On Integrability of QFTs in AdS_2

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Punchline(s)

- There exist correlators of "QFTs in AdS_2" which exhibit integrable features — No particle-production

- BUT: It is impossible to non-trivially deform a) Free fields in AdS_2, b) CFTs in AdS_2 While preserving Higher-Spin Charges*

Outline :

I) Motivation: QFT in AdS & "integrable" solutions

II) Higher-Spin Currents/Charges

III) Illustration in GFF

IV) No-Go Theorems

V) Outlook

Motivation: QFT in AdS_{d+1} is conceptually simpler but technically harder than $\mathbb{R}^{d+1}$

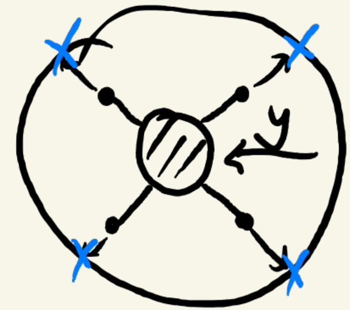
- Box/IR Regulator
- Maximally Symmetric (as $\mathbb{R}^{d+1}$)

Isometries act as Conformal Transf. @ ∂AdS

- Assymptotic States

$$\Phi(x_i, y) \sim y^{\Delta_i} \mathcal{O}(x_i) + \cdots$$

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle$$



- Boundary Conformal Theory

- $SO(d+1,1)$ covariance
- Unitary (no $T_{\mu\nu}$ on ∂) $\quad m^2 R^2 = \Delta(\Delta-1)$
- State-Op. map
- Convergent OPE

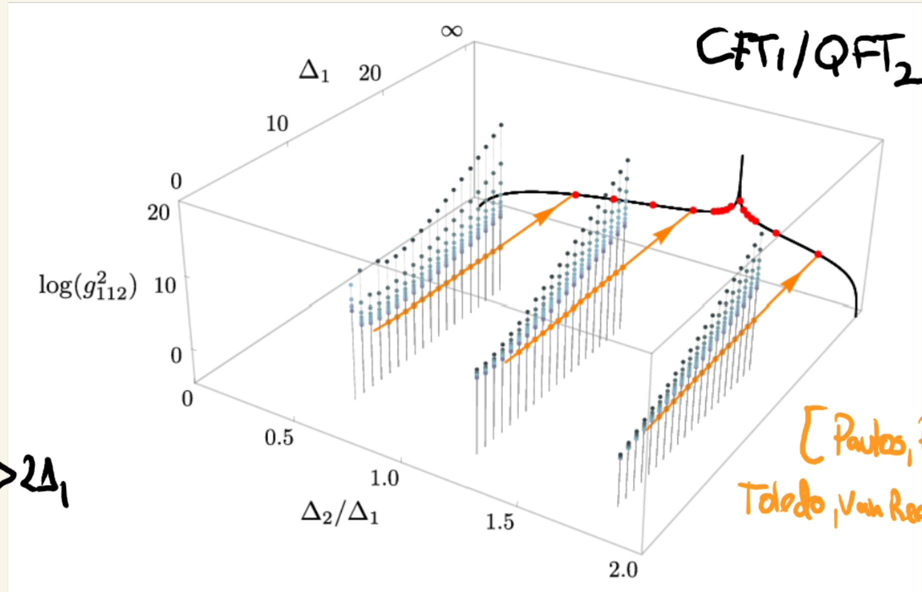
Bootstrap!

$\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle$ $\quad CFT_1$

$$\mathcal{O}_1 \times \mathcal{O}_1 \sim \mathbb{1} + C_{112}\,\mathcal{O}_2 + \Delta > 2\Delta_1$$

- No momentum space
- Harder Feyn Rules
- Curved space w/ ∂

- Proof of concept (Flat space limit $\Delta_i \to \infty$, $L_{AdS} \to \infty$)



CFT_1/QFT_2

[Paulos, Penedones, Toledo, Van Rees, Vieira '16]

• 1D CFT in $L_{AdS} \to \infty$ limit reconstructs exactly solvable 2d QFT (sine-Gordon)

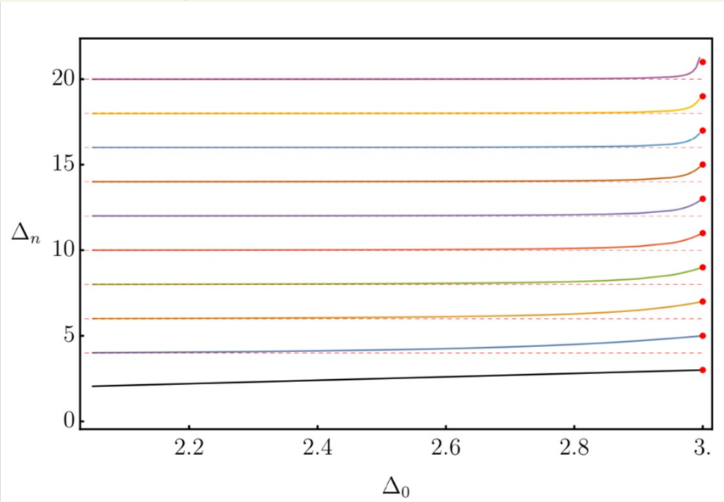
• Are there exactly solvable examples @ finite L_{AdS}?

• Integrable solutions?

Technical insight. cf. $O(N)$ @ $N \to \infty$

[Komatsu, di Pietro, Carmi 18']

$\langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \rangle$



• "Extremal correlator"

• Sparse spectrum

• Maximizes $C_{\phi\phi\phi^2}$ $\quad \phi \times \phi \sim 1 + C_{\phi\phi\phi^2}\, \phi^2 + \dots$

• $C_{\phi\phi\phi^n} = 0 \; ; \; n \geq 3$

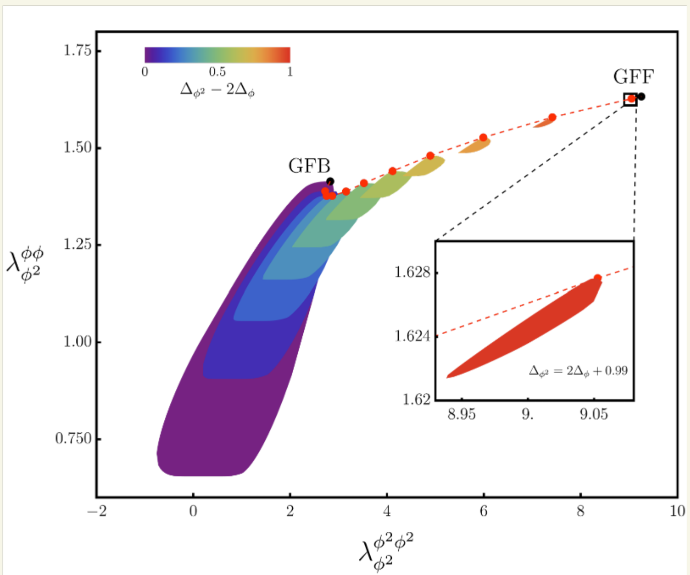
• No "particle-production"

[Paulos, Zan] 19'

- Perhaps Artifact of Single-correlator only.
- Generally expect: $\phi\times\phi \sim \phi^2 + \phi^4 + \ldots$
  $\phi\times\phi^2 \sim \phi^3 + \ldots$
  $\phi^2\times\phi^2 \sim \phi^2 + \phi^4 + \ldots$

Mixed Correlator Conditions
$\langle\phi\phi\phi\phi\rangle, \langle\phi\phi\phi^2\phi^2\rangle, \langle\phi^2\phi^2\phi^2\phi^2\rangle$
[Paulos, Kaviraj, Ghosh 24']

But, a "miracle" happens:



- Same Max $C_{\phi\phi\phi^2}$

→ No Particle Production!!

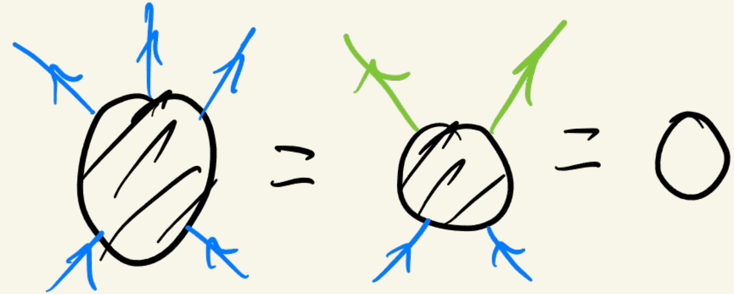
- Integrability?
- Dynamical Principle?

→ Recall Flatspace

Refresher on integrable QFTs in $\mathbb{R}^2$

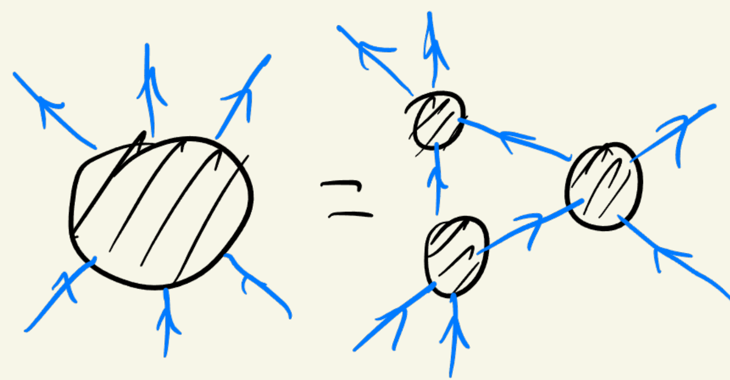
Integrable S-Matrices have Remarkable properties

i) No particle-production



$$S_{aa\to aaa} = S_{aa\to bb} = 0$$

ii) Factorization (+ Yang-Baxter)



$$S_{aaa\to aaa}(p_1,p_2,\ldots) = S_{aa\to aa}(p_1,p_2)\, S_{aa\to aa}(p_1,p_3)\, S_{aa\to aa}(p_2,p_3)$$

iii) Exactly Solvable $S_{2\to 2}(s)$ $\quad s = (p_1 + p_2)^2$

e.g. Simplest "breather" in sine-Gordon:

$$\boxed{S_{aa\to aa}(s) = \frac{\sqrt{s(s-4m^2)} + \sqrt{m_b^2(m_b^2-4m^2)}}{\sqrt{s(s-4m^2)} - \sqrt{m_b^2(m_b^2-4m^2)}}}$$

- 1st Principles origin: Existence of additional (∞-many) conserved quantities.

Demystifying Integrable QFTs in AdS_2

Integrability $\longleftrightarrow$ ∞-many commuting conserved charges

- non-local ?
- $\boxed{\text{Higher-spin}}$

$$\boxed{\partial^\mu T_{\mu zzz} = 0}$$

$$Q_s \Rightarrow \sum_i p_i^s = 0 \quad \Rightarrow$$

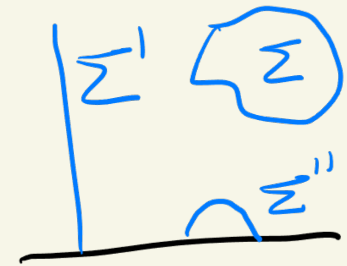
- Factorized Scattering
- Non-Trivial $S_{aa\to aa}$

What are the Consequenses of Higher-Spin Charges for QFTs in AdS_2 ?

II) Higher Spin Currents/Charges in AdS

- Define a HS charge from a HS current

$$Q_{\zeta}^{(y)} = \int_{\Sigma_{d-1}} d\Sigma^{\mu}\, \underbrace{\zeta^{\mu_2 \cdots \mu_y} T_{\mu \cdots \mu_y}}_{j_\mu^{(\zeta)}}$$



- ζ Killing Tensor $\quad \nabla_{(\mu} \zeta_{\nu_1 \cdots \nu_{y-1})} = 0$

Flat Space

$$\left(\zeta^{\mu_1 \cdots \mu_{y-1}} = \int_z^{\mu_1} \cdots \int_z^{\mu_{y-1}} \right)$$

- Minimal Assumption : $\nabla^{\mu} j_{\mu}^{(\zeta)} = 0$

- We showed that in AdS_2

$$\boxed{\nabla^\mu j_\mu^{(\xi)} \Rightarrow \nabla^\mu T_{\mu\mu_2\cdots\mu_J} = 0}$$

The current must be fully conserved, more restrictive than flat space!

[Thompson] 86'

$$\xi_{\nu_1\cdots\nu_{J-1}} = \sum_{A_i} d_{A_1\cdots A_{J-1}} \xi^{A_1}_{\nu_1} \cdots \xi^{A_{J-1}}_{\nu_{J-1}}$$

Sketch of Proof:

$A_i = P, K, d$

Killing Vectors ξ^A of Poincaré AdS_2

$$ds^2 = \frac{L^2}{y^2}(dx^2 + dy^2)$$

$$P = \partial_x, \quad d = x\partial_x + y\partial_y, \quad K = (x^2 - y^2)\partial_x + 2xy\partial_y$$

- We act on
  with isometries ξ

$$\nabla^{\mu}\left(\xi^{\mu_2 \cdots \mu_{J-1}} T_{\mu \mu_2 \cdots \mu_{J-1}}\right) = 0$$

- Leads to new conservation

$$\boxed{\left(\mathcal{L}_{\xi}\, \xi^{\mu_2 \cdots \mu_{J-1}}\right) \nabla^{\mu} T_{\mu \mu_2 \cdots \mu_{J-1}} = 0}$$

- This allows us to span
  space of Killing Tensors

$$\underbrace{P^{J-1} \underset{P}{\overset{K}{\rightleftarrows}} P^{J-2} L \underset{P}{\overset{K}{\rightleftarrows}} \cdots \underset{P}{\overset{K}{\rightleftarrows}} K^{J-1}}_{2J-1}$$

Locally span
J-dim space
of symmetric Tensors

$\Rightarrow$ Fully Conserved

- Same logic allows us to write conservation of charges as Heisenberg's e.o.m:

if $\xi \equiv \frac{\partial}{\partial \mu}$

$$\boxed{\frac{dQ_\xi}{d\mu} = Q_{\frac{\partial \xi}{\partial \mu}} + [Q_\xi, Q_\xi] = 0}$$

Examples ($J=4$)

$$\xi_{\mu\nu\rho} = p^\mu p^\nu p^\rho = \delta^\mu_x \delta^\nu_x \delta^\rho_x$$

$$[P, Q_{p^3}] = 0 \qquad [D, Q_{p^3}] = 3 Q_{p^3}$$

III) Illustration in GFF

- Consider a free massive scalar Φ (dual to GFF ϕ)

$$S = \frac{1}{2}\int d^2x\sqrt{g}\left((\partial_\mu\Phi)^2 + m^2\Phi^2\right) \qquad m^2L^2 = \Delta_\phi(\Delta_\phi - 1)$$

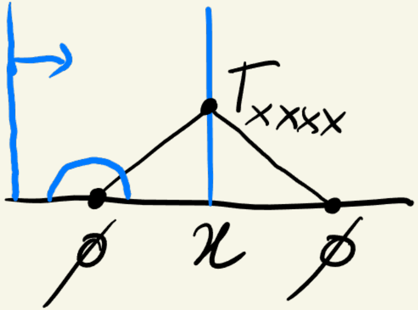
- This theory has HS currents for all even J. e.g.

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\rho\Phi\partial^\rho\Phi + m^2\Phi^2\right)$$

[Bekaert, Meunier] 10'

$$T_{\mu\nu\rho\sigma} = \Phi\overleftrightarrow{\nabla}_{(\mu}\overleftrightarrow{\nabla}_\nu\overleftrightarrow{\nabla}_\rho\overleftrightarrow{\nabla}_{\sigma)}\Phi + \frac{14}{L^2}g_{(\mu\nu}\Phi\overleftrightarrow{\nabla}_\rho\overleftrightarrow{\nabla}_{\sigma)}\Phi + \frac{9}{L^4}g_{(\mu\nu}g_{\rho\sigma)}\vec{\Phi}^2$$

- <u>Build Charge</u> $\quad Q_{p^3} = \int_0^\infty dy\, T_{xxxx}(x,y)$



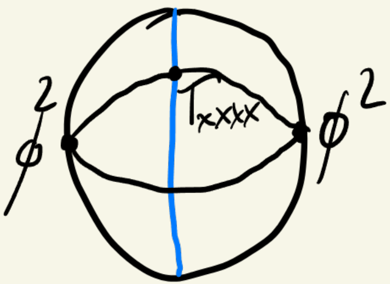
- Determine <u>action</u> on local <u>boundary Operators</u> (simplest $\not\phi$)

$$\langle \not\phi(x_1)\, Q_{p^3}(x)\, \not\phi(x_2)\rangle = \int_0^\infty dy\, \langle \not\phi(x_1)\, T_{xxxx}(x,y)\, \not\phi(x_2)\rangle$$

$$= \begin{cases} 16\, \partial_{x_2}^3 \langle \not\phi(x_1)\, \not\phi(x_2)\rangle & x_1 < x < x_2 \\ 0 & x < x_1 \text{ and } x > x_2 \end{cases} \longrightarrow \boxed{[Q_{p^3}, \not\phi(x)] = 16\, \partial^3 \not\phi(x)}$$

Is This Simply ϕ^3? No, much more powerful!

$$[Q_{\phi^3}, \phi^2(x)] = \kappa_\phi \, \phi \partial^3 \phi = \alpha_\phi \partial^3(\phi^2) + \beta_\phi \partial(\underbrace{\phi \Box \phi \cdots}_{\text{Primary!}})$$



- HS Charges imply integer spacing in Primary Spectrum $\Delta = n\Delta_\phi + m$

(HS multiplets)

- Full set of charges Q_s forms $hs[\lambda]$

[Vasiliev]'91 [Bekaert, Alkalaev]'19

- Same for Majorana Fermion

IV) NO-GO Theorems for deformations

Physics-Mathematics Dictionary

by Hrvoje Nikolić

physical language	mathematical language
physics	applied mathematics
mathematics	applied mathematics
abstract nonsense	mathematics
conjecture	vague idea
theorem	conjecture
rigorous theorem	theorem
proof	sketch of the proof
basic operations $+, -, \cdot, :$	commutative field theory
classical field theory	multi-variable calculus
mathematical analysis	calculus
linear algebra	matrix calculus
group theory	representation theory
abstract group theory	group theory
quantum mechanics	mambo jumbo Dirac notation
fancy schmancy functional analysis	quantum mechanics
Hilbert space	rigged Hilbert space
Riemannian geometry	pseudo-Riemannian geometry
tensor	components of a tensor
contravariant vector	components of a vector
covariant vector	components of a one-form
fancy schmancy index-free notation	differential geometry

IV) NO-GO Theorems for deformations

Theorem: Continuous Interacting deformations of a GFF cannot preserve Q_{p^3}.

Corollary: Continuous Interacting deformations of a Free field in AdS cannot preserve HS currents

- Theorem implies Corollary: $\nabla^\mu j_\mu \Rightarrow \nabla^\mu T_{\mu\nu\rho\sigma} \Rightarrow \dot{Q}_{p^3} = 0$

For spin J currents, $Q_\xi^{(J)} \xrightarrow{hs[\lambda]} Q_{p^3}$

- Note that Q_{p^3} + Isom. conserved implies all other Q_ξ conserved.

For example: $[Q_{p^3}, Q_{p^2 d}] = Q_{p^5}$ by Jacobi identities

Proof now proceeds in two steps:

1) Show that $[Q_\xi, \phi(x)]$ cannot get modified

2) Show that $[Q_\xi, \phi(x)]$ fixes GFF correlators by Ward + Continuity

1) Rigidity of The charges' Action

$\theta_0 \to \phi + \theta(\tilde{\lambda})$

GFF $+\tilde{\lambda}$ $\quad [Q_{p^3}, \theta_0] = \partial^3\theta_0 + \alpha(\tilde{\lambda})\partial(\theta_2) + \beta(\tilde{\lambda})\theta_3$

Deforming $(\tilde{\lambda})$ (Non-perturbative form of Action)

- Note $\Delta[\theta_2] = \Delta_\phi + 2 \qquad \Delta[\theta_3] = \Delta_\phi + 3$ Primaries

- However for generic Δ_ϕ There are simply no candidates.

- Same applies for $\quad [Q_{p^{J-1}}, \phi] = \partial^{J-1}\phi + \ldots$

2) Fixing Correlators w/ Ward Identity

Cf. [Maldacena, Zhiboedov] 10'

When charge is preserved $Q_\zeta|0\rangle = 0$. Hence,

$$0 = \langle 0| Q_\zeta \mathcal{O}_0(x_1)\mathcal{O}_0(x_2)\mathcal{O}_0(x_3)\mathcal{O}_0(x_4)|0\rangle = \langle 0| [Q_{\zeta}, \mathcal{O}_0(x_1)]\mathcal{O}_0(x_2)\cdots|0\rangle$$

+ Perms.

Fix 4-Point Function

$$\left(\partial^3_{x_1}+\partial^3_{x_2}+\partial^3_{x_3}+\partial^3_{x_4}\right)\langle \mathcal{O}_0(x_1)\mathcal{O}_0(x_2)\mathcal{O}_0(x_3)\mathcal{O}_0(x_4)\rangle = 0$$

$$\Rightarrow \langle \mathcal{O}_0\mathcal{O}_0\mathcal{O}_0\mathcal{O}_0\rangle = \eta\, GF\text{Boson} + (1-\eta)\, GF\text{Fermion}$$

By continuity, cannot introduce new states.

Remains GFB or GFF

2') Fixing Higher Point Functions

- To show theory remains free, we focus on n-point functions of θ_0.

Now using $Q_p^{\,y-1} \longrightarrow \left(\sum_{i=1}^{N} \partial_{x_i}^{\,y-1}\right)\langle \theta_0(x_1)\cdots\theta_0(x_n)\rangle = 0$

- In momentum space: $\left(\sum_{i=1}^{N} p_i^{\,y-1}\right)\langle \tilde{\theta}_0(p_1)\cdots\tilde{\theta}_0(p_n)\rangle = 0$

- By famous 2d S-Matrix argument: Pairwise Factorization

$$\langle \tilde{\theta}_0(p_1)\cdots\tilde{\theta}_0(p_n)\rangle = \delta(p_1+p_2)\,\delta(p_3+p_4)\cdots\delta(p_{n-1}+p_n)\,f(p_1,\cdots,p_n)$$

- Going back to position space and using conformality

$$\langle \mathcal{O}_0(x_1) \cdots \mathcal{O}_0(x_n) \rangle = \sum_{\text{Perms}} \frac{C_{(12)\cdots(n-1\,n)}}{x_{12}^{2\Delta_0} \cdots x_{n-1\,n}^{2\Delta_0}}$$

- Then Statistics/continuity + OPE Fixes

$$\boxed{\langle \mathcal{O}_0(x_1) \cdots \mathcal{O}_0(x_n) \rangle = \text{GFF or GFB}}$$

- Special values of Δ_ϕ can be checked to leading order
- Interesting leeway for many integer spaced ϕ_i

Similar Theorem for deformations of CFTs in AdS

- HS charges are not Virasoro L_n's but live in UEA.

E.g. $Q_{p^3} = \int (TT)(z)\, dz = \boxed{2\sum_{n=-1}^{\infty} L_{-3-n} L_n}$

- Any CFT in AdS has displacement family $D, D^2, \ldots$

$\Delta_D = 2$

$\Delta_{D^2} = 4$

$[Q_{p^3}, D] = \alpha_c \partial^3 D + \beta_c \partial D^2 \rightarrow \gamma_D = \gamma_{D^2}$

[Lauria, Van Rees 23', 24' | Milam AA]

But

$$\gamma_{D^2} = \frac{20c + 25\Delta_\Phi(\Delta_\Phi - 2) + 64}{2(5c+22)}\gamma_D$$

Only Unitary Solution

$c = 1/2 \quad \Delta_\Phi = 1$!

Non-perturbative Theorem for Long-Range Models

$$\text{Long-Range Ising} = \int_{\mathbb{R}} dx\,dy\, \frac{\phi(x)\phi(y)}{|x-y|^{2(1-\Delta_\phi)}} + \lambda \int_{\mathbb{R}} dx\, \phi^4$$

Family of 1d = Ts

w/ $\frac{1}{4} \geq \Delta_\phi \geq 0$

$$= \frac{1}{2}\int_{AdS_2} d^2x\, (\partial_\mu \Phi)^2 + m^2 \Phi^2 + \lambda \int_{\partial AdS} dx\, \phi^4$$

Theorem: Interacting fixed points of LR models cannot preserve HS charges and must contain O_J w $\Delta = J = 4, 6, 8 \ldots$

(Breaking of HS Charges by Boundary Conditions)

V) Summary and Outlook

- $\exists$ Solutions to crossing (CFT$_1$) which do not have particle production. But:

- GFF + CFTs in AdS have HS charges that cannot be preserved by deformations.

no time

- We also developed the theory of when these charges are finite (Cardy-like conditions, Improvements)

- Sum rules for BOE were also derived

Outlook

- Weakly broken currents & Flat space limit

Is there a First Principle?

- Non-local charges (Quantum Groups?)

- Can we construct the exact correlators?

Thank you for your Attention!