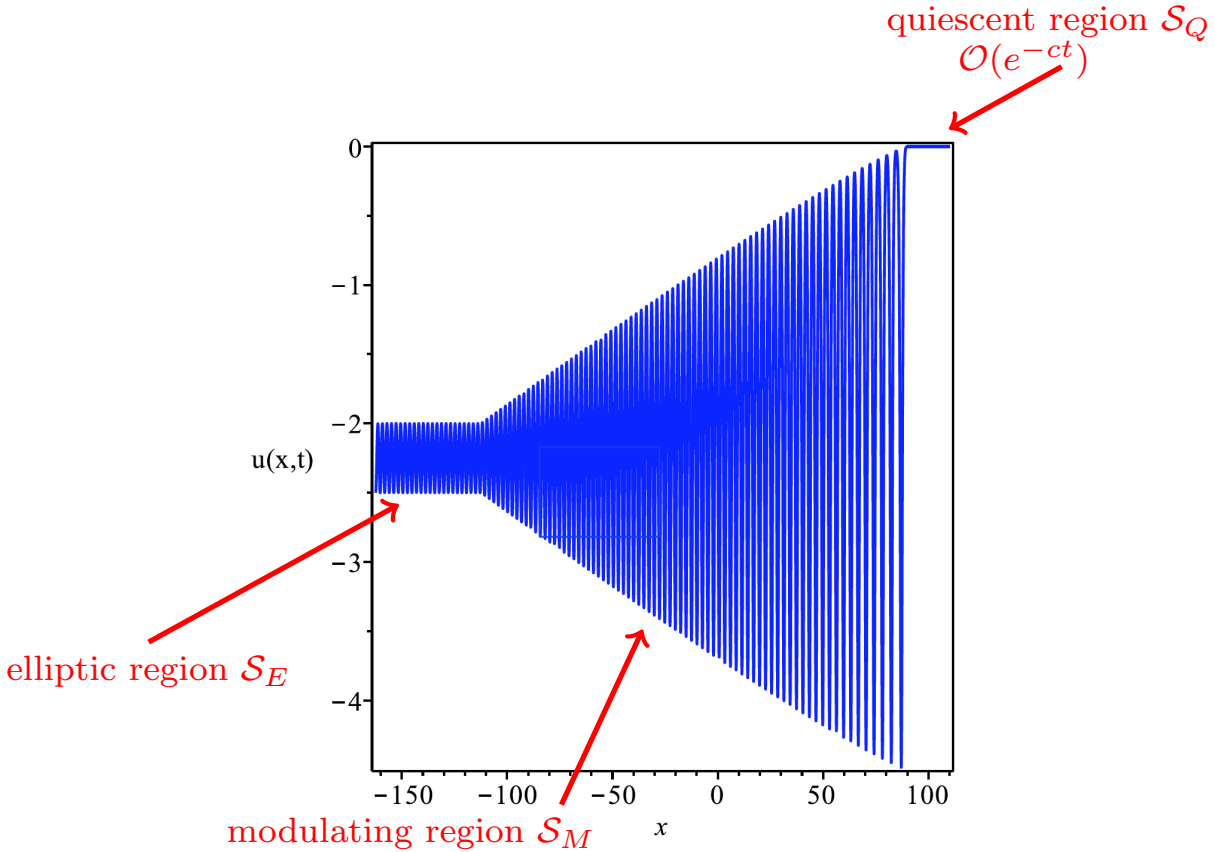
$$K(k,z) = \frac{\sqrt{r(k)}\sqrt{r(z)}\, e^{\mathrm{i}\,(\theta(k;x,t)+\theta(z;x,t))}}{2\pi\mathrm{i}(k+z)}, \qquad k,z \in [\mathrm{i}\eta_1, \mathrm{i}\eta_2]\ .$$

- this gas is *smooth* and *dense*, and *deterministic*.
- it belongs to a sub-class of *primitive/Bargmann potentials* [Dyachenko, Nabelek, Zakharov, Zakharov, '16–'20].
- similar expressions have been derived in
  - [Tracy, Widom, '96] for the solution of the defocusing mKDV.
  - [Baik, Bothner, '20], [Krajenbrink, Le Doussal, '21] for solving the weak noise theory of the KPZ equation
  - [Bothner,'22] in the study of Fredholm determinants of a class of Hankel composition operators.

3. "Step-like" initial profile:

$$q(x,0) = \mathcal{O}(e^{-C|x|}) \qquad \text{as } x \to +\infty \ ,$$
$$q(x,0) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \,\mathrm{dn}^2(\eta_2(x+\phi) + K(m), m) + \mathcal{O}(x^{-1}) \ , \qquad \text{as } x \to -\infty \ .$$

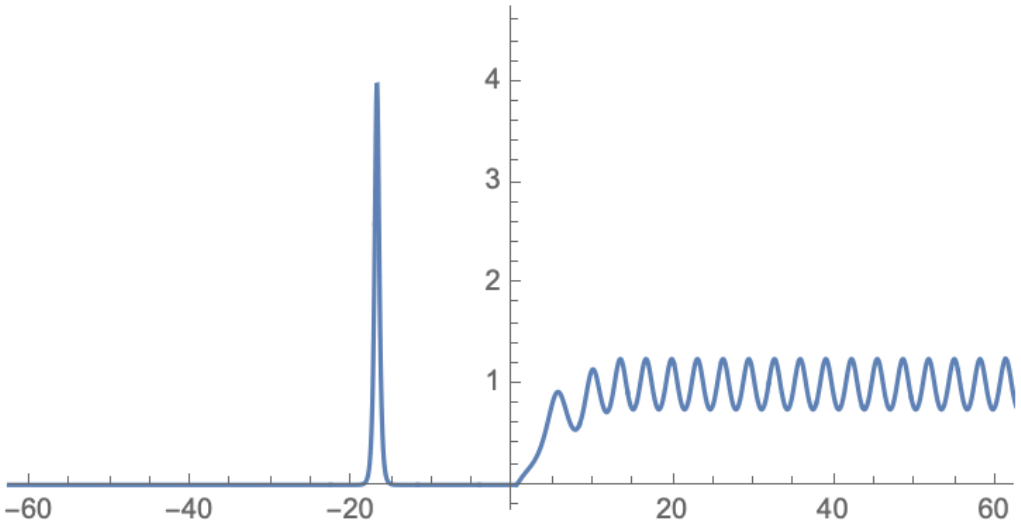
4. Long-time asymptotic analysis [G., Grava, Jenkins, McLaughlin, '21]:



$$q_{\mathrm{ell}}(x,t) = \eta_2^2 - \alpha^2 - 2\eta_2^2 \,\mathrm{dn}^2\left(\eta_2(x - 2(\alpha^2+\eta_2^2)t + \widetilde{\phi}) + K(m_\alpha) \mid m_\alpha\right) + \mathcal{O}(t^{-1})$$

The soliton VS the gas

4. Gas+trial soliton long-time asymptotics (for mKdV; [G., Grava, Jenkins, McLaughlin, Minakov, '23]):



At large time, the solution decouples

$$q(x,t) = q_{\mathrm{bg}}(x,t) + q_{\mathrm{sol}}(x,t) + \mathcal{O}\left(t^{-1}\right),$$

where

$$q_{\mathrm{bg}}(x,t) = (\alpha+\eta_1)\,\mathrm{dn}\left((\alpha+\eta_1)(x-2(\eta_1^2+\alpha^2)t - x^{(\pm)}) \mid m_1\right),$$

$m_1 = \frac{4\alpha\eta_1}{(\alpha+\eta_1)^2}$, and $q_{\mathrm{sol}}(x,t)$ is a localised profile.

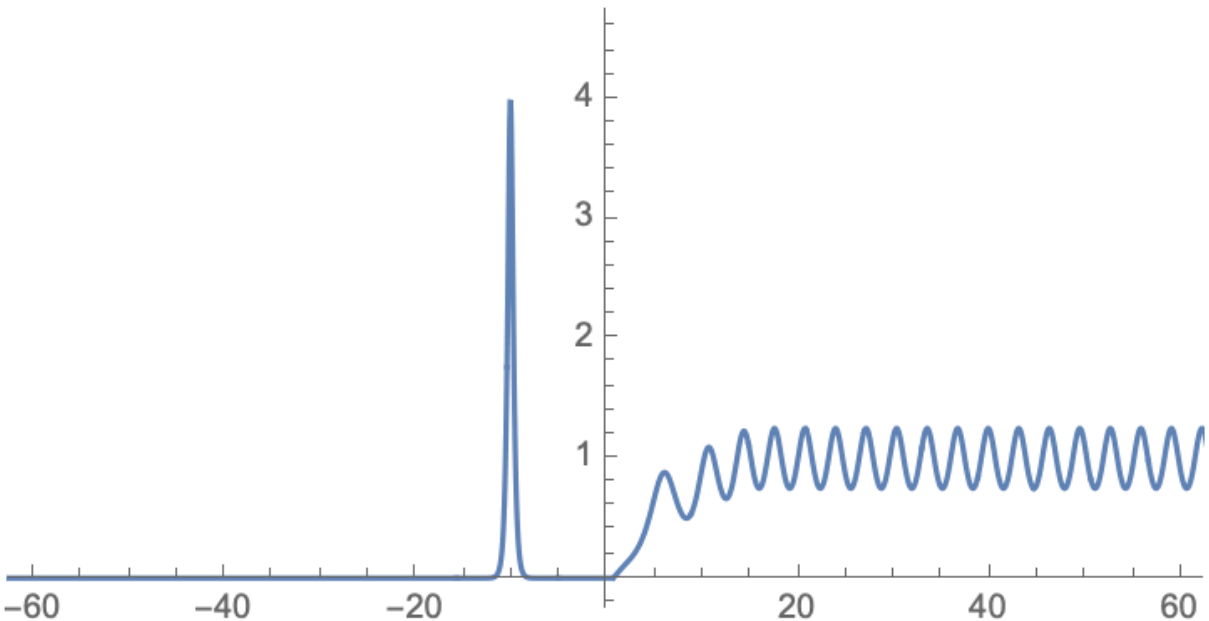
4. Dynamics [G., Grava, Jenkins, McLaughlin, Minakov, '23]:

$$\dot{x}_{\text{peak}} = -\frac{2\varphi_t(\mathrm{i}\kappa_0) - \partial_t \ln \Psi(x,t;\kappa_0,\eta_1)}{2\varphi_x(\mathrm{i}\kappa_0) - \partial_x \ln \Psi(x,t;\kappa_0,\eta_1)}\Big|_{x=x_{\text{peak}}(t)} + \mathcal{O}\left(t^{-1}\right)$$

and

$$\bar{v}_{\text{sol}}(\kappa_0) = 4\kappa_0^2 + \frac{1}{\kappa_0}\int_{\eta_1}^{\alpha} \ln\left|\frac{\kappa_0 - s}{\kappa_0 + s}\right| (v_{\text{group}}(s) - \bar{v}_{\text{sol}}(\kappa_0))\, \partial_x \varrho(\mathrm{i}s)\, \mathrm{d}s \ .$$

where $\bar{v}_{\text{sol}}$ is the *average velocity* of the soliton peak over one period of the background.
This is the kinetic equation posited by [El *el al.*, '03+]!



Some ingredients of the proof

The core of the RH problem asymptotic analysis lies on

1. **Small Norm Theorem**: if we have a RHP of the type

$$\boldsymbol{X}_+(z) = \boldsymbol{X}_-(z) \underbrace{\left(\boldsymbol{I} + \{\text{small terms}\}\right)}_{\text{(almost) no jumps}} \qquad \text{on the contours}$$

$$\boldsymbol{X}(z) = \boldsymbol{I} + \mathcal{O}\left(z^{-1}\right) \qquad z \to \infty.$$

then, we can conclude that the solution is $\boldsymbol{X} = \boldsymbol{I} + \{\text{small terms}\}$ (the approximation can be explicitly estimated!).

2. **Deift–Zhou Steepest Descent method**: perform a sequence of invertible transformations of the original RH problem $\boldsymbol{X}$

$$\boldsymbol{X} \mapsto \boldsymbol{T} \mapsto \boldsymbol{U} \mapsto \ldots \mapsto \boldsymbol{S}$$

in such away that, in the appropriate regime, the final RH problem $\boldsymbol{S}$ can be solved by an approximating solution $\boldsymbol{W}$ (the "model problem"):

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The phase in the jumps

$$\theta(z;x,t) = xz + 4z^3 t = tz\left(4z^2 + \frac{x}{t}\right)$$

shows different sign.

There are three main domains:

- $4\eta_2^2 < \frac{x}{t}$: the phases are exponentially decaying as $t \to +\infty$, therefore (by a small-norm argument)

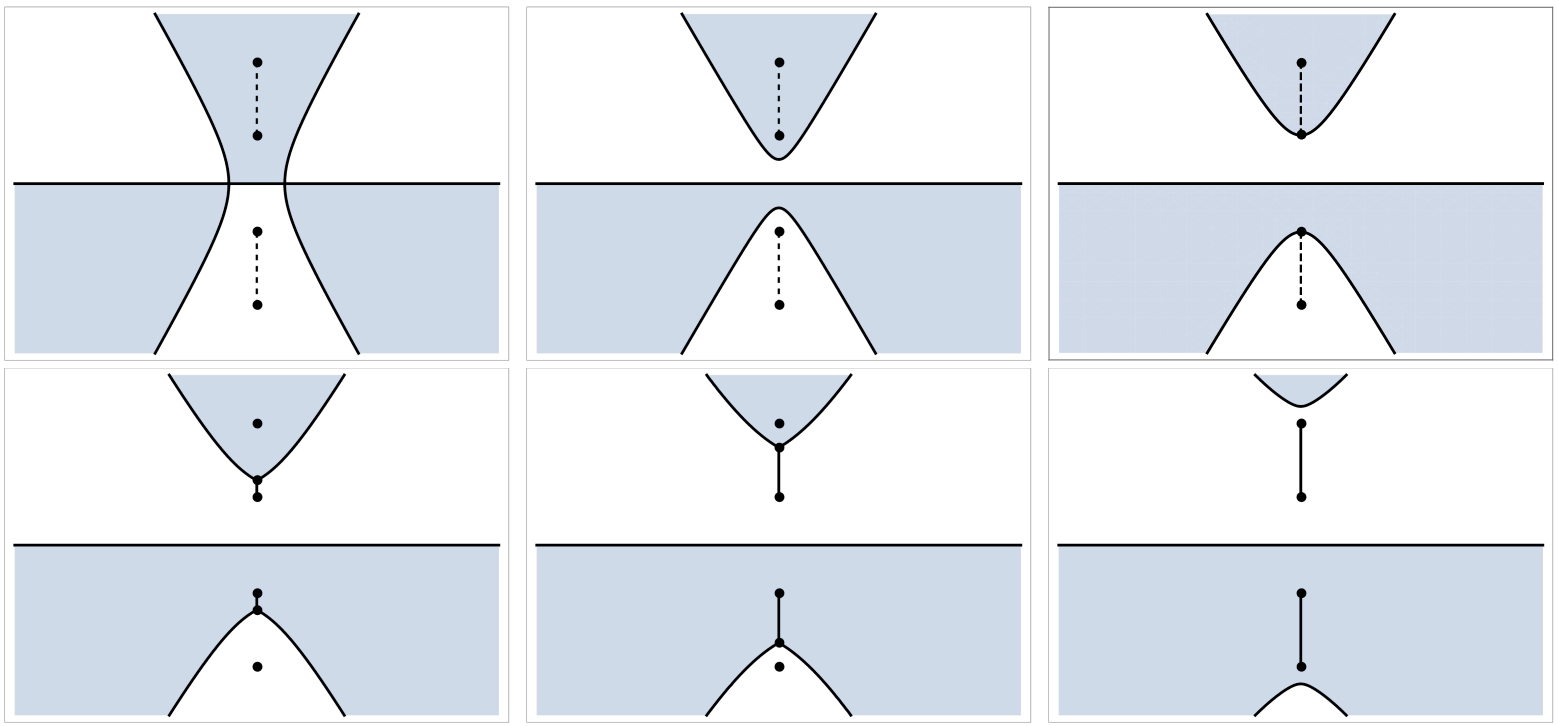
$$q(x,t) = \mathcal{O}\left(\mathrm{e}^{-Ct}\right) \qquad (\textit{quiescent region } \mathcal{S}_Q).$$

- $v_{\text{crit}} < \frac{x}{t} < 4\eta_2^2$: $q(x,t)$ is a periodic travelling wave with slowly varying parameter $\alpha = \alpha(x/t)$, solving the Whitham modulation equations (*modulated region* $\mathcal{S}_M$).
- $\frac{x}{t} < v_{\text{crit}}$: $q(x,t)$ is a periodic travelling wave with fixed parameter $\alpha \equiv \eta_1$ (*elliptic region* $\mathcal{S}_E$).

In the modulated and elliptic regions, the asymptotic is dictated by the wave phase

$$\begin{aligned} \varphi(k;x,t) &= g(z;x,t) + zx + 4z^3 t \ , \\ g(z;x,t) &= \int_{\mathrm{i}\eta_2}^{\mathrm{i}\alpha} \log\left(\frac{z-s}{z+s}\right) \varrho(s;x,t)\mathrm{d}s \ . \end{aligned}$$

and ϱ satisfies a variational equation (density of states).



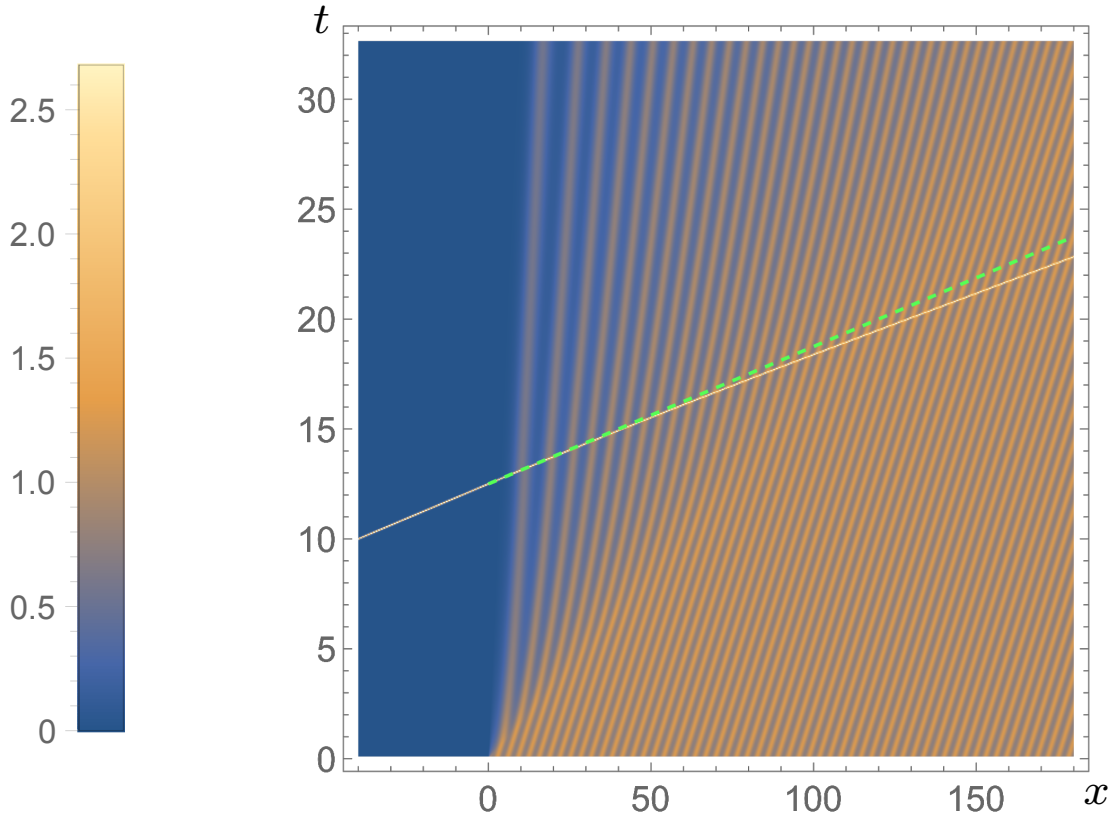


Figure: Color map plotting the leading order behaviour of a trial soliton traveling through a soliton gas. The dashed green line shows the position the trial soliton in a vacuum. Here $r \equiv 1$, $\eta_1 = 0.25$, $\eta_2 = 1$, $\kappa_0 = 2$, and $\chi = 4e^{-800}$.

Note: check out also [Ballew, Bilman, Trogdon, '25] for cool videos!

Overlook so far

- Full description of the soliton gas in the large time regime, over the whole spatial domain.
- Rigorous description of how a particle (= soliton) interacts with a complex medium (= gas): how the medium affects the propagation of the particle (the soliton velocity) and how the particle affects the evolution of the complex medium (the phase shift in the soliton gas).
- This -regular, dense- gas is *deterministic*!
  However, the interaction dynamics is governed by the same kinetic equations of El, *et al.*
- Next → introduce randomness

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Random initial data

Consider N-soliton solutions to the KdV equation $q_t - 6qq_x + q_{xxx} = 0$ with *random* initial data.

Note: given $q_N(x,t)$ random, in general $\mathbb{E}[q_N(x,t)]$ does *not* solve KdV!

On the other hand, given *random scattering data* $\mathcal{S}(t)$, we consider its (deterministic, limiting) *expectation*

$$\lim_{N\to\infty} \mathbb{E}[\mathcal{S}(t)]$$

which still solves the linear spectral problem and defines a KdV solution (the gas).

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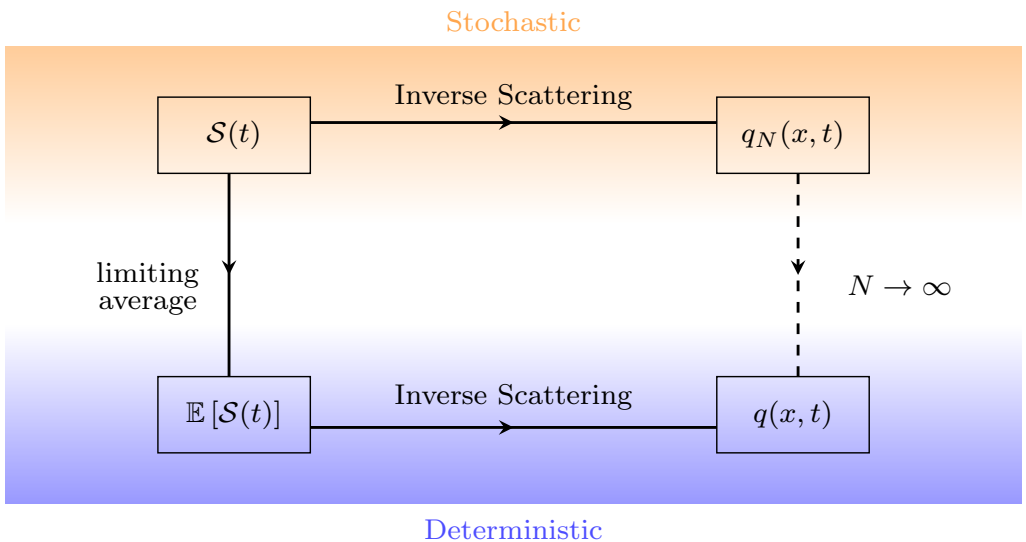
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The randomness is introduced directly into the spectral data:

- The poles/solitons $\{\mathrm{i}\kappa_j\}$ are sampled from a given distribution

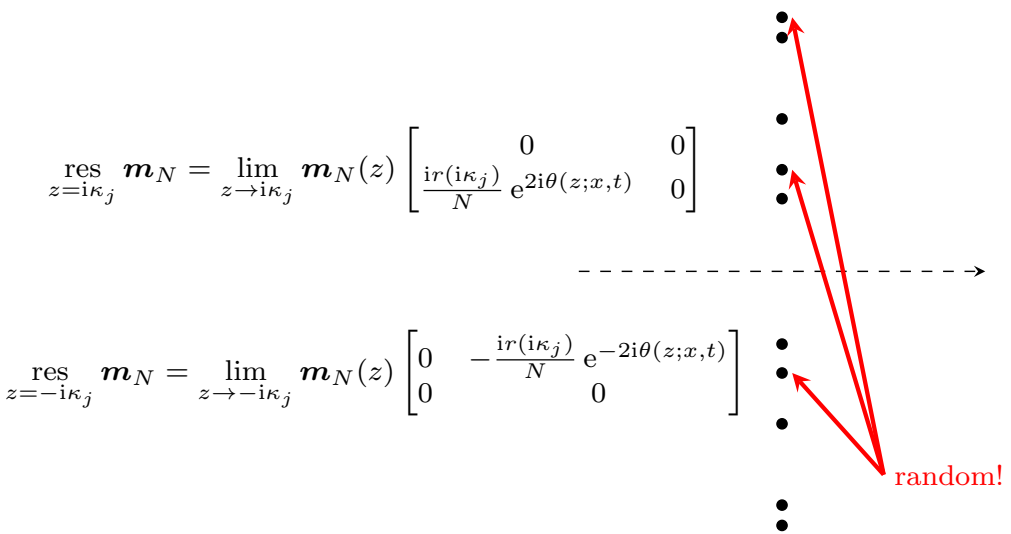
$$\{\kappa_j\} \sim \mathcal{D}$$

  (extended to $\mathrm{i}\mathbb{R}_-$ by symmetry).

- The norming constants are determined by the $\mathrm{i}\kappa_j$'s, sampled from a smooth function

$$c_j = \frac{r(\mathrm{i}\kappa_j)}{N}$$

  (their distribution is correlated to $\mathcal{D}$).



Which distribution can we choose? (part I)

Consider an N-soliton solution q_N to the KdV equation $q_t - 6qq_x + q_{xxx} = 0$.

The set of N-soliton solutions is a finite dimensional space

$$\mathcal{M} = \Big((\mathrm{i}\mathbb{R}_+)^N \setminus D \Big) \times \mathrm{i}\mathbb{R}_+^N$$

The Hamiltonian takes the form

$$H[q] = -\int_{\mathbb{R}} \left[q(x)^3 + \frac{1}{2} q(x)^2 \right] \mathrm{d}x = -\frac{32}{5} \sum_{j=1}^{N} (\kappa_j)^5$$

so that

$$q_t = \left(\frac{\delta H}{\delta q} \right)_x .$$

With the canonical variables $(\kappa_j^2, -2\log c_j)$, the dynamics is simply

$$\dot{\kappa}_j = 0 \qquad \dot{c}_j = 8 c_j (\kappa_j)^3$$

Despite the existence of infinitely many conserved quantities, there is no normalizable (Gibbs) invariant measure on the phase space of N-soliton solutions.

For example, the density

$$e^{-\beta H(\boldsymbol{\kappa})} \prod_{j=1}^{N} \mathrm{d}\kappa_j \ \mathrm{d}c_j \ ,$$

is not normalizable without a cutoff.

Note: [Forlano, Killip, Visan, '24] constructed dynamics with initial data sampled from the Gibbs measure for (m)KdV on infinite volume, and not-decaying IC.

Which distribution can we choose? (part II)

- *i.i.d.* poles, sampled from the uniform distribution on $[\mathrm{i}\eta_1, \mathrm{i}\eta_2]$



- *random matrix theory*/determinantal point process on $\mathrm{i}\mathbb{R}_+$:



e.g. Laguerre–Wishart ensemble

$$\frac{1}{Z_N} \prod_{j=1}^{N} x_j^{N\alpha} \prod_{i<j} |x_i - x_j|^2 e^{-\frac{\beta N}{2} \sum_{j=1}^{N} x_j} \, \mathrm{d}x_1 \ldots \mathrm{d}x_N$$

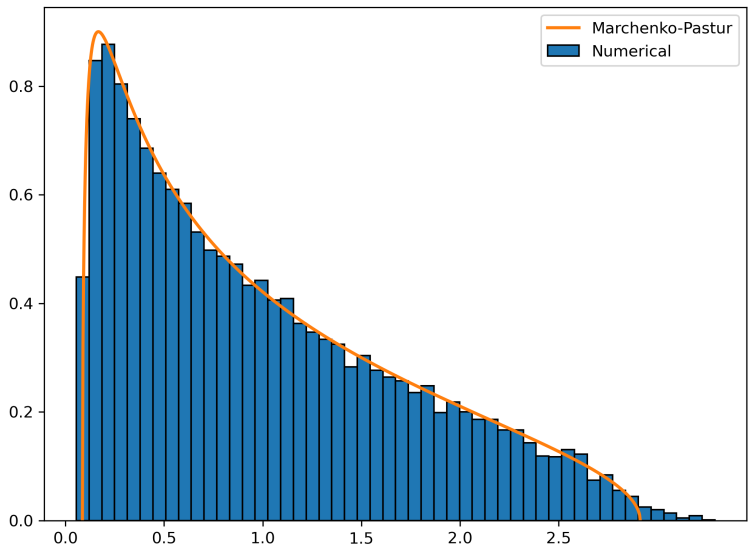
this is the joint probability density function of the eigenvalues of the Wishart random matrix ensemble

$$\frac{1}{N} \boldsymbol{Y}^\dagger \boldsymbol{Y} \ ,$$

$\boldsymbol{Y}$ is a $M \times N$ Gaussian random matrix (real or complex).

As $N \to \infty$, we obtain

- the uniform distribution over $[\mathrm{i}\eta_1, \mathrm{i}\eta_2]$
- a universal limiting distribution: Marchenko–Pastur law



$$\psi_{\mathrm{MP}} = \frac{\sqrt{(\eta_2 - s)(s - \eta_1)}}{2\pi\gamma s} \mathbf{1}_{[\eta_1, \eta_2]}(s)\, \mathrm{d}s$$

with $\eta_j = \left(1 + (-1)^j \sqrt{\gamma}\right)^2$.

To compare the two potentials $q_N - q$ we need to compare RH problems:

$$\boldsymbol{\mathcal{E}}(z) = \boldsymbol{m}_N(z)\boldsymbol{M}_\infty^{-1}(z)$$

where

- the finite-N RH problem $\boldsymbol{m}_N$ is *random*
- the limit-average RH problem $\boldsymbol{m}_\infty$ is the RH problem for the *gas*

$$q_N(x,t;\boldsymbol{\kappa}) - q(x,t) = \partial_x \lim_{z\to\infty} z^2 \boldsymbol{\mathcal{E}}_2(z;x,t)$$

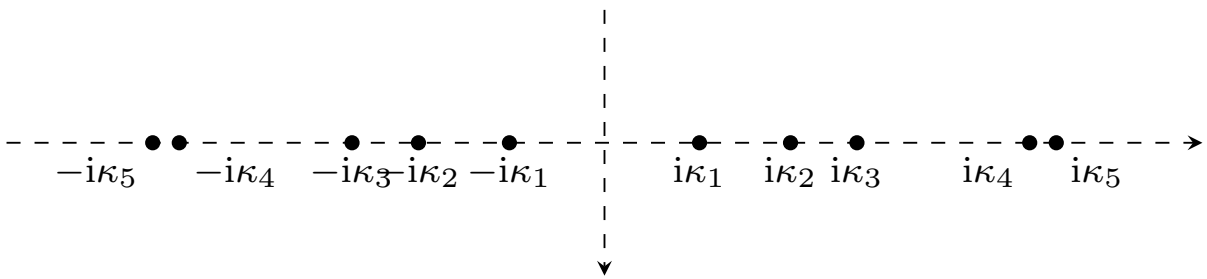
We use a *probabilistic* Small Norm Argument.

Roadmap (part 1):

1. Large deviation estimates (RMT case)

$$\mathbb{P}\left(\kappa_{\max}^{(N)} > 2\eta_2\right) = \mathcal{O}(e^{-c(\eta_2)\,N})\,, \quad \mathbb{P}\left(\kappa_{\min}^{(N)} < \tfrac{\eta_1}{2}\right) = \mathcal{O}(e^{-c(\eta_1)\,N})\,.$$

2. Transform the residue condition into a jump condition



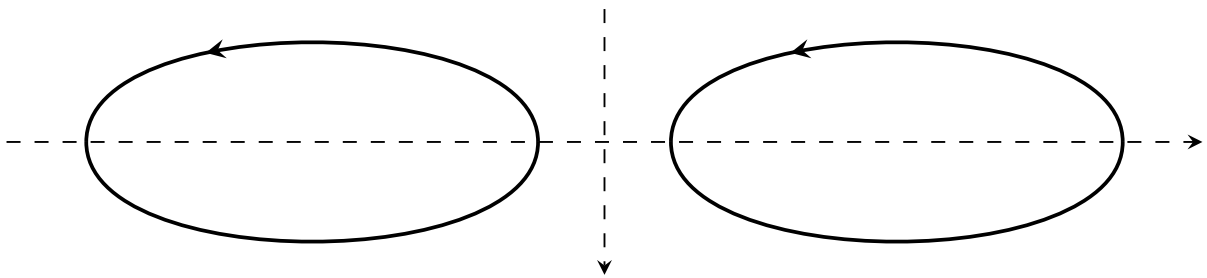
$$\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\frac{r(\mathrm{i}\kappa_j)e^{2\mathrm{i}\theta(z;x,t)}}{z-\mathrm{i}\kappa_j}\right] \longrightarrow e^{2\mathrm{i}\theta(z;x,t)}\int_{\mathrm{i}\eta_1}^{\mathrm{i}\eta_2}\frac{r(s)}{z-s}\mathrm{d}\mu_\infty(s)$$

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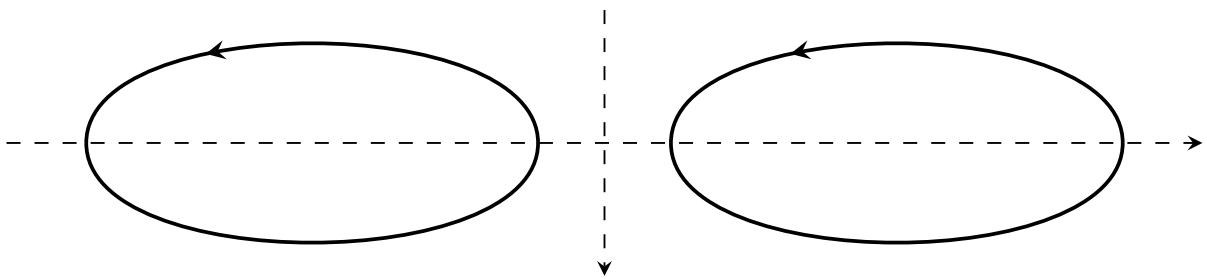
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so that

$$\boldsymbol{\mathcal{E}} = \begin{bmatrix}1 & 0\end{bmatrix} + \frac{\boldsymbol{\mathcal{E}}^{(1)}(x,t)}{z} + \frac{\boldsymbol{\mathcal{E}}^{(2)}(x,t)}{z^2} + \{\text{small random terms}\}$$

* the randomness in the jump is encoded in the quantity

$$X_N^f := \sum_{j=1}^{N} \frac{r(\kappa_j)}{z - \mathrm{i}\kappa_j} - N\int_{\mathrm{i}\eta_1}^{\mathrm{i}\eta_2} \frac{r(s)}{z-s}\mathrm{d}\mu_\infty(s)$$

Roadmap (part 2):

1. identify the probabilistic event where the RH problem $\boldsymbol{\mathcal{E}}$ is in a small-norm setting as $N \to \infty$:

$$\boldsymbol{\mathcal{E}}(z) = \boldsymbol{I} + \{\text{small terms}\}\ , \quad \text{implying} \quad q_N(x,t;\boldsymbol{\kappa}) - q(x,t) \to 0\ ;$$

2. the probability of the complement of the "good" event vanishes as $N \to \infty$.

LLN and CLT [G., Grava, McLaughlin, Najnudel, '26+]

Theorem (Convergence in L^1)

The limiting average of the random N-soliton solution is the KdV soliton gas:

$$\lim_{N\to\infty} \mathbb{E}\Big[\, |q_N(x,t;\boldsymbol{\kappa}) - q(x,t)| \,\Big] = 0 \ .$$

uniformly for x, t growing as $\mathcal{O}(\log N)$.

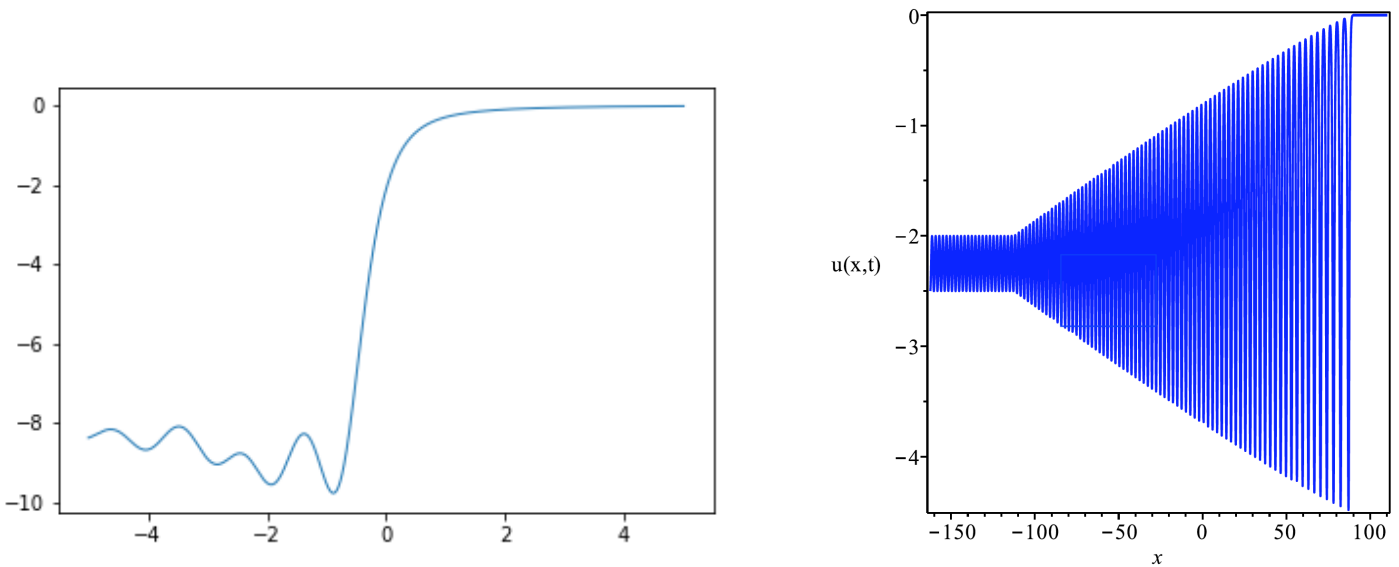


Figure: A 200-KdV-soliton solution with poles sampled from a Laguerre–Wishart distribution with $\gamma = \frac{1}{2}$ (left), and the limiting soliton condensate.

Theorem (Local fluctuations; mean field regime)

The centered random variable

$$N\left(q_N(x,t;\boldsymbol{\kappa}) - q(x,t)\right) = X_N^G(x,t) + \mathcal{O}_{\mathbb{P},N}(\epsilon)\ ,$$

where the linear statistics

$$X_N^G(x,t) = \sum_{j=1}^{N} G(\kappa_j;x,t) - N\int_\Sigma G(w;x,t)\mathrm{d}\mu_{\mathrm{MP}}(w)$$

$$G(z;x,t) = \partial_x\left(r(z)e^{2\mathrm{i}\theta(z;x,t)}\boldsymbol{m}^2_{\infty;2}(z;x,t)\right)$$

converges in distribution to a real Gaussian random variable with zero mean and prescribed variance, uniformly for x,t growing as $\mathcal{O}(\log N)$.

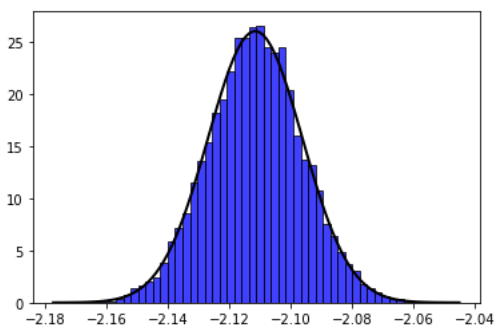


Figure: Histogram of $10,000$ realizations of a 200-KdV-soliton solution of evaluated at $(x,t) = (0,0)$, with poles sampled from a Laguerre–Wishart distribution with $\gamma = \frac{1}{2}$.

A heuristic interpretation of the function G (uniform distribution)

Given that $q_N(x,t) = -4\sum_{j=1}^{N} \kappa_j \psi^2(\mathrm{i}\kappa_j; x, t, \boldsymbol{\kappa})$, the empirical soliton density is

$$\rho_N(z;x,t) = \sum_{j=1}^{N} \delta(z - \mathrm{i}\kappa_j)\psi^2(z;x,t,\boldsymbol{\kappa}) = \frac{1}{N}\sum_{j=1}^{N} \delta(z-\mathrm{i}\kappa_j) r(z) \boldsymbol{m}^2_{N;2}(z;x,t,\boldsymbol{\kappa}) e^{2\mathrm{i}\theta(z;x,t)}$$

so that

$\rho_N(z;x,t)\mathrm{d}z\mathrm{d}x = \{\#$ of solitons in $(x, x+\mathrm{d}x)$ with spectral parameter in $(z, z+\mathrm{d}z)\}$

and

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \rho_N(z;x,t)\,\mathrm{d}z\mathrm{d}x = N\ .$$

Define the limiting average soliton density as

$$\rho_\infty(z;x,t) = \lim_{N\to\infty} \mathbb{E}\left[\rho_N(z;x,t)\right] = r(z)\boldsymbol{m}^2_{\infty;2}(z;x,t)e^{2\mathrm{i}\theta(z;x,t)}\ .$$

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(Technical) ingredients in the proofs:

- concentration estimates on the eigenvalues; → strong correlation among eigenvalues

- uniform bound on the N-potential [Gesztesy, Karwowski, Zhao, '92] and on the gas potential [G., Grava, Jenkins, McLaughlin, '23] → no rogue waves

$$\sup_{\mathbb{R}\times\mathbb{R}_+} |q_N(x,t;\boldsymbol{\kappa})| \leq 2\left(\kappa_{\max}^{(N)}\right)^2 , \qquad \sup_{\mathbb{R}\times\mathbb{R}_+} |q(x,t)| \leq K_0 ;$$

- the jumps in the RH problem $\boldsymbol{\mathcal{E}}$ are determined by the *limit* linear statistics

$$X_N^f := \sum_{j=1}^N f(\mathrm{i}\kappa_j^{(N)};z) - N\int_{\mathrm{i}\eta_1}^{\mathrm{i}\eta_2} f(s;z)\mathrm{d}\mu_{\mathrm{MP}}(s) , \qquad f(s;z) = \frac{r(s)}{z-s};$$

  - z-uniform large deviation estimates on the jump entries;
  - fine estimates on $X_N^f - L_N^f$, where L_N^f is the linear statistics

$$L_N^f := \sum_{j=1}^N f(\mathrm{i}\kappa_j^{(N)}) - N\int_0^{\mathrm{i}\infty} f(s)\mathrm{d}\mu_N(s) .$$

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  - z-uniform large deviation estimates on the jump entries;
  - fine estimates on $X_N^f - L_N^f$, where L_N^f is the linear statistics

$$L_N^f := \sum_{j=1}^N f(\mathrm{i}\kappa_j^{(N)}) - N\int_0^{\mathrm{i}\infty} f(s)\mathrm{d}\mu_N(s) .$$

Updated overlook...

What we have so far:

- Full description of a regular, dense soliton gas in the large time regime, over the whole spatial domain.
- Rigorous description of how a particle (= soliton) interacts with a complex medium (= gas): how the medium affects the propagation of the particle (the soliton velocity) and how the particle affects the evolution of the complex medium (the phase shift in the soliton gas).
  We recover (and improve!) the kinetic equations of El, *et al.*.
- The gas is *deterministic*, but it is the *universal* limit of a large class of random (RMT and other distributions) N-soliton potentials.

Remark (G., Grava, McLaughlin, Najnudel, '26)

Similar LLN/CLT results (+ correlation functions) holds also for fNLS with rescaled Ginibre/i.i.d. distribution of the poles over a domain $\mathcal{D} \subset \mathbb{C}_+$.

Updated overlook and upcoming efforts

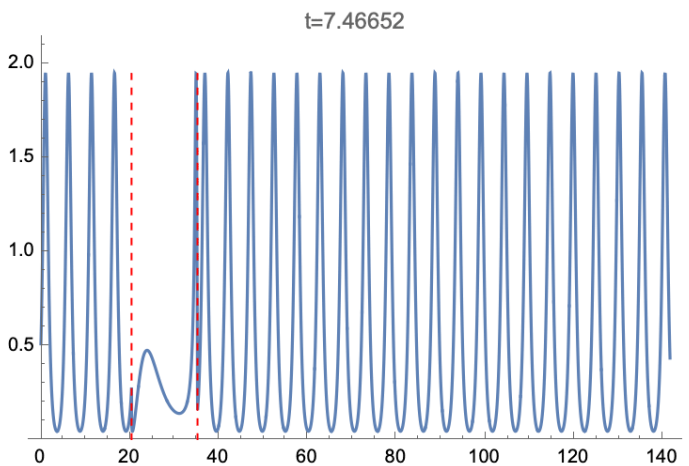
Next:

- more *randomness*: adding (independent) randomness to norming constants and reflection coefficient
- derivation of *NDR eq's*

$$v(\kappa) = 4\kappa^2 + \frac{1}{\kappa}\int_0^\infty \ln\left|\frac{s+\kappa}{s-\kappa}\right| (v(\kappa) - v(s))\, f(s;x,t)\,\mathrm{d}s\ , \qquad f_t + (vf)_x = 0$$

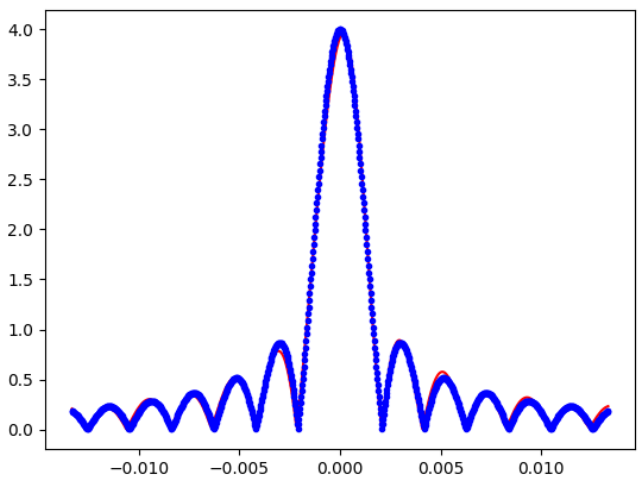
  for random soliton ensembles.

- *generalized/primitive gas* [G., Jenkins, McLaughlin, '26+]: long time behaviour, NDR eq's, phase shifts, etc.



Conclusive overlook and upcoming efforts

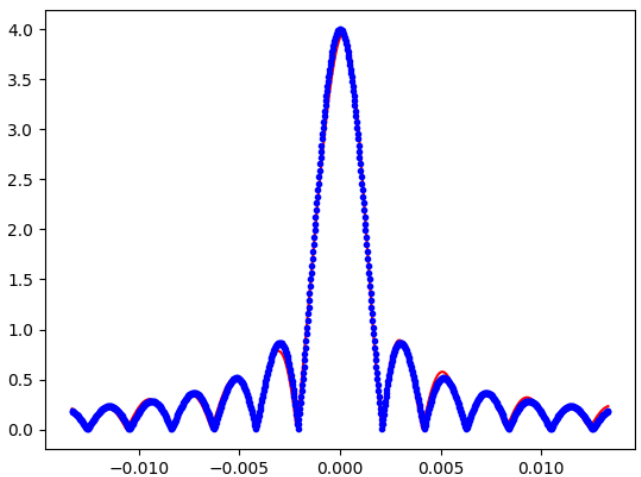
- construct *new gasses* from different scaling limits as $N \to \infty$
  e.g. reflection coefficients *do not* scale like $\mathcal{O}(N^{-1})$, $(x,t) = \mathcal{O}((\log N)^{\alpha})$
  - derivation of NDR eq's for gasses that are *not* condensate
  - asymptotic behaviour will be different
- new random configurations, rogue waves and *universal profiles*:
  [G., Grava, Jenkins, McLaughlin, Mazzuca, Yattselev, '25]
  [Gkogkou, McLaughlin, Mazzuca, '26]



Thank you! – Questions?

Conclusive overlook and upcoming efforts

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Thank you! – Questions?