Regularization by noise

Paul Gassiat

LAMA, Université Gustave Eiffel & IUF

Les Probabilités de Demain
April 15, 2026

Regularization by noise

- General principle : adding an irregular / stochastic term to an equation improves (in some sense) its behaviour compared to the deterministic equation.
- Physical interpretation/motivation : microscopic structures have a stabilizing influence on macroscopic dynamics.

Regularization by noise

- General principle : adding an irregular / stochastic term to an equation improves (in some sense) its behaviour compared to the deterministic equation.
- Physical interpretation/motivation : microscopic structures have a stabilizing influence on macroscopic dynamics.
- Can take many forms :
  - Restoring well-posedness to an ill-posed equation, which can lead to selection by noise
  - Delaying / Preventing blow-up,
  - Improving properties (e.g. regularity, convergence to equilibrium) of solutions,

  and in many mathematical contexts (ODE, PDE, ...), domains of applications (fluid dynamics,...).

Regularization by noise

- General principle : adding an irregular / stochastic term to an equation improves (in some sense) its behaviour compared to the deterministic equation.
- Physical interpretation/motivation : microscopic structures have a stabilizing influence on macroscopic dynamics.
- Can take many forms :
  - Restoring well-posedness to an ill-posed equation, which can lead to selection by noise
  - Delaying / Preventing blow-up,
  - Improving properties (e.g. regularity, convergence to equilibrium) of solutions,

  and in many mathematical contexts (ODE, PDE, ...), domains of applications (fluid dynamics,...).
- Mathematical analysis typically based on stochastic calculus

Regularization by noise

- General principle : adding an irregular / stochastic term to an equation improves (in some sense) its behaviour compared to the deterministic equation.
- Physical interpretation/motivation : microscopic structures have a stabilizing influence on macroscopic dynamics.
- Can take many forms :
  - Restoring well-posedness to an ill-posed equation, which can lead to selection by noise
  - Delaying / Preventing blow-up,
  - Improving properties (e.g. regularity, convergence to equilibrium) of solutions,

  and in many mathematical contexts (ODE, PDE, ...), domains of applications (fluid dynamics,...).
- Mathematical analysis typically based on stochastic calculus

1. The case of ODE
   - Classical results
   - Regularization by fractional noise

2. Further examples
   - Selection by noise for mean curvature flow
   - Regularization for the PDE of fluid dynamics
   - Irregular paths as controls

Outline

1. The case of ODE
   - Classical results
   - Regularization by fractional noise
2. Further examples
   - Selection by noise for mean curvature flow
   - Regularization for the PDE of fluid dynamics
   - Irregular paths as controls

Case of ODE

Recall that, by the classical Cauchy-Lipschitz theory, the ODE

$$dx_t = b(x_t)dt, \quad x_0 \in \mathbb{R}^N$$

is classically well-posed for b **Lipschitz continuous**.

In contrast, consider the case of SDE :

$$dX_t = b(X_t)dt + dW_t, \quad X_0 \in \mathbb{R}^N, \tag{SDE}$$

where W is a (standard) Brownian motion.
We have the following result due to Zvonkine '74, Veretennikov '81 (adapted solutions), Davie '07 (path-by-path uniqueness).

Theorem

Le $b : \mathbb{R}^N \to \mathbb{R}^N$ be bounded. Then, a.s., (SDE) admits a unique solution.

Case of ODE

Recall that, by the classical Cauchy-Lipschitz theory, the ODE

$$dx_t = b(x_t)dt, \quad x_0 \in \mathbb{R}^N$$

is classically well-posed for b **Lipschitz continuous**.

In contrast, consider the case of SDE :

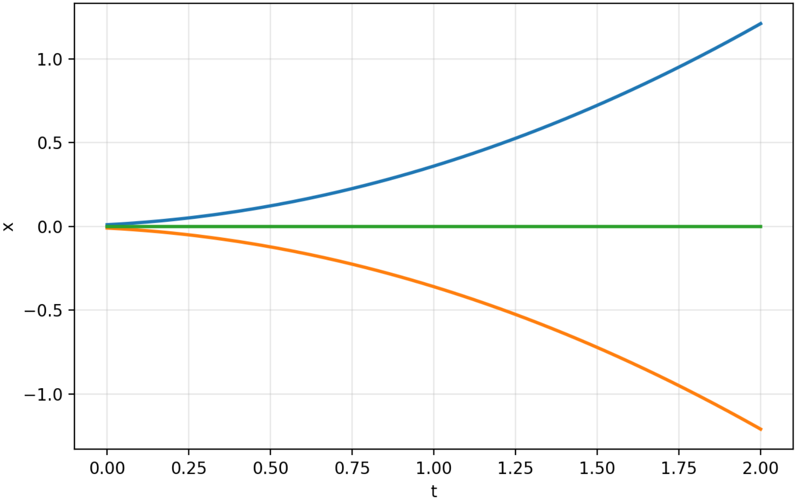
$$dX_t = b(X_t)dt + dW_t, \quad X_0 \in \mathbb{R}^N, \tag{SDE}$$

where W is a (standard) Brownian motion.
We have the following result due to Zvonkine '74, Veretennikov '81 (adapted solutions), Davie '07 (path-by-path uniqueness).

Theorem

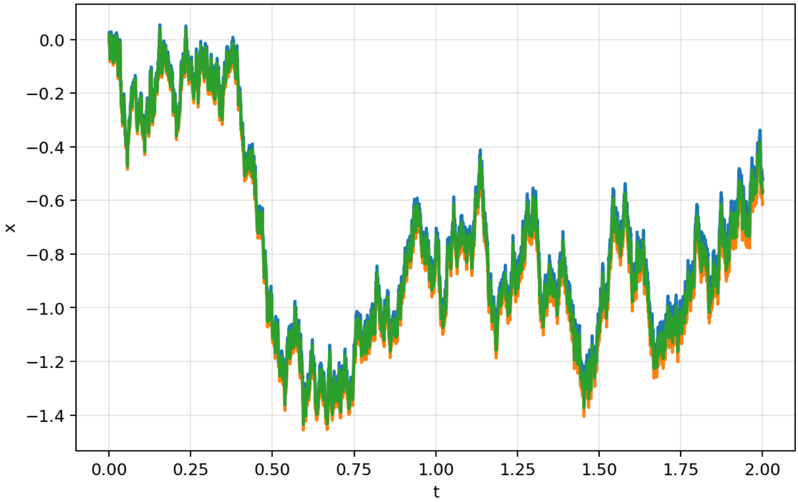
Le $b : \mathbb{R}^N \to \mathbb{R}^N$ be bounded. Then, a.s., (SDE) *admits a unique solution.*

Illustration : $dx_t = sgn(x_t)|x_t|^{1/2}dt$



For $x_0 = .01, -.01$ and 0 (constant solution).

Illustration : $dX_t = sgn(X_t)|X_t|^{1/2}dt + dW_t$



For $x_0 = .01, -.01$ and 0.

Intuition

$$X_t = X_0 + \int_0^t b(X_s)ds + W_t, \quad X_0 \in \mathbb{R}^N,$$

Letting $\theta = X - W$, the SDE is equivalent to

$$\theta_t = \theta_0 + \int_0^t b(\theta_s + W_s)ds.$$

This is a random ODE, close to an ODE along

$$T^{W;[0,t]}b \quad : \quad x \mapsto \int_0^t b(x + W_s)ds$$

(note that θ evolves at a slower time scale than W).

$T^{W;[0,t]}b$ is much more regular w.r.t. x than b !
In fact : $T^{W;[0,t]}b = b * \mu_{W;[0,t]}$ where $\mu_{W;[0,t]}$ is the occupation measure, and roughly speaking,

$$\text{irregularity of } W \leftrightarrow \text{regularity of } \mu_{W;[0,t]}.$$

Intuition

$$X_t = X_0 + \int_0^t b(X_s)ds + W_t, \quad X_0 \in \mathbb{R}^N,$$

Letting $\theta = X - W$, the SDE is equivalent to

$$\theta_t = \theta_0 + \int_0^t b(\theta_s + W_s)ds.$$

This is a random ODE, close to an ODE along

$$T^{W;[0,t]}b \quad : \quad x \mapsto \int_0^t b(x + W_s)ds$$

(note that θ evolves at a slower time scale than W).

$T^{W;[0,t]}b$ is much more regular w.r.t. x than b !
In fact : $T^{W;[0,t]}b = b * \mu_{W;[0,t]}$ where $\mu_{W;[0,t]}$ is the occupation measure, and roughly speaking,

$$\text{irregularity of } W \leftrightarrow \text{regularity of } \mu_{W;[0,t]}.$$

The Itô trick

Regularity of $T^W b$ is easy to prove using stochastic calculus.

For simplicity, assume we are in the scalar case.
Let u satisfy $\frac{1}{2}u'' = b$, then by Itô's formula :

$$\int_0^t b(x+W_s)ds = u(x+W_t) - u(x) - \int_0^t u'(x+W_s)dW_s.$$

Note that u' is now Lipschitz, and by Itô isometry it follows

$$\left\|(T^{W;[0,t]}b)(x) - (T^{W;[0,t]}b)(y)\right\|_{L^2(\Omega)} \leqslant C|x-y|.$$

With a little more work (and a slightly different test function), a similar argument proves existence/uniqueness of adapted solutions.

Some more comments

- The same argument works in any dimensions (considering PDE $\frac{1}{2}\Delta u = f$)
- Idem for multiplicative noise $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ ($\sigma\sigma^T$ elliptic)
- Time-dependent, other function spaces (L^p,...), numerical schemes, other Markovian processes (Lévy).

Selection by noise

Let b be non-Lipschitz and X^ε solve

$$dX_t = b(X_t)dt + \varepsilon dW_t, \quad X_0 \in \mathbb{R}^N,$$

what can we say about the behaviour of X^ε as $\varepsilon \to 0$?

Hope : convergence to one (or more) particular solution(s) to the ODE $\dot{x} = b(x)$, which could be interpreted as the natural "physical" solutions. (**Selection by noise**).

Theorem (Bafico-Baldi ('82))

In the scalar case : as $\varepsilon \to 0$, the law of X^ε concentrates on the "extremal" solutions.

Note : spontaneous stochasticity !

Further works : dynamical approach (Delarue-Flandoli '14), Large deviations (Gradinaru-Herrmann-Roynette '00), Higher dimensional examples (Delarue-Maurelli '19)

Selection by noise

Let b be non-Lipschitz and X^ε solve

$$dX_t = b(X_t)dt + \varepsilon dW_t, \quad X_0 \in \mathbb{R}^N,$$

what can we say about the behaviour of X^ε as $\varepsilon \to 0$?

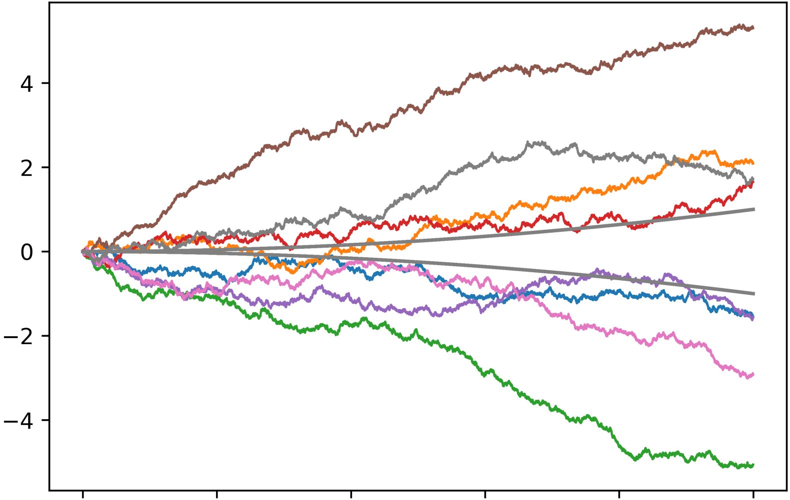
Hope : convergence to one (or more) particular solution(s) to the ODE $\dot{x} = b(x)$, which could be interpreted as the natural "physical" solutions. (**Selection by noise**).

Theorem (Bafico-Baldi ('82))

In the scalar case : as $\varepsilon \to 0$, the law of X^ε concentrates on the ''extremal'' solutions.

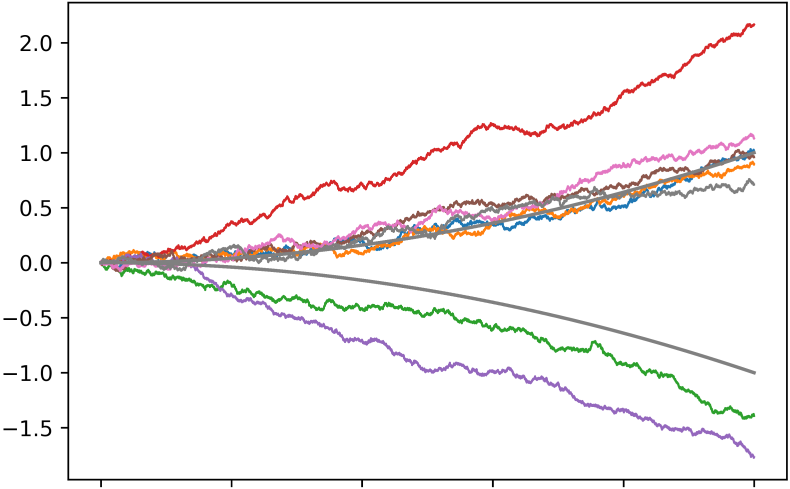
Note : spontaneous stochasticity !
Further works : dynamical approach (Delarue-Flandoli '14), Large deviations (Gradinaru-Herrmann-Roynette '00), Higher dimensional examples (Delarue-Maurelli '19)

Illustration $dX_t = sgn(X_t)|X_t|^{1/2}dt + \varepsilon dW_t$



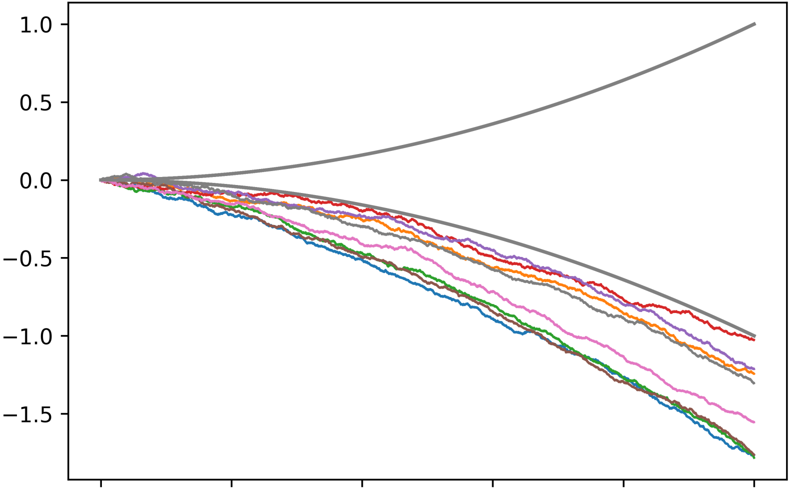
$\varepsilon = 1.0$

Illustration $dX_t = sgn(X_t)|X_t|^{1/2}dt + \varepsilon dW_t$



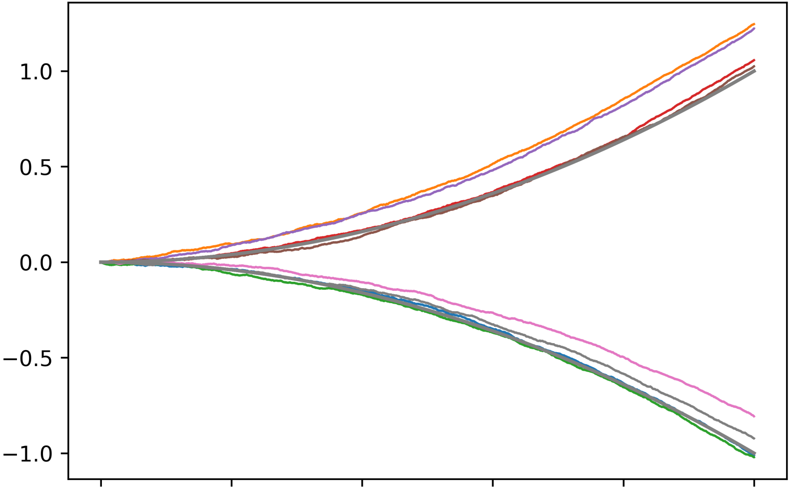
$\varepsilon = 0.3$

Illustration $dX_t = sgn(X_t)|X_t|^{1/2}dt + \varepsilon dW_t$



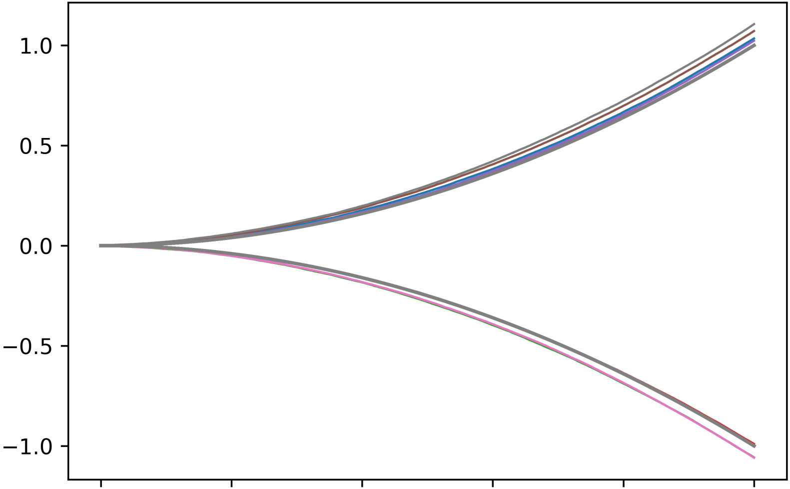
$\varepsilon = 0.1$

Illustration $dX_t = sgn(X_t)|X_t|^{1/2}dt + \varepsilon dW_t$



$\varepsilon = 0.03$

Illustration $dX_t = sgn(X_t)|X_t|^{1/2}dt + \varepsilon dW_t$



$\varepsilon = 0.005$

Fractional noise : motivation

Replace dW by dW^H, *fractional* Brownian motion (fBm).

Why ?

- Intuition : irregularity implies regularity. Want to make this quantitative.
  fBm comes with a **Hurst parameter** $H \in (0,1)$ allowing to tune (ir)regularity.
  (classical BM corresponds to $H = \frac{1}{2}$).
- Other reasons :
  - develop more flexible methods, which can then be used in more complicated contexts (e.g. PDE)
  - Stochastic heat equation $u(t,0)$ is a fBm with index $\frac{1}{4}$.

Fractional Brownian motion

$W = (W_t)_{t \geqslant 0}$ **fractional Brownian motion** (fBm) with **Hurst parameter** $H \in (0,1)$.

- Gaussian process, stationary increments, $W_0 = 0$ and

$$\|W_t - W_s\|_{L^2(\Omega)} = |t-s|^H,$$

- sample paths are $(H - \varepsilon)$-Hölder continuous
- NOT a semimartingale, NOT a Markov process (for $H \neq \frac{1}{2}$).
  ($\longrightarrow$ Itô's formula, PDE methods are not applicable)

Fractional Brownian motion

$W = (W_t)_{t \geqslant 0}$ **fractional Brownian motion** (fBm) with **Hurst parameter** $H \in (0, 1)$.

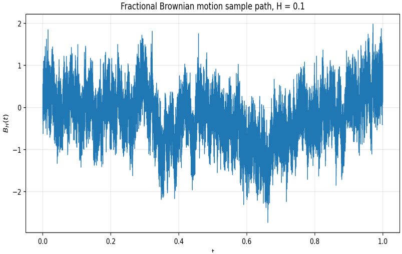
- Gaussian process, stationary increments, $W_0 = 0$ and

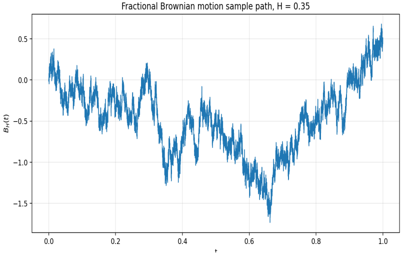
$$\|W_t - W_s\|_{L^2(\Omega)} = |t - s|^H,$$

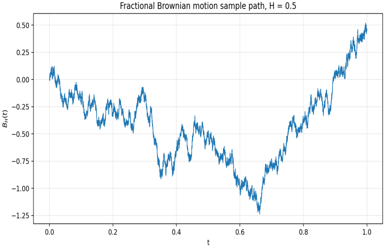
- sample paths are $(H - \varepsilon)$-Hölder continuous
- NOT a semimartingale, NOT a Markov process (for $H \neq \frac{1}{2}$).
  ($\longrightarrow$ Itô's formula, PDE methods are not applicable)

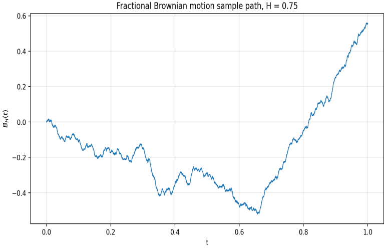
Fractional Brownian motion : sample paths

Simulated sample paths with $H \in \{0.1, 0.35, 0.5, 0.75\}$:

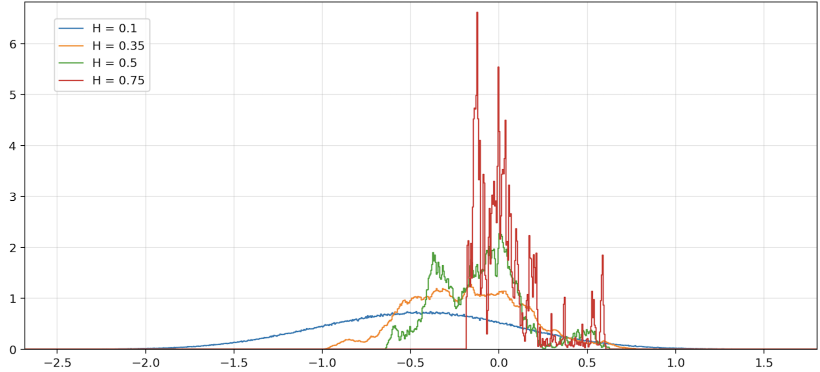








Fractional Brownian motion : occupation measures



Occupation density for the previous fBm paths

Well-posedness by fractional noise for ODE

Consider the equation

$$X_t = X_0 + \int_0^t b(X_s)ds + W_t^H \qquad \text{(SDE-H)}$$

with W^H a fBm.

Theorem (Catellier-Gubinelli '16)

Assume that $b \in C^\alpha$, with

$$\alpha > 1 - \frac{1}{2H}.$$

Then (SDE-H) *admits a unique strong solution.*

- One may have $\alpha < 0$, in which case $\int_0^t b(X_s)ds$ is suitably interpreted (e.g. as $\lim_n \int_0^t b_n(X_s)ds$)
- Note that the regularity requirement goes to $-\infty$ as $H \to 0$!

Ingredients 1 : LND property

As before, the main ingredient is the regularity of $x \mapsto \int_0^t f(x+W_s^H)ds$. How do we obtain it ?

- **Local non-determinism** property of fBm :

$$\forall s < t, Var((W_t - W_s)|\mathcal{F}_s) \geqslant c(t-s)^{2H}.$$

As a result :

$$\mathbb{E}[f(x+W_t)|\mathcal{F}_s] = (P_{\sigma(s,t)}f)(x+m_{s,t}(\omega))$$

wihere $(P_u)_{u \geqslant 0}$ is the heat kernel and $\sigma_H(s,t) \geqslant C(t-s)^H$.

Ingredients 2 : stochastic sewing

Sewing : allows to obtain estimates for

$$\mathcal{A}_t - \mathcal{A}_s = \int_s^t A_{u,u+du} := \lim_{(u_i) \text{ partition of } [s,t]} \sum_i A_{u_i,u_{i+1}},$$

given local "almost-increments" $A_{u,v}$ for $u < v$.

- Deterministic sewing : assuming

$$(\delta A)_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t} \leqslant C_A(t-s)^{1+\epsilon},$$

then the limit exists and

$$|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}| \leqslant c_\epsilon C_A(t-s)^{1+\epsilon}.$$

- **Stochastic sewing** (Lê '20) Idem, assuming

$$\| \mathbb{E}[(\delta A)_{s,u,t}|\mathcal{F}_s]\|_{L^m} \lesssim (t-s)^{1+\epsilon_2}, \quad \|(\delta A)_{s,u,t}\|_{L^m} \lesssim (t-s)^{1/2+\epsilon_1}.$$

get bounds on $\|\mathcal{A}_t - \mathcal{A}_s\|_{L^m}$.

Ingredients 3 : putting them together

We then want to apply this to

$$\int_0^t b(x+W_s)ds = \int_0^t \mathbb{E}[b(x+W_s)\mathcal{F}_s]ds \approx \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E}[b(x+W_s)|\mathcal{F}_{t_i}]ds.$$

Letting $A_{s,t} = \int_s^t \mathbb{E}[b(x+W_u) - b(y+W_u)|\mathcal{F}_s]du$, one has

$$\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s] \equiv 0 \text{ (by the tower property)}$$

and

$$|A_{s,t}| \leqslant \int_s^t \|P_{\sigma_H(s,u)}b\|_{C^1}du \leqslant \int_s^t (s-u)^{-H(1-\alpha)}\|b\|_{C^\alpha}du \lesssim (t-s)^{1/2+\epsilon_2}$$

with $\epsilon_2 := H(\alpha - \frac{1}{2H}) > 0$.
By stochastic sewing, we obtain

$$\left\| \int_0^t b(x+W_s)ds - \int_0^t b(y+W_s)ds \right\|_{L^m} \lesssim |x-y|.$$

Stochastic sewing and regularization by (fractional) noise

Many papers in the last decade using techniques of the type described above. For instance :

- Stochastic heat equations (Athreya-Butkovsky-Lê-Mytnik '26)
- Convergence rates for numerical schemes (Butkovsky-Dareiotis-Gerencser '21, Lê-Ling '25)
- Multiplicative noise (Catellier-Duboscq '25, Dareiotis-Gerencser '24, Matsuda-Mayorcas '23)
- Selection by noise (Pilipenko-Proske '17, G.-Mądry '26)
- Weak well-posedness : assuming

$$\alpha > \frac{1}{2} - \frac{1}{2H},$$

one has weak existence (Anzeletti-Richard-Tanré '23, Galeati-Gerencser '25) and **weak uniqueness** (Butkovsky-Mytnik '24+)

Outline

Mean curvature flow with noise

- Mean curvature flow is an evolution for hypersurfaces $(\Gamma(t))_{t \geqslant 0}$ defined by
$$V = -\kappa.$$

- Well-developed weak solution theory (in particular level set equations via viscosity PDE theory)

Mean curvature flow with noise

- Mean curvature flow is an evolution for hypersurfaces $(\Gamma(t))_{t \geqslant 0}$ defined by
$$V = -\kappa.$$
- Well-developed weak solution theory (in particular level set equations via viscosity PDE theory)
- For some configurations, the solution may not be uniquely defined (**fattening** phenomenon) : only gets $\Gamma^-(t) \subset \Gamma(t) \subset \Gamma^+(t)$.
- A particular example is when Γ_0 is two circles touching in $\mathbb{R}^2$.

Mean curvature flow with noise

- Mean curvature flow is an evolution for hypersurfaces $(\Gamma(t))_{t \geqslant 0}$ defined by

$$V = -\kappa.$$

- Well-developed weak solution theory (in particular level set equations via viscosity PDE theory)
- For some configurations, the solution may not be uniquely defined (**fattening** phenomenon) : only gets $\Gamma^{-}(t) \subset \Gamma(t) \subset \Gamma^{+}(t)$.
- A particular example is when Γ_0 is two circles touching in $\mathbb{R}^2$.

Theorem (Dirr-Luckhaus-Novaga '01, Souganidis-Yip '02)

Consider Γ^{ϵ} evolving via mcf with noise :

$$V = -\kappa + \epsilon \dot{W}(t).$$

Then, as $\epsilon \to 0$, $\Gamma^{\epsilon} \to \Gamma^{+}$, the outer solution.

Mean curvature flow with noise

- Mean curvature flow is an evolution for hypersurfaces $(\Gamma(t))_{t \geqslant 0}$ defined by
$$V = -\kappa.$$
- Well-developed weak solution theory (in particular level set equations via viscosity PDE theory)
- For some configurations, the solution may not be uniquely defined (**fattening** phenomenon) : only gets $\Gamma^-(t) \subset \Gamma(t) \subset \Gamma^+(t)$.
- A particular example is when Γ_0 is two circles touching in $\mathbb{R}^2$.

Theorem (Dirr-Luckhaus-Novaga '01, Souganidis-Yip '02)

Consider Γ^ϵ evolving via mcf with noise :
$$V = -\kappa + \epsilon \dot{W}(t).$$
Then, as $\epsilon \to 0$, $\Gamma^\epsilon \to \Gamma^+$, the outer solution.

Transport noise in fluid dynamics

Since 2020, series of works by Flandoli, Luo, Galeati, Pappalettera,... on regularization of PDE from fluid dynamics by **transport noise**.
An example :

Theorem (Flandoli-Luo '21)

Consider the 3D Navier-Stokes equation in vorticity form, perturbed by transport noise :

$$\partial_t \xi + (u \cdot \nabla \xi - \xi \cdot \nabla u) = \Delta \xi + \Pi(\circ \eta(t,x) \cdot \nabla \xi), \quad \xi = curl(u)$$

Then, for any initial condition ξ_0, any $\epsilon > 0$, for some choice of noise :

$$\mathbb{P}\left(\tau_{blow-up} = \infty\right) \geqslant 1 - \epsilon.$$

- Without noise term : existence only known for sufficiently small i.c. (or high enough viscosity coefficient)

Transport noise in fluid dynamics

Since 2020, series of works by Flandoli, Luo, Galeati, Pappalettera,... on regularization of PDE from fluid dynamics by **transport noise**.
An example :

Theorem (Flandoli-Luo '21)

Consider the 3D Navier-Stokes equation in vorticity form, perturbed by transport noise :

$$\partial_t \xi + (u \cdot \nabla \xi - \xi \cdot \nabla u) = \Delta \xi + \Pi(\circ \eta(t,x) \cdot \nabla \xi), \quad \xi = curl(u)$$

Then, for any initial condition ξ_0, any $\epsilon > 0$, for some choice of noise :

$$\mathbb{P}\left(\tau_{blow-up} = \infty\right) \geqslant 1 - \epsilon.$$

- Without noise term : existence only known for sufficiently small i.c. (or high enough viscosity coefficient)
- Idea : the equation with suitable noise is close to a NS equation with **enhanced viscosity**

$$\partial_t \xi + (u \cdot \nabla \xi - \xi \cdot \nabla u) = (1 + \nu_\eta) \Delta \xi$$

Transport noise in fluid dynamics

Since 2020, series of works by Flandoli, Luo, Galeati, Pappalettera,... on regularization of PDE from fluid dynamics by **transport noise**.
An example :

Theorem (Flandoli-Luo '21)

Consider the 3D Navier-Stokes equation in vorticity form, perturbed by transport noise :

$$\partial_t \xi + (u \cdot \nabla \xi - \xi \cdot \nabla u) = \Delta \xi + \Pi(\circ \eta(t,x) \cdot \nabla \xi), \quad \xi = curl(u)$$

Then, for any initial condition ξ_0*, any* $\epsilon > 0$*, for some choice of noise :*

$$\mathbb{P}\left(\tau_{blow-up} = \infty\right) \geqslant 1 - \epsilon.$$

- Without noise term : existence only known for sufficiently small i.c. (or high enough viscosity coefficient)
- Idea : the equation with suitable noise is close to a NS equation with **enhanced viscosity**

$$\partial_t \xi + (u \cdot \nabla \xi - \xi \cdot \nabla u) = (1 + \nu_\eta)\Delta \xi$$

Irregular controls

Control problem :

$$dX_t = \sum_i V_i(X_t) u^i_t dt, \quad X_0 = x. \text{ Find } u \in L^2([0,1], \mathbb{R}^d) \text{ s.t. } X^u_1 = y.$$

Under **bracket-generating (Hörmander)** condition on V_i, solution exists.

Issue : Possible degenerate points of the endoint-map. (leads in particular to saddle-points when trying to solve by gradient flow).

Theorem (G.-Suciu '26)

Let W be rough, and consider

$$dX_t = \sum_i V_i(X_t)(u^i_t dt + dW^i_t), \quad X_0 = x.$$

- *Degenerate points are moved to infinity,*
- *In the step-2 nilpotent case, convergence of the gradient flow holds almost surely if W is a Brownian path (or rougher).*

Irregular controls

Control problem :

$$dX_t = \sum_i V_i(X_t) u_t^i dt, \quad X_0 = x. \text{ Find } u \in L^2([0,1], \mathbb{R}^d) \text{ s.t. } X_1^u = y.$$

Under **bracket-generating (Hörmander)** condition on V_i, solution exists.
Issue : Possible degenerate points of the endoint-map. (leads in particular to saddle-points when trying to solve by gradient flow).

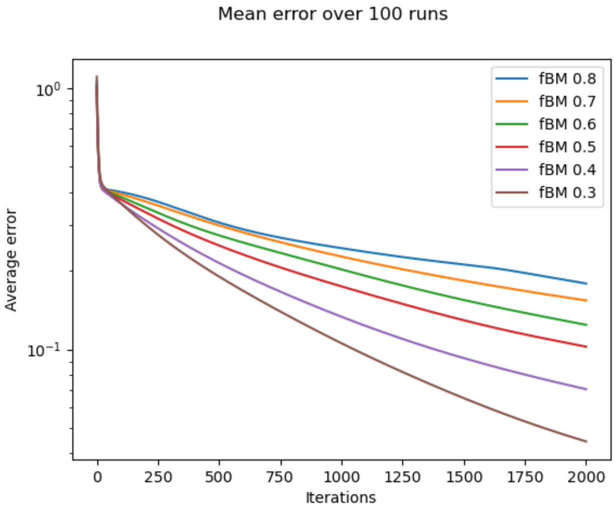
Theorem (G.-Suciu '26)

Let W be rough, and consider

$$dX_t = \sum_i V_i(X_t)(u_t^i dt + dW_t^i), \quad X_0 = x.$$

- *Degenerate points are moved to infinity,*
- *In the step-2 nilpotent case, convergence of the gradient flow holds almost surely if W is a Brownian path (or rougher).*

Irregular controls : a numerical illustration



(rank $d = 10$, $n = 55$ (step 2 nilpotent), 100 time points, learning rate$= 0.1$)

Conclusion / summary

Regularization by noise :
Adding an irregular / stochastic perturbation can improve the behaviour of an equation compared to the deterministic case.

Possible effects include :

- Restoring well-posedness / uniqueness,
- Selection of distinguished solutions,
- Delaying or preventing singularities,
- Improving stability or convergence properties.

Intuition : irregularity of the driving signal creates an averaging / smoothing effect on the dynamics.

Appears in many settings : ODE, geometric evolutions, PDE of fluid dynamics, control problems, ...

Selected (pedagogical) references

- F. Flandoli. *Random Perturbation of PDEs and Fluid Dynamic Models*. St-Flour lecture notes. 2010.
- B. Gess *Regularization and well-posedness by noise for ODE and PDE*. 2016.
- F. Flandoli, M. Rehmeier. *Remarks on regularization by noise, convex integration and spontaneous stochasticity.* arXiv 2024
- O. Butkovsky. *Lectures on stochastic sewing with applications*. arXiv 2025.