

Inhomogeneous Chiral Phases and Finite-Momentum Instabilities in the QCD Phase Diagram

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Inhomogeneous Phases - NJL analysis

- NJL analysis

- Series expansion approach

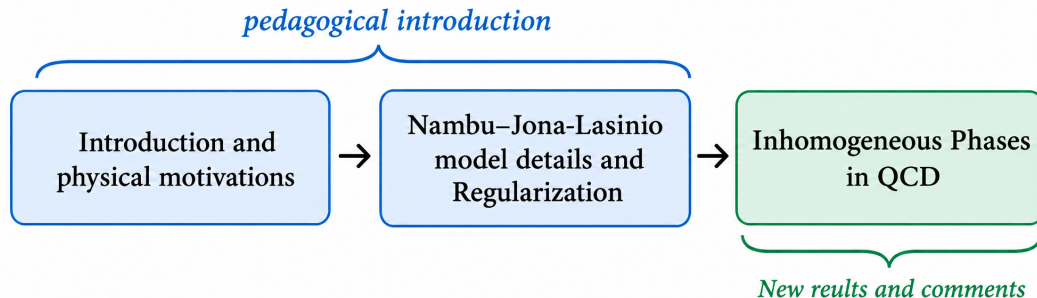
Results

- Previous Results

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Conclusions

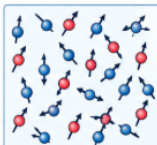
Structure of the seminar



Analogy with Ferromagnetism

Ferromagnetism: spontaneous symmetry breaking and the role of the external magnetic field

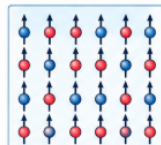
Paramagnetic phase ($T > T_c$)



- No net magnetization
 $M = 0$
- Rotational symmetry is preserved
($O(3)$)

$T < T_c$
Spontaneous
symmetry breaking

Ferromagnetic phase ($T < T_c$)



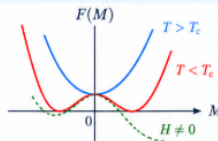
- Nonzero magnetization
 $M \neq 0$
- Rotational symmetry is broken
($O(3) \rightarrow O(2)$)

The Landau-Ginzburg picture

The free energy (or effective potential) can be written as

$$F(M) = a(T)M^2 + bM^4 - HM,$$

where M is the magnetization and H is an external magnetic field (explicit symmetry breaking).



Order parameter and external field

Order parameter: $M = \langle S \rangle$

External field: H

Susceptibility: $\chi_M = \frac{\partial M}{\partial H}$

Basics of Quantum Chromodynamics

QCD is the accepted theory of the strong interactions.

- ▶ At large momentum transfer the theory is perturbative due to asymptotic freedom.
- ▶ In the infrared, the theory becomes nonperturbative.

Consider the QCD Lagrangian, keeping only the three lightest quarks, u , d and s :

$$\mathcal{L} = \sum_{i=1}^3 (\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i) - \frac{1}{2} \text{Tr} G_{\mu\nu} G^{\mu\nu}, \quad (1)$$

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative, $A_\mu = A_\mu^a T_a$ are the eight gluon fields with T_a being $SU(3)$ generators in the 3 representation, and $G_{\mu\nu}$ being the gluon field strength.

Symmetries of QCD

- Global $U_V(1)$ i.e., the lagrangian $\mathcal{L}$ should be invariant to

$$\psi \rightarrow e^{-i\alpha} \psi \quad \text{and} \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha} \Rightarrow \quad j_\mu = \bar{\psi} \gamma_\mu \psi, \quad \alpha \in \mathbb{R} \quad \text{Baryon Number Conservation} \quad (2)$$

- Global $SU_V(2)$ i.e., the lagrangian $\mathcal{L}$ should be invariant to

$$\psi \rightarrow e^{-i\vec{\tau}\vec{\theta}/2} \psi \quad \text{and} \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\vec{\tau}\vec{\theta}/2} \Rightarrow \quad j_\mu^a = \bar{\psi} \gamma_\mu \tau^a \psi, \quad \vec{\theta} \in \mathbb{R}^3 \quad \text{Isospin Conservation} \quad (3)$$

Good symmetry if $m_u = m_d$.

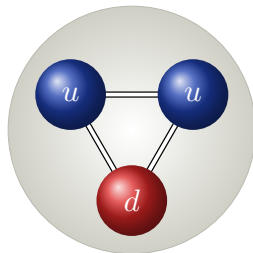
- Global $SU_A(2)$ i.e., the lagrangian $\mathcal{L}$ should be invariant to

$$\psi \rightarrow e^{-i\gamma_5 \vec{\tau}\vec{\theta}/2} \psi \quad \text{and} \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\gamma_5 \vec{\tau}\vec{\theta}/2} \Rightarrow \quad j_\mu^a = \bar{\psi} \gamma_5 \gamma_\mu \tau^a \psi, \quad \vec{\theta} \in \mathbb{R}^3 \quad (4)$$

This symmetry is true if $m = 0$.

Partial Chiral Symmetry Restoration

$$M \approx 938 \text{ MeV}$$



Proton

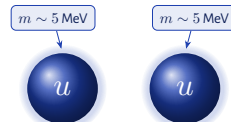
$$\langle \bar{\psi}\psi \rangle \neq 0$$

*partial chiral
symmetry restoration*



$$(T, \mu)$$

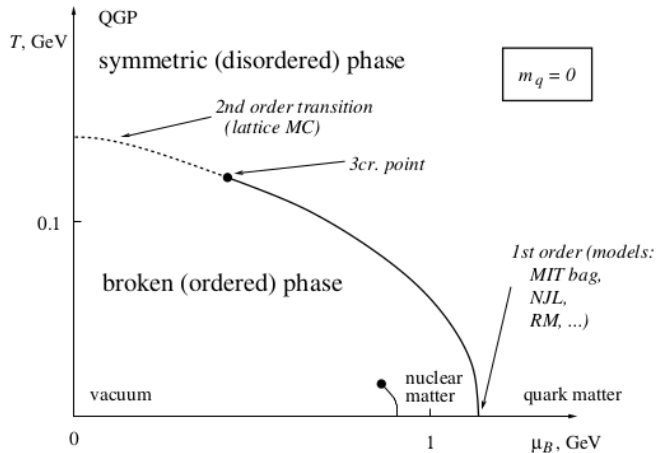
What is the phase
structure of the QCD?



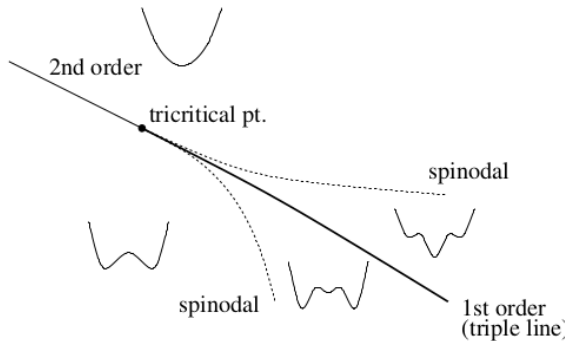
Free quarks

$$\langle \bar{\psi}\psi \rangle \sim 0$$

QCD Phase Diagram



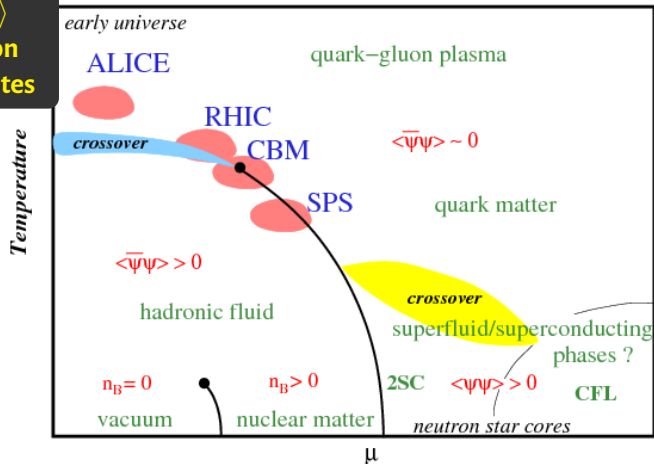
QCD Phase Diagram



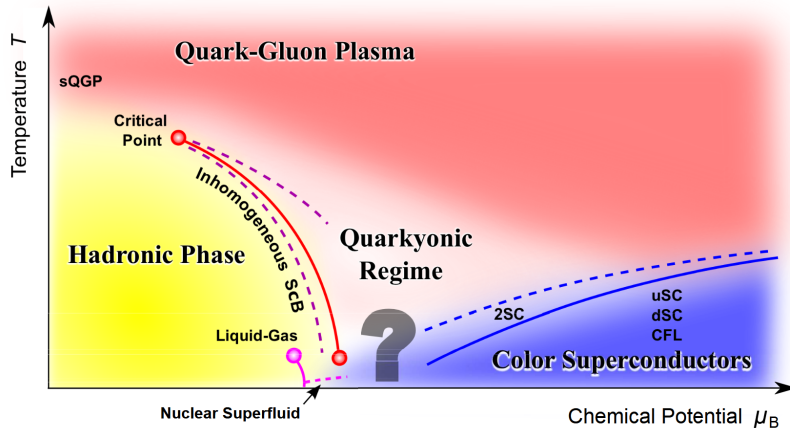
Source: Mikhail A. Stephanov. Prog.Theor.Phys.Suppl. 153 (2004) 139-156, Int.J.Mod.Phys.A 20 (2005) 4387-4392

QCD Phase Diagram

$\langle \bar{\psi}\psi \rangle \equiv \langle |0\bar{\psi}\psi|0 \rangle$
does not depend on
the spatial coordinates



QCD Phase Diagram



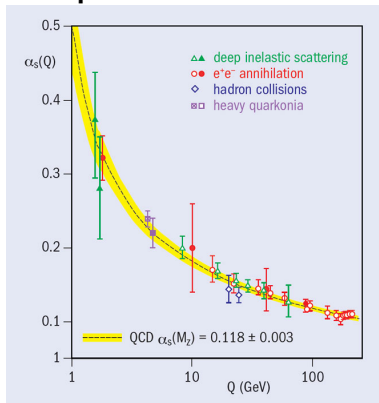
Inhomogeneous phases?

Inhomogeneous phases are phases where in addition to chiral symmetry translational invariance is also broken, i.e., the chiral condensate $\langle \bar{\psi}\psi \rangle$ is a function of the spatial coordinates $\vec{x} = (x_1, x_2, x_3)$.

- ▶ Relevant to nonvanishing chemical potential and low to intermediate temperatures.
- ▶ Quarkyonic matter;
- ▶ color-superconducting phases
- ▶ liquid-crystal like behavior
- ▶ Phases with spatially modulated chiral currents, which are observed in holographic models fitted to lattice QCD data, could possibly coincide or be an alternative scenario to an IP.

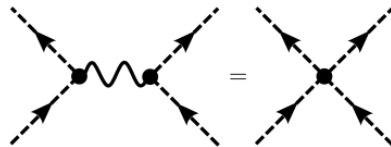
Nonperturbative QCD

► Asymptotic freedom



Nonperturbative QCD

- Lattice QCD
- Effective theories
- Effective models:
Nambu–Jona-Lasinio model



Nambu–Jona-Lasinio model SU(2)

The lagrangian of the two-flavor NJL model within chiral imbalanced medium is given by

$$\mathcal{L} = \bar{\psi} [i\gamma^\mu \partial_\mu - \tilde{m}] \psi + G [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \vec{\tau}\psi)^2], \quad (5)$$

- **Basic definitions:** the current quark masses matrix $\tilde{m} = \text{diag}(m_u, m_d)$ in which $m_u = m_d = m_0$; τ are the Pauli matrices; ψ is the spinor representing the quark fields $\psi = (\psi_u \quad \psi_d)^T$; G is the coupling constant and μ is the quark chemical potential.
- **This is a non-renormalizable model:** the coupling constant G has dimension of MeV^{-2} .
- **This model has three parameters to set:** the cutoff Λ , the current quark mass m_0 and the coupling G .

The parameters are chosen in order to obtain the vacuum values of the pion decay constant, $f_\pi \approx 93 \text{ MeV}$; the pion mass $m_\pi \approx 135 \text{ MeV}$ and the quark condensate $\langle \bar{u}u \rangle \approx (-250 \text{ MeV})^3$.

Nambu–Jona-Lasinio SU(2): some details

To find the quark dispersion relation we start by looking to the functional generator:

$$Z = \mathcal{N} \int D\bar{\psi} D\psi \exp \left[i \int d^4x \mathcal{L}(\bar{\psi}, \psi) \right] \Rightarrow \text{H.S.} \Rightarrow \begin{cases} \sigma = -2G\bar{\psi}\psi \rightarrow -2G\langle\bar{\psi}\psi\rangle \\ \vec{\pi} = -2G\bar{\psi}i\gamma_5\vec{\tau}\psi \end{cases}$$

where we performed the Hubbard-Stratanovich transformation (H.S.) with $\vec{\pi} = 0$ due to isospin symmetry. The partition function becomes

$$Z = \exp \left[-i \frac{\sigma^2}{4G} \int d^4x \right] \exp \left[N_c \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{Tr}_{f,D} \ln(\not{k} - m_c - \sigma) \right]. \quad (6)$$

with $N_c = 3$. Evaluating the trace over the flavor and Dirac indices using $\text{Tr} \ln \mathcal{O} \equiv \ln \text{Det } \mathcal{O}$

$$\omega(k) = \sqrt{k^2 + M^2} \quad \text{where} \quad \begin{cases} M = m_0 + \sigma \end{cases}$$

Thermodynamic Potential

We will include temperature by means of the Matsubara formalism

$$\int_{\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \Rightarrow iT \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3}, \quad \text{where} \quad \begin{cases} (k_0, \vec{k}) \Rightarrow (i\omega_n, \vec{k}) \\ \omega_n = (2n+1)\pi T \end{cases} \quad (7)$$

The thermodynamics of this model will be evaluated through the Landau Free energy, $\mathcal{F}$,

$$\mathcal{Z} = \exp \left[-\beta \int d^3 x \mathcal{F} \right], \quad \beta = \frac{1}{T} \quad (8)$$

Then the thermodynamic potential will be given by

$$\mathcal{F}(T) = \frac{(M - m_0)}{4G} - \underbrace{2N_c N_f \int_{\Lambda} \frac{d^3 k}{(2\pi)^3} \omega(k)}_{\text{needs regularization!}} - 4N_c N_f T \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3} \log \left(1 + e^{-\frac{\omega(k)}{T}} \right) \quad (9)$$

Λ is the 3D Sharp cutoff $\Rightarrow$ **Traditional Regularization Scheme - TRS**

The gap equation

The effective quark masses can be obtained by the minimization of the free energy, i.e.,

$$\frac{\partial \mathcal{F}}{\partial M} = 0, \quad \text{which gives} \quad (10)$$

$$M = m_c + \underbrace{4N_f N_c G \int_{-\infty}^{\infty} \frac{d^3 p}{(2\pi)^3} \frac{1}{\omega(k)}}_{\text{needs regularization!}} - 8N_f N_c G M \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega(k)} \frac{1}{e^{\omega_s(k)/T} + 1}. \quad (11)$$

We can also rewrite the above equation as

$$M = m_c - 2G \langle \bar{\psi} \psi \rangle (T), \quad (12)$$

where we defined $\langle \bar{\psi} \psi \rangle$ as the chiral condensate.

Usual Regularization Schemes

In the literature one can find several different regularization procedures. For example, we can comment some of these procedures

- ▶ 3D-Sharp cutoff regularization

$$\int_0^\infty \frac{d^3k}{(2\pi)^3} \omega(k) \Rightarrow \int_0^\Lambda \frac{dk k^2}{2\pi^2} \omega(k) \quad (13)$$

- ▶ Proper-time regularization

$$\int_0^\infty \frac{d^3k}{(2\pi)^3} \omega(k) \Rightarrow \int_{1/\Lambda^2}^\infty \frac{d\tau}{2\pi^2} \frac{e^{-\tau M^2}}{\tau^3} \quad (14)$$

- ▶ Pauli-Villars regularization

$$\int_0^\infty \frac{d^3k}{(2\pi)^3} \omega(k) \Rightarrow \sum_{i=0}^N c_i \int_0^\Lambda \frac{dk k^2}{2\pi^2} \omega_i(k), \quad \omega_i = \sqrt{M_i^2 + k^2} \quad \begin{cases} M_i^2 = M^2 + \alpha_i \Lambda^2 \\ N = 2, 3 \dots \end{cases} \quad (15)$$

Different quark dispersion relations

The quark energy dispersion relation can present several variations across the multiple diverse environments, for example:

$$\omega(k) \Rightarrow \omega_s(k, \mu_5) = \sqrt{(k \pm \mu_5)^2 + M^2}, \quad \text{Chiral imbalanced medium}^1$$

$$\omega(k) \Rightarrow \omega_s(k, \Delta, \mu_l^2) = \sqrt{(\omega(k) \pm \frac{\mu_l}{2})^2 + \Delta^2}, \quad \text{Isospin imbalanced medium}^2$$

Therefore, the vacuum contribution of the thermodynamic potential is now entangled with medium effects as

$$\mathcal{F}_{vac} \sim \int_0^\Lambda \frac{d^3k}{(2\pi)^3} \omega_s(k), \quad (16)$$

and cutoff effects could be the source of unphysical behavior.

¹ R. Farias, D. Duarte, G. Krein & R. Ramos. Phys.Rev.D 94, 074011 (2016)

² Jens O. Andersen and Lars Kyllingstad. J.Phys.G 37 (2009) 015003

Example

Example: Chiral imbalanced medium

- ▶ The chiral imbalance is the difference between right- and left-handed quarks.
- ▶ No sign problem in LQCD!
- ▶ Event-by-event C and CP violating processes in heavy-ion collisions; (Adler-Jackiw anomaly $\rightarrow$ chiral imbalanced medium)²;
- ▶ chiral anomaly transport model³:

$$\sim \mu_5 = 2.1T + \sqrt{eB} > 300\text{MeV}$$

¹K. Fukushima, D. Kharzeev, H. Warringa. Phys.Rev.D 78 (2008) 074033

²D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

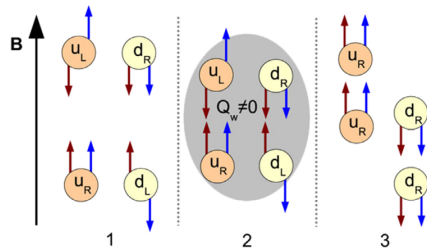
² H. Aoyama, H. Goldberg, and Z. Ryzak, Phys. Rev. Lett. 60, 1902 (1988).

³Zilin Yuan, Anping Huang, Wen-Hao Zhou, Guo-Liang Ma and Mei Huang. Phys. Rev. C 109, L031903

Figure: D. Kharzeev, L. McLerran and H. Warringa. Nucl.Phys.A 803 (2008) 227-253

- ▶ Chiral Magnetic Effect¹:

$$\vec{J} = \frac{e\mu_5}{2\pi^2} \int d^3x \vec{B}$$



Inverse Chiral catalysis?

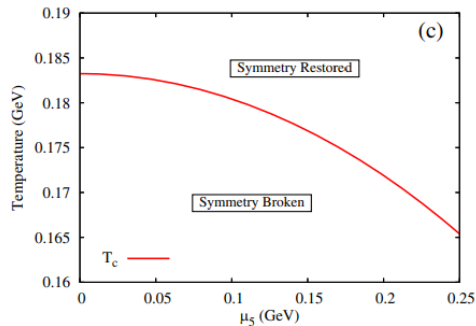
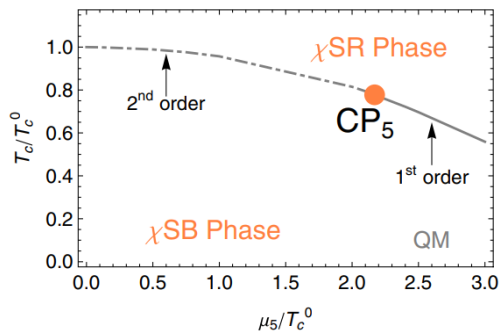


Figure: Left: Pseudocritical temperatures of chiral transition as a function of the chiral chemical potential in the quark-meson model⁴ and Nambu–Jona-Lasinio model⁵.

Left Plot: M. Ruggieri. Phys. Rev. D 84, 014011 (2011)

Right Plot: Snigdha Ghosh et al. Phys. Rev. D 109, 016021 (2024)

Phase structure of a chiral imbalanced medium

Inverse Chiral Catalysis?

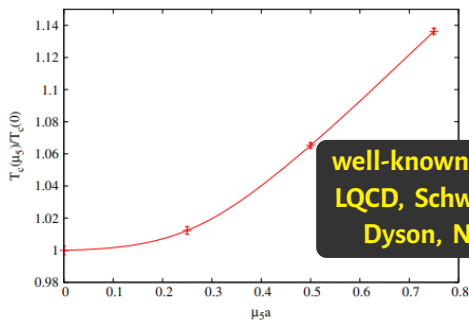
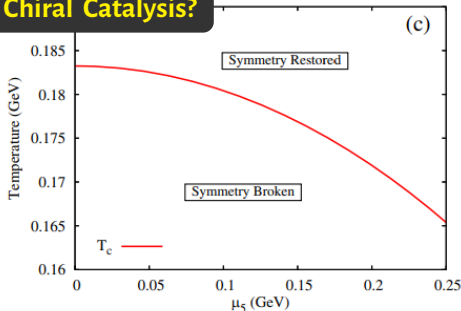


Figure: Pseudocritical temperatures of chiral transition as a function of the chiral chemical potential in Nambu–Jona-Lasinio model (left) and LQCD (right).

Left Plot: Snigdha Ghosh et al. Phys. Rev. D 109, 016021 (2024)

Right Plot: V. V. Braguta, E.-M. Ilgenfritz, A. Yu. Kotov, B. Petersson and S. A. Skinderov. D 93, 034509 (2016)

Medium separation Scheme (MSS)

First, we can work with a more treatable integral by evaluating (euclidean space)

$$\frac{\partial}{\partial M^2} \left[\int^\Lambda \frac{d^3 k}{(2\pi)^3} \omega_s(k) \right] = \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} \int^\Lambda \frac{d^3 k}{(2\pi)^3} \frac{1}{k_4^2 + \omega_s^2(k)} \equiv \int_k^\Lambda \frac{1}{k_4^2 + \omega_s^2(k)} \quad (17)$$

(18)

Then we can split the integration function as

$$\frac{1}{k_4^2 + \omega_s^2(k)} = \frac{1}{k_4^2 + \omega_0^2(k)} + \frac{k^2 + M_0^2 - \omega_s(k)^2}{(k_4^2 + \omega_0^2(k)) [k_4^2 + \omega_s^2(k)]}, \quad (19)$$

where

$$A_s(k) = \mu_5 + 2sk\mu_5 + M^2 - M_0^2, \quad (20)$$

where M_0 is the effective quark mass in the vacuum and $\omega_0(k) = \sqrt{k^2 + M_0^2}$.

R. Farias, D. Duarte, G. Krein & R. Ramos. Phys.Rev.D 94, 074011 (2016)

Medium separation scheme (MSS)

Now, by using three times the same identity Eq. (19) identity three times

$$\frac{1}{k_4^2 + \omega_s^2(k)} = \frac{1}{k_4^2 + \omega_0^2(k)} - \frac{A_s(k)}{(k_4^2 + \omega_0^2(k))^2} + \frac{A_s^2(k)}{(k_4^2 + \omega_0^2(k))^3} - \frac{A_s^3(k)}{(k_4^2 + \omega_0^2(k))^3 [k_4^2 + \omega_s^2(k)]},$$

therefore,

$$\int_k^\Lambda \frac{1}{k_4^2 + \omega_s^2(k)} = \underbrace{\int_k^\Lambda \frac{1}{k_4^2 + \omega_0^2(k)}}_{\text{Quad. Div.}} - \underbrace{\int_k^\Lambda \frac{A_s(k)}{(k_4^2 + \omega_0^2(k))^2}}_{\text{Log.Div.}} + \underbrace{\int_k^\Lambda \frac{A_s^2(k)}{(k_4^2 + \omega_0^2(k))^3}}_{\text{Log.Div.}} - \underbrace{\int_k^\infty \frac{A_s^3(k)}{(k_4^2 + \omega_0^2(k))^3 [k_4^2 + \omega_s^2(k)]}}_{\text{Convergent!}}, \quad (21)$$

however, we regulate just the divergent contributions.

R. Farias, D. Duarte, G. Krein and R. Ramos. Phys.Rev.D 94, 074011 (2016)

Medium Separation Scheme

The effective quark masses can be obtained by the minimization of the free energy, i.e., $\frac{\partial F}{\partial M} = 0$, which is

$$\frac{M - m_c}{4N_f N_c G M} = \overbrace{l_{\text{quad}}(\Lambda, M_0)}^{\text{Quad. Div.}} + (2\mu_5^2 - M^2 + M_0^2) \overbrace{l_{\text{log}}(\Lambda, M_0)}^{\text{Log. Div.}} - \frac{2\mu_5^2 + M^2 - M_0^2}{8\pi^2} + \frac{M^2 - 2\mu_5^2}{8\pi^2} \ln \left(\frac{M^2}{M_0^2} \right) - \sum_{s=\pm 1} \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega_s(k)} \frac{1}{e^{\omega_s(k)/T} + 1}, \quad (22)$$

where $l_{\text{quad}}(\Lambda, M_0)$ and $l_{\text{log}}(\Lambda, M_0)$ denote the quadratically and logarithmically UV divergent integrals, respectively,

$$l_{\text{quad}}(\Lambda, M_0) = 2 \int_0^\Lambda \frac{d^4 k}{(2\pi)^4} \frac{1}{k_4^2 + k^2 + M_0^2}, \quad l_{\text{log}}(\Lambda, M_0) = -\frac{\partial}{\partial M_0^2} l_{\text{quad}}(\Lambda, M_0), \quad (23)$$

MSS versus TRS results

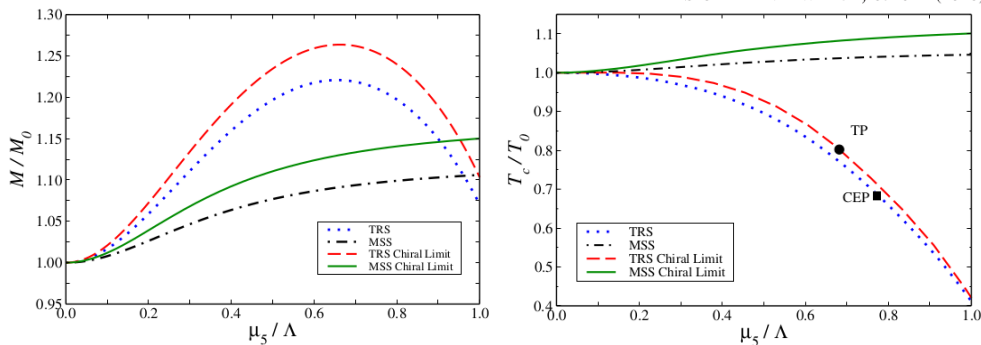


Figure: Left: Effective quark masses as a function of the normalized chiral chemical potential for both TRS and MSS regularizations. Right: Transition temperature as a function of the normalized chiral chemical potential for both MSS and TRS regularizations.

Cutoff Independent Regularization

In perturbative QCD the two-flavor spin-0 superconducting gap Δ can be written as¹

$$\Delta \sim \frac{\mu}{g^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right) \quad (24)$$

This result is inapplicable for densities typically found in the interiors of neutron stars. However, four-fermion models at the one-loop level

$$1 = \lambda Gi \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \left[\frac{1}{k_0^2 - (k + \mu)^2 - \Delta^2} + (\mu \rightarrow -\mu) \right],$$

predict vanishing superconducting gaps at high densities, a feature that is caused by the use of a regularizing momentum cutoff Λ

PHYSICAL REVIEW C 73, 018201 (2006)

Cutoff-independent regularization of four-fermion interactions for color superconductivity

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(Received 31 October 2005; published 27 January 2006)

We implement a cutoff-independent regularization of four-fermion interactions to calculate the color-superconducting gap parameter in quark matter. The traditional cutoff regularization has difficulties for chemical potentials μ of the order of the cutoff Λ , predicting in particular a vanishing gap at $\mu \sim \Lambda$. The proposed cutoff-independent regularization predicts a finite gap at high densities and indicates a smooth matching with the weak coupling QCD prediction for the gap at asymptotically high densities.

DOI: 10.1103/PhysRevC.73.018201

PACS number(s): 24.85.+p, 12.38.Mh, 21.65.+f, 26.60.+c

¹D. T. Son, Phys. Rev. D 59, 094019 (1999).

Farias R., Dallabona G., Krein G., Battistel O. Phys. Rev. C 73, 018201 (2006)

Cutoff Independent Regularization

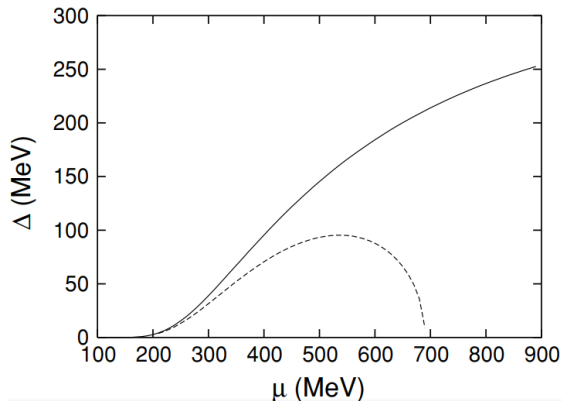
$$1 = \lambda G i \int \frac{d^4 k}{(2\pi)^4} \left[\frac{1}{k_0^2 - (k + \mu)^2 - \Delta^2} + (\mu \rightarrow -\mu) \right],$$

$$1 = 8\lambda G \{ 2[i I_{\text{quad}}(\Delta^2)] - 4\mu^2 [i I_{\text{log}}(\Delta^2)] + I_{\text{fin}}(\Delta^2, \mu) + I_{\text{fin}}(\Delta^2, -\mu) \},$$

where

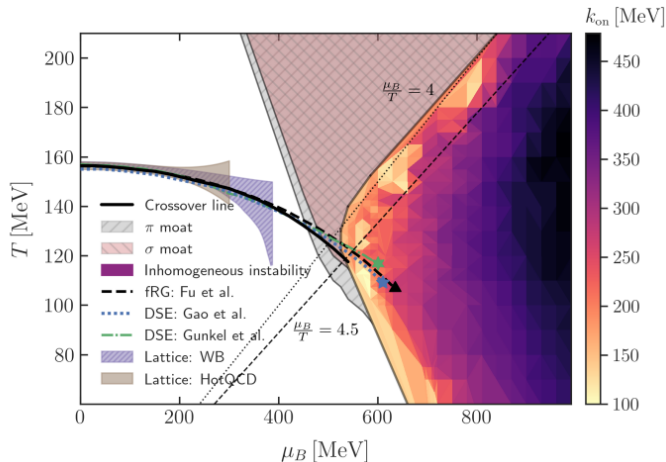
$$I_{\text{quad}}(\Delta^2) = \int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{k_0^2 - k^2 - \Delta^2},$$

$$I_{\text{log}}(\Delta^2) = \int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{(k_0^2 - k^2 - \Delta^2)^2},$$



Inhomogeneous Phases in QCD

Inhomogeneous Phases



Inhomogeneous Phases - NJL analysis

Robust Finite-Momentum Instabilities in Dense Matter

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The predicted extent of inhomogeneous chiral phases in effective models of quantum chromodynamics is notoriously sensitive to ultraviolet regularization. We show that this sensitivity is largely artificial. In the two-flavor Nambu–Jona-Lasinio model, conventional implementations of three-dimensional cutoff, Pauli–Villars, and proper-time regularization produce strongly different finite-momentum instability regions. Once ultraviolet regulators are restricted to genuinely divergent vacuum contributions, however, all three prescriptions yield nearly identical stability diagrams. Both the onset of the moat regime and the subsequent finite-momentum instability become quantitatively robust. The apparent scheme dependence originates from regulating ultraviolet-finite medium contributions associated with the Fermi-surface response. Our results identify spatially modulated chiral correlations as a genuine property of the dense medium rather than an artifact of the ultraviolet prescription.

Inhomogeneous Phases - NJL analysis

Starting from the NJL model effective Lagrangian

$$\mathcal{L} = \bar{\psi}(\not{p} - m - i\gamma_5 \vec{\sigma} \cdot \vec{\pi})\psi - \frac{\sigma^2 + \vec{\pi}^2}{4G}, \quad (25)$$

by defining inhomogeneous perturbations to the boson fields¹

$$\sigma(x) = \bar{\sigma} + \delta\sigma(x), \quad \pi(x) = \bar{\pi} + \delta\pi(x), \quad (26)$$

which, after defining the perturbed Dirac operator, we can use to obtain the effective action as a power series in the inhomogeneous perturbation

$$S_{eff} = \sum_{n=0}^{\infty} S_{eff}^{(n)}. \quad (27)$$

¹In the absence of isospin chemical potential we can easily define $\pi_1 = \pi_2 = 0$ and $\pi_3 = \pi$

For a discussion of inhomogeneous phases in the physical limit see: Michael Buballa and Stefano Carignano.
Phys.Lett.B 791 (2019) 361-366

Inhomogeneous Phases - NJL analysis

The first 3 terms are proportional to the homogeneous effective action, the gap equations, and integrals of the two point functions, which are interpreted as the curvature of the effective action in the direction of inhomogeneous perturbations

$$S_{\text{eff}}^{(2)} = \frac{\beta}{2} \int \frac{d^3 q}{(2\pi)^3} \left[|\delta \bar{\sigma}^2(\vec{q})|^2 \Gamma_{\sigma}^{(2)} + |\delta \pi(\vec{q})|^2 \Gamma_{\pi}^{(2)} \right], \quad (28)$$

up to $\mathcal{O}(\delta \phi_i^2)$

$$\left. \frac{\delta^2 S_{\text{eff}}}{\delta \phi_i^2} \right|_{\phi=\bar{\phi}} = \Gamma_{\phi_i}^{(2)}. \quad (29)$$

Evaluating the traces and Matsubara summations, etc., we arrive at the following form

$$\Gamma_{\phi_i}^{(2)} = \frac{1}{2G} - \bar{N} \ell_1(\bar{\sigma}, T, \mu) + \bar{N} L_{2,\phi_i}(\bar{\sigma}, \vec{q}, T, \mu). \quad (30)$$

Inhomogeneous Phases - NJL analysis

We define the moat regime as the region in which the second derivative of the two-point function at zero momentum, which is proportional to the wave-function renormalization, is negative ,

$$\left. \frac{\partial^2 \Gamma^{(2)}}{\partial q^2} \right|_{q=0} < 0, \quad (31)$$

whereas a genuine finite-momentum instability is signaled by

$$q_{\min} \neq 0, \quad \Gamma^{(2)}(q_{\min}) < 0. \quad (32)$$

Thus, the moat regime should be understood as a precursor of spatial modulation: the homogeneous phase might remain stable, but its softest static mode already carries a finite momentum.

L. Pannullo, M. Wagner, and M. Winstel, Phys. Rev. D 110, 076006 (2024), arXiv:2406.11312 [hep-ph].

R. D. Pisarski and F. Rennecke, Phys. Rev. Lett. 127, 152302 (2021), arXiv:2103.06890 [hep-ph]

Inhomogeneous Phases - NJL analysis

ℓ_1 is the gap equation integral

$$\ell_1(\bar{\sigma}, T, \mu) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_{\vec{p}}} [1 - n(E_{\vec{p}}) - \bar{n}(E_{\vec{p}})] , \quad (33)$$

while L_{2,ϕ_i} is an integral that depends on the external momentum $\vec{q}$

$$L_{2,\phi_i}(\bar{\sigma}, \vec{q}, T, \mu) = \int \frac{d^3 p}{(2\pi)^3} \frac{\vec{p} \cdot \vec{q} + \vec{q}^2 + 2\delta_{i1}\bar{\sigma}^2}{2\vec{p} \cdot \vec{q} + \vec{q}^2} \left[\frac{1 - n(E_{\vec{p}}) - \bar{n}(E_{\vec{p}})}{2E_{\vec{p}}} - \frac{1 - n(E_{\vec{p}+\vec{q}}) - \bar{n}(E_{\vec{p}+\vec{q}})}{2E_{\vec{p}+\vec{q}}} \right] . \quad (34)$$

- ▶ Regularization of the medium integrals was shown to violate thermodynamics foundational relations, such as the Boltzmann limit and causality;
- ▶ Meanwhile, regularization of medium dependent vacuum integrals (or medium dependent divergent integrals) was seen to lead to results that don't match perturbative QCD and lattice QCD;

Series expansion approach

The second approach relies on series expansion of the integrand. We start with

$$L_{2,\phi_i}(\vec{\sigma}, \vec{q}, T, \mu) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p} \cdot \vec{q} + \vec{q}^2 + 2M^2}{2\vec{p} \cdot \vec{q} + \vec{q}^2} \left[\frac{1 - n(E_{\vec{p}}) - \bar{n}(E_{\vec{p}})}{2E_{\vec{p}}} - \frac{1 - n(E_{\vec{p}+\vec{q}}) - \bar{n}(E_{\vec{p}+\vec{q}})}{2E_{\vec{p}+\vec{q}}} \right]. \quad (35)$$

We perform a momentum shift² such that $\vec{p} = \vec{k} - \vec{q}/2$ and

$$\begin{aligned} \vec{p} \cdot \vec{q} + \vec{q}^2 &= \vec{k} \cdot \vec{q} - \frac{\vec{q}^2}{2} + \vec{q}^2 = \vec{k} \cdot \vec{q} + \frac{\vec{q}^2}{2}, \\ 2\vec{p} \cdot \vec{q} + \vec{q}^2 &= 2\vec{k} \cdot \vec{q} - \vec{q}^2 + \vec{q}^2 = 2\vec{k} \cdot \vec{q}, \\ \frac{\vec{p} \cdot \vec{q} + \vec{q}^2 + 2M^2}{2\vec{p} \cdot \vec{q} + \vec{q}^2} &= \frac{1}{2} + \frac{\vec{q}^2}{4\vec{k} \cdot \vec{q}} + \frac{M^2}{\vec{k} \cdot \vec{q}}. \end{aligned} \quad (36)$$

²Not an identity in cutoff regularization

Series expansion approach

Correspondingly, $E_p \rightarrow E_{k-q/2}$ and $E_{p+q} \rightarrow E_{k+q/2}$. Since the first term has no $\vec{k}$ dependence (aside from the Jacobian) it also happens that³

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left(\frac{1}{2E_{k-q/2}} - \frac{1}{2E_{k+q/2}} \right) = 0. \quad (37)$$

Then, we arrive at the momentum shifted integral

$$L_{2,vac} = \int \frac{d^3k}{(2\pi)^3} \frac{q^2 + 4M^2}{4\vec{k} \cdot \vec{q}} \left(\frac{1}{2E_{k-q/2}} - \frac{1}{2E_{k+q/2}} \right). \quad (38)$$

We begin by making use of the proper-time representation

$$\frac{1}{E} = \frac{1}{\Gamma(1/2)} \int_0^\infty d\tau \tau^{-1/2} e^{-\tau E^2}, \quad (39)$$

³Only true on unregulated integrals or with translation invariant regularizations

Series expansion approach

In the PT parametrization, we will have to deal with integrations like

$$J_1 = \frac{q}{16\pi^{5/2}} \int_{-1}^1 dx \int_0^\infty dk \int_0^\infty d\tau \tau^{-1/2} \frac{k}{x} e^{-\tau(k^2 + M^2 + q^2/4)} \sinh(\tau k q x) \quad (40)$$

Expanding the hyperbolic sine into a series we can extract the first term, while all others are convergent integrals. The result of this is

$$J_1 = \frac{q^2}{32\pi^2} \int_0^\infty \frac{d\tau}{\tau} e^{-\tau(M^2 + q^2/4)} + \frac{q}{16\pi^{5/2}} \int_{-1}^1 dx \int_0^\infty dk \int_0^\infty d\tau \tau^{-1/2} \frac{k}{x} e^{-\tau(k^2 + M^2 + q^2/4)} \sum_{n=1}^{\infty} \frac{(\tau k q x)^{2n+1}}{(2n+1)!}. \quad (41)$$

Conveniently, the second integral sum can be evaluated by noting that sum part of the integrand is equal to $\sinh(\tau k q x) - \tau k q x$.

Series expansion approach

► Cutoff 3D:

$$\int \frac{d^3p}{(2\pi)^3} \longrightarrow \int \frac{d^3p}{(2\pi)^3} \Theta(\Lambda - |\vec{p}|).$$

► Proper-time:

$$\frac{1}{A^n} = \frac{1}{\Gamma(N)} \int d\tau \tau^{-1+n} e^{-\tau A} \Rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \longrightarrow \frac{1}{\Gamma(1/2)} \int_{1/\Lambda^2}^{\infty} d\tau \tau^{1/2} \int \frac{d^3p}{(2\pi)^3} e^{-\tau E_p^2}.$$

► Pauli-Villars:

$$\int \frac{d^3p}{(2\pi)^3} f(E_p) \longrightarrow \int \frac{d^3p}{(2\pi)^3} \sum_{j=0}^3 c_j f(E_j).$$

Series expansion approach

In the PT parametrization, we obtain

$$L_{2,\text{vac}}(\mathbf{q}^2) = \frac{\mathbf{q}^2}{32\pi^2} \int \frac{d\tau}{\tau} e^{-\tau \mathbf{q}^2/4} + \frac{|\mathbf{q}|}{16\pi^{5/2}} \int_0^\infty d\tau \tau^{1/2} e^{-\tau \mathbf{q}^2/4} \quad (42)$$

$$\times \int_{-1}^1 dx \int_0^\infty dp \frac{p}{x} e^{-\tau p^2} [\sinh(\tau p q x) - \tau p q x].$$

Performing the integration of the second term we arrive at the separated integral

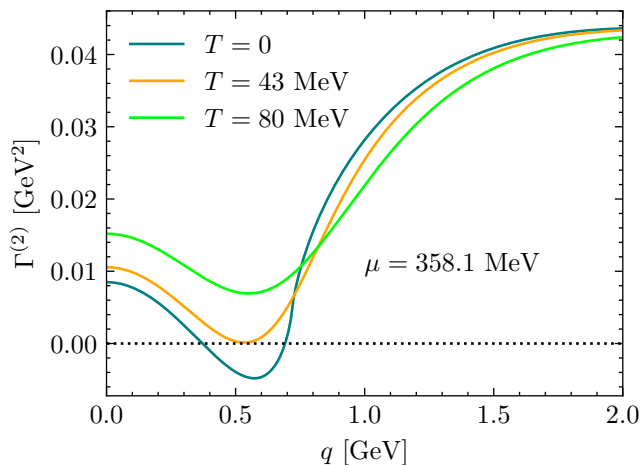
$$L_{2,\text{vac}}(\mathbf{q}^2) = \frac{\mathbf{q}^2}{32\pi^2} \left[l^{(\log)}(\mathbf{q}^2/4) + 2 - \ln 4 \right], \quad (43)$$

where

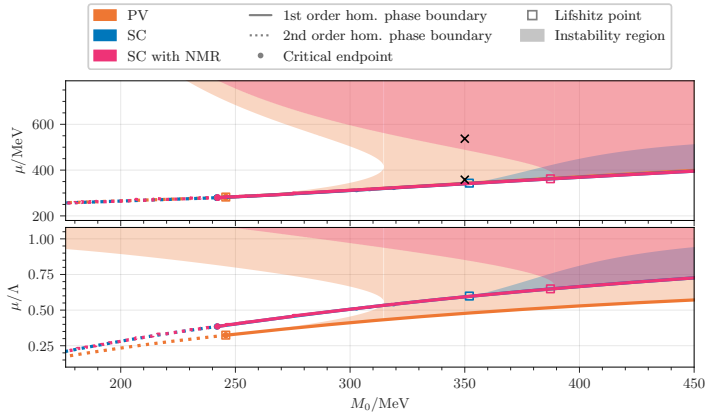
$$l^{(\log)}(\mathbf{q}^2/4) = \int_\Lambda \frac{d\tau}{\tau} e^{-\tau \mathbf{q}^2/4} \quad (44)$$

is a logarithmically divergent integral to be regulated via three-dimensional cutoff, Pauli–Villars regularization, and proper-time regularization.

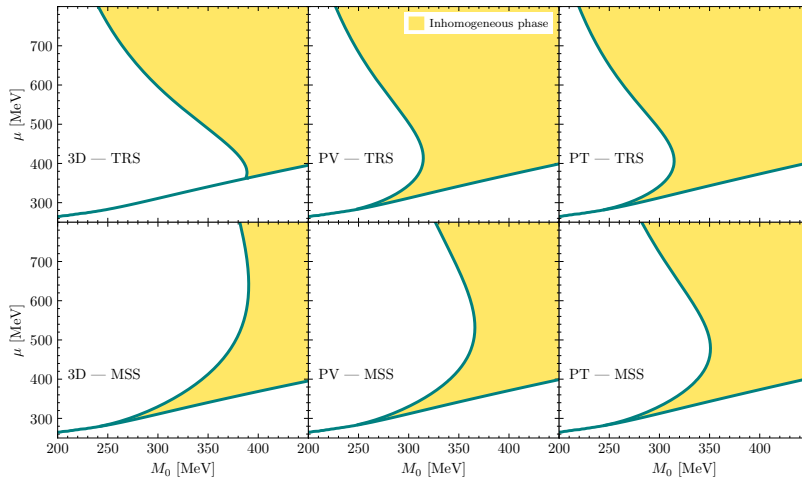
Two point correlation function



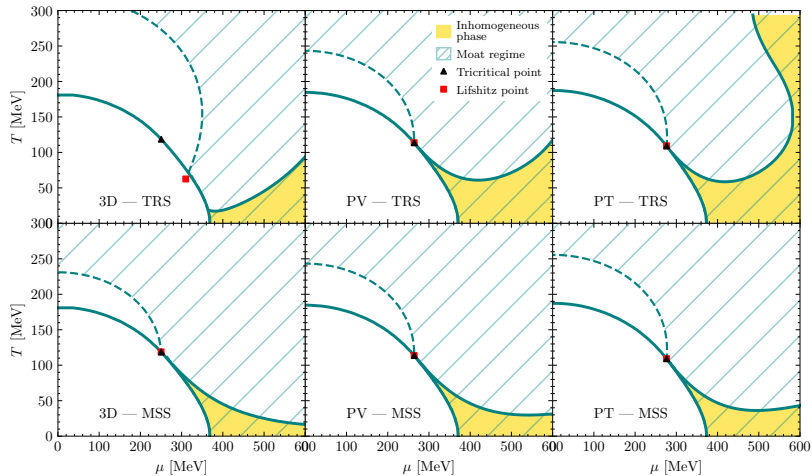
Previous Results



Results



Results



Summary

- ▶ We have shown that the pronounced regulator dependence of finite-momentum instabilities in the two-flavor NJL model is largely generated by applying ultraviolet regularization to contributions that are finite and medium dependent.
- ▶ Traditional prescriptions lead to qualitatively different instability domains and moat boundaries, whereas the medium separation scheme yields a common phase structure for three-dimensional cutoff, Pauli–Villars, and proper-time vacuum regularizations.
- ▶ The robustness of both the finite-momentum minimum and the subsequent loss of stability demonstrates that these structures are controlled primarily by the medium response rather than by the ultraviolet prescription.

Conclusions

- ▶ Our analysis establishes the local onset of spatial modulation, while determining the fully developed inhomogeneous ground state and its fluctuation corrections remains an important next step.
- ▶ More broadly, these results indicate that finitemomentum structures in dense effective theories can remain quantitatively predictive even in nonrenormalizable frameworks, provided ultraviolet and medium physics are consistently disentangled.

Merci!!

Collaborators

- ▶ Ricardo L. S. Farias (UFSM/Kent University)
- ▶ André Gonçalves (master's student at UFSM)
- ▶ Theo Motta (IFT/ICTP-SAIFR)

Acknowledgements

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- ▶ **Main Reference:** André G. da Silva, Ricardo L.S. Farias, Theo Motta, **William R. Tavares**.2608.25857 [hep-ph].