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Part I: Introduction to Finite Element Exterior Calculus

Abstract:

In this lecture we will introduce the geometrical concepts needed for Finite Element Exterior Calculus, the theory introduced by Arnold, Falk and Winther for the discretization of differential forms. We will in particular introduce the notions of manifolds, tangent vectors, differential forms, wedge product, interior product, pullback, metric and a scalar product for differential forms and show how they can be discretized using the Finite Element concept. This will lead to a discrete Hilbert complex of differential forms that can be reproduced at the discrete level yielding natural stability properties.
We will also see how Finite Element spaces of differential forms relate to more classical finite element spaces.
We will focus on spline based Finite Element, which are in particular used in isogeometric analysis and very convenient for the coupling with particle methods for the Vlasov equation.

Part II: A discrete variational framework for the Vlasov-Maxwell equations

Abstract:

Starting either from the action principle proposed by Low or the non canonical hamiltonian formulation of the Vlasov-Maxwell system by Morrison, Marsden and Weinstein, we shall show that using a particle discretization of the Vlasov equation and a compatible Finite Element approximation of Maxwell's equations based on Arnold, Falk and Winther's Finite Element Exterior Calculus, the semi-discrete equations form a finite dimensional Poisson system, which is the non canonical extension of a symplectic system. This system features Casimir invariants, which are the Gauss law and div B, as well as energy and momentum conservation, which yields enhanced stability properties compared to standard Particle In Cell codes.
The derivation and analysis of the discrete system will be presented as well as numerical results illustrating its properties.

Part III: Modern gyrokinetic theory and its implementation

Abstract:

The Vlasov-Maxwell equations can be derived from an action principle, which allows to derive conservation laws from symmetries of the Lagrangian. Gyrokinetic theory is an asymptotic reduction of the Vlasov-Maxwell equations in the presence of a large magnetic field. What is called modern gyrokinetic theory, following Brizard, Qin, Scott, Sugama and others, aims at retaining the same structure. More precisely, starting from the Vlasov-Maxwell Lagrangian consisting of a particle and a field part, the asymptotic reduction is performed on the single particle Lagrangian via a series of near identity coordinate transforms, called Lie transforms, and possibly on other terms of the Lagrangian. Then the gyrokinetic equations are derived as the Euler-Lagrange equations of the modified Lagrangian without further approximations. This yields gyrokinetic conservation laws, which can be compared to the Vlasov-Maxwell conservation laws.
In this lecture we will introduce the derivation of modern gyrokinetic theory detailing a few of its variants and see how this can be related to models implemented in actual gyrokinetic codes. A code verification effort based on this theory will also be presented.