Exercise 1 (Getting familiar with category theory)**.**

1. *Verify that there is a category* Set *whose objects are sets and morphisms are functions.*
2. *Check that the product of sets X, Y in the category of sets is given by their cartesian product $X \times Y$.*

   hint*: Don't forget that you need to specify the two functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$*
3. *Verify that there is a category* Cat *whose objects are categories and whose morphisms are functors between them.*
4. *Given a category C, and an object x of C, show that the functor $\mathsf{Yon}(x) = C(x, -) \colon C \to \mathsf{Psh}(C)$ preserves products.*

Exercise 2 (Natural transformations and presheaves)**.**

1. *Recall the example of multigraphs: we consider the category*

$$\mathbb{G} = \begin{tikzcd} 0 & 1 \arrow[l, shift right, "\sigma"'] \arrow[l, shift left, "\tau"] \end{tikzcd}$$

   and the category of presheaves $\mathsf{Psh}(\mathbb{G})$, whose objects are functors $\mathbb{G} \to \mathsf{Set}$ and morphisms are natural transformations.

   (a) *Unfold the definition to describe explicitly the objects of $\mathsf{Psh}(\mathbb{G})$*

   (b) *Unfold the definition to describe explicitly the morphisms of $\mathsf{Psh}(\mathbb{G})$*
2. *This question aims at proving the Yoneda lemma. To this end, we consider a category C, and a presheaf $F \colon C \to \mathsf{Set}$*

   (a) *Define a function $\mathsf{Nat}(\mathsf{Yon}(x), F) \to F(x)$*

   hint*: You don't need the naturality at this point, there is only one way to go...*

   (b) *Show that this function is injective*

   hint*: Now is the time to use the naturality*

   (c) *Show that this function is surjective*

Exercise 3 (The theory of monoids)**.** *We recall the rules of the theory of monoids $\mathcal{T}_{\mathsf{Mon}}$:*

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{e} : \star} \qquad \frac{\Gamma \vdash u : \star \qquad \Gamma \vdash v : \star}{\Gamma \vdash u * v : \star}$$

$$\frac{\Gamma \vdash u : \star}{\Gamma \vdash u * e \equiv u} \qquad \frac{\Gamma \vdash u : \star}{\Gamma \vdash e * u \equiv u} \qquad \frac{\Gamma \vdash u : \star \qquad \Gamma \vdash v : \star \qquad \Gamma \vdash w : \star}{\Gamma \vdash u * (v * w) \equiv (u * v) * w}$$

1. *Build a derivation tree for each of the following judgment in the theory of monoids*

$$x : \star \vdash x * (\mathsf{e} * \mathsf{e}) \qquad x : \star, y : \star \vdash y * (x * y) \qquad x : \star, y : \star \vdash ((y * \mathsf{e}) * x) * x$$

   hint*: You do not need to use the equality to build the derivation trees*

2. *Among the terms presented above, which ones are equal in the theory*

Note*: This shows something important about these theories, a single term (up to quotient by $\equiv$) may have several derivation trees.*

Exercise 4 (Lawvere theories and algebraic theories)**.** *This exercise aims at proving the central correspondences between Lawvere theories and algebraic theories.*

1. *We first construct the syntactic category associated to an algebraic theory $\mathcal{T}$*
   (a) *Given a substitution $\Gamma \vdash \sigma : \Delta$ and a term $\Delta \vdash u : \star$, define a term $\Gamma \vdash u[\sigma] : \star$*
   hint*: If it helps, do it first for the theory of monoids. You will need induction somewhere.*
   (b) *Using the action of substitution on terms, define a composition of substitutions: Given a substitution $\Gamma \vdash \sigma : \Delta$ and a substitution $\Delta \vdash \tau : \Theta$, define a substitution $\Gamma \vdash \tau \circ \sigma : \Theta$*
   (c) *Define an identity substitution $\Gamma \vdash \mathsf{id} : \Gamma$*
   (d) ($\star$) *Verify that the composition is unital*
   hint*: You may need to introduce a lemma characterising the action of the identity substitution on a term*
   (e) ($\star\star$) *Verify that the composition is associative*
   hint*: You may need to introduce a lemma characterising the action of a composite substitution on a term*

   Summary*: Given an algebraic theory $\mathcal{T}$, we have built a category $\mathsf{Syn}(\mathcal{T})$ whose objects are contexts and whose morphisms are substitutions.*

2. *Show that the syntactic category of $\mathcal{T}$ is a Lawvere theory.*
   hint*: We already know the object, so you only need to check that products are correct*
3. ($\star$) *Show that the models of $\mathsf{Syn}(\mathcal{T})$ are the same as the models of $\mathcal{T}$*

Exercise 5 (Free monoids)**.** *Given a set X, remember that we define $X^\star$ as the set of lists of elements of X*

1. *Show that $X^\star$ is a monoid*
2. *Show that $X^\star$ satisfies the universal property of the free monoid on X: i.e. Given a monoid M, morphisms of monoids $X^\star \to M$ are in bijective correspondence with functions $X \to M$*
3. *Show that any monoid that satisfies the same property is isomorphic (as a monoid) to $M^\star$*

We now consider the Lawvere theory of monoids $\mathbb{T}_{\mathsf{Mon}} = \mathsf{Syn}(\mathcal{T}_{\mathsf{Mon}})$, and the Yoneda embedding $\mathsf{Yon}(n) : \mathbb{T}_{\mathsf{Mon}}(n, -)$.

4. *Using the Yoneda Lemma, show that $\mathsf{Yon}(n)$ satisfies the universal property of the free monoid on $\{0, \dots, n-1\}$*

Note*: Combined with the previous results, this shows that $\mathsf{Yon}(n)$ is isomorphic to $\{0, \dots, n-1\}^\star$. We now check this manually for $\mathsf{Yon}(4)$.*

5. *Describe the monoid structure on $\mathsf{Yon}(4)$*
6. *Exhibit 4 elements in $\mathsf{Yon}(4)$ that generate it as a monoid*
7. *Show that these 4 elements define a free family*