

# Renormalisation in the flow approach for singular SPDEs

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Probabilités de demain

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Let us consider the following general subcritical SPDE

$$(\partial_t + (-\Delta)^{\frac{\sigma}{2}})\varphi = g(\varphi, \nabla\varphi, \dots) + f(\varphi, \nabla\varphi, \dots)\xi. \quad (1)$$

$\xi$  is a space-time white noise *i.e.* a mean-zero Gaussian field with covariance structure given by

$$\mathbb{E}[(\xi, \varphi)(\xi, \psi)] = \langle \varphi, \psi \rangle_{L^2(\mathbb{R}_+ \times \mathbb{T}^d)}. \quad (2)$$

Belongs to  $\mathcal{C}_s^{-1-d/2-\kappa}$ .

## Examples:

- $\varphi^4$  equation:

$$(\partial_t - \Delta)\varphi = -\varphi^3 + \xi, \quad (3)$$

- KPZ equation:

$$(\partial_t - \Delta)u = (\partial_x u)^2 + \xi, \quad (4)$$

- Multiplicative heat equation:

$$(\partial_t - \Delta)u = u\xi, \quad (5)$$

- Sine-Gordon equation:

$$(\partial_t - \Delta)u = \sin(\beta u) + \xi \quad (6)$$

- Generalized KPZ equation:

$$(\partial_t - \Delta)u = g(\varphi)(\partial_x u)^2 + f(u)\xi. \quad (7)$$

## An elementary example: The Da Prato Debussche trick

Want to solve the  $\varphi_2^4$  equation.

Strategy: Mollify the noise, *i.e.* convolve it with

$$\rho_\varepsilon(t, x) = \varepsilon^{-2-d} \rho(\varepsilon^{-2}t, \varepsilon^{-d}x), \quad (8)$$

solve the equation for some  $\varphi_\varepsilon$ , and take the limit. Very generally ill-defined  $\rightarrow$  **Need for renormalisation.**

Define the solution to the linear equation  $\partial_t \varphi = \Delta \varphi + \xi$

$$\mathfrak{I}(t, x) = \int p_{t-s}(x-y) \xi(ds, dy), \quad (9)$$

and its mollified version  $\mathfrak{I}_\varepsilon$ .

Fact: By Schauder estimate,  $\mathfrak{I}$  is in  $C_s^{1-d/2-\kappa}$ .

Let  $\varphi_\varepsilon = \mathfrak{I}_\varepsilon + v_\varepsilon$ . We expect  $v$  to be more regular and run a fixed point argument. Naively, it solves

$$\partial_t v_\varepsilon = \Delta v_\varepsilon - v_\varepsilon^3 - 3\mathfrak{I}_\varepsilon v_\varepsilon^2 - 3\mathfrak{I}_\varepsilon^2 v_\varepsilon - \mathfrak{I}_\varepsilon^3. \quad (10)$$

Powers of  $\mathfrak{I}$  a priori ill-defined! But, taking the renormalised equation

$$(\partial_t - \Delta)\varphi_\varepsilon = -\varphi_\varepsilon^3 + 3C_\varepsilon\varphi_\varepsilon + \xi_\varepsilon, \quad (11)$$

we can write

$$\partial_t v_\varepsilon = \Delta v_\varepsilon - v_\varepsilon^3 - 3\mathfrak{I}_\varepsilon v_\varepsilon^2 - 3(\mathfrak{I}_\varepsilon^2 - C_\varepsilon)v_\varepsilon - (\mathfrak{I}_\varepsilon^3 - 3C_\varepsilon\mathfrak{I}_\varepsilon). \quad (12)$$

**Wick renormalisation:** We can show that, taking  $C_\varepsilon = \int (G * \rho_\varepsilon)(t, x)^2 dt dx$ , there are objects such that

$$\heartsuit = \lim_{\varepsilon \rightarrow 0} \mathfrak{I}_\varepsilon^2 - C_\varepsilon \quad \text{and} \quad \spadesuit = \lim_{\varepsilon \rightarrow 0} \mathfrak{I}_\varepsilon^3 - 3C_\varepsilon. \quad (13)$$

For  $d = 2$ , they are both in  $\mathcal{C}_s^{-\kappa}$ . We then try to solve the equation

$$v = -G * (v^3 + 3\heartsuit v^2 + 3\heartsuit v + \spadesuit) \quad (14)$$

Fact: Let  $f \in \mathcal{C}_s^\alpha$  and  $g \in \mathcal{C}_s^\beta$ . If  $\alpha + \beta > 0$ , then  $fg$  is well defined in  $\mathcal{C}_s^{\alpha \wedge \beta}$ .

Allows to solve (14) in  $\mathcal{C}_s^{2-\kappa}$ ! But breaks down in dimension 3.

It does not work to try

$$\varphi_\varepsilon = \mathfrak{I}_\varepsilon + \heartsuit_\varepsilon + v_\varepsilon \quad (15)$$

## Solution theories for SPDEs:

- Regularity structures, Hairer, 2014: Local expansions  
→ General ansatz available
- Paracontrolled calculus, Gubinelli-Imkeller-Perkowski, 2015:  
Harmonic analysis  
→ No general ansatz available
- Renormalisation group, Kupiainen 2016 and Polchinski-like  
flow: Duch 2021, Duch-Gubinelli-Rinaldi 2023, and  
Chandra-Ferdinand 2024  
→ General ansatz available (**this work**)

Focus on a simpler equation in dimension  $d = 5$ ,  $\sigma \in (d/3, d/2]$ .

$$(1 - \Delta)^{\frac{\sigma}{2}} = \lambda\varphi^3 + \sum_{i=1}^{i_\star} \lambda^i c_\varepsilon^i \varphi + \xi_\varepsilon, \quad (16)$$

where  $\xi$  is a space white noise,  $i_\star = \lfloor \sigma/(3\sigma - d) \rfloor$ , and the  $c_\varepsilon^i$  are renormalisation constants. Let

$$F_\varepsilon[\varphi] = \lambda\varphi^3 + \sum_{i=1}^{i_\star} \lambda^i c_\varepsilon^i \varphi + \xi_\varepsilon. \quad (17)$$

Rewrite (16) as

$$\varphi_\varepsilon = G * F_\varepsilon[\varphi_\varepsilon]. \quad (18)$$

Introduce

$$G_\mu(x) = \chi_\mu(|x|^\sigma)G(x), \quad (19)$$

with good support properties on  $\chi_\mu$  such that  $G_0 = G$  and  $G_1 = 0$ . Define the coarse-grained process

$$\varphi_{\varepsilon,\mu} = G_\mu * F_\varepsilon[\varphi_\varepsilon], \quad (20)$$

and an effective force

$$F_\varepsilon[\varphi_\varepsilon] = F_{\varepsilon,\mu}[\varphi_{\varepsilon,\mu}]. \quad (21)$$

It solves the Polchinski-like equation

$$\partial_\mu F_{\varepsilon,\mu}[\varphi_{\varepsilon,\mu}] + DF_{\varepsilon,\mu}[\varphi_{\varepsilon,\mu}] \cdot \dot{G}_\mu * F_{\varepsilon,\mu}[\varphi_{\varepsilon,\mu}] = 0. \quad (22)$$

Allows to construct global stochastic terms.

Denote graphically

$$\varphi^3 = \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \quad \text{and} \quad \xi_\varepsilon = \bullet \quad (23)$$

We get

$$D \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \cdot (\dot{G}_\mu * \bullet) = 3\varphi(x)^2 \int_{\mathbf{T}^d} \dot{G}_\mu(x-y)\xi_\varepsilon(y)dy \quad (24)$$

$$= 3 \begin{array}{c} \bullet \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \mu. \quad (25)$$

Reintegrating gives

$$- \int_0^\mu \begin{array}{c} \bullet \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \nu d\nu = \begin{array}{c} \bullet \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \mu + \begin{array}{c} \bullet \\ \downarrow \\ \downarrow \\ \downarrow \end{array} 0. \quad (26)$$

## Other examples

$$D \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} \cdot (\dot{G}_\mu * \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array}) = \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array}, \quad (27)$$

$$D \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} \cdot (\dot{G}_\mu * \bullet) = \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} \quad (28)$$

This reads

$$\begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} = ((G - G_\mu) * \xi_\varepsilon)(x)^2 \int_{\mathbf{T}^d} \dot{G}_\mu(x-y) ((G - G_\mu) * \xi_\varepsilon)(y)^2 \varphi(y) dy. \quad (29)$$

Also works the other way around by splitting

$$\partial_\mu \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} = - \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} - 2 \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} - 2 \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \quad \bullet \end{array} \quad (30)$$

Ansatz

$$F_{\varepsilon,\mu}[\varphi](x) = \sum_{i=0}^{i_*} \sum_{m=0}^{3i} \lambda^i \int_{(\mathbf{T}^d)^m} F_{\varepsilon,\mu}^{i,m}(x; y_1, \dots, y_m) \varphi(y_1) \dots \varphi(y_m) dy, \quad (31)$$

with initial conditions

$$F_{\varepsilon,\mu}^{0,0}(x) = \xi_\varepsilon(x), \quad (32)$$

$$F_{\varepsilon,\mu}^{1,3}(x, y_1, y_2, y_3) = \delta(x - y_1) \delta(x - y_2) \delta(x - y_3), \quad (33)$$

$$F_{\varepsilon,0}^{i,1}(x, y_1) = c_\varepsilon^i \delta(x - y_1). \quad (34)$$

The flow equation reads on the coefficients

$$\begin{aligned} \partial_\mu F_{\varepsilon,\mu}^{i,m}(x, y_1, \dots, y_m) = \\ - \sum_{j=0}^i \sum_{k=0}^m (1+k) \sum_{l=1}^{1+k} \int_{(\mathbf{T}^d)^2} F_{\varepsilon,\mu}^{j,1+k}(x; y_1, \dots, y_{1+k}) \times \\ \dot{G}_\mu(y_l - y) F_{\varepsilon,\mu}^{i-j,m-k}(y; y_{2+k}, \dots, y_m) dy_l dy. \end{aligned} \quad (35)$$

Allows to construct inductively the force terms. For instance

$$\int_{\mathbf{T}^d} F_{\varepsilon,\mu}^{1,1}(x; y_1) \varphi(y_1) dy_1 = \text{V} + c_\varepsilon^1 | \quad (36)$$

The renormalisation is implemented inductively.

$$D\left(\text{V} + c_\varepsilon^1 \text{I}\right) \cdot \left(\dot{G}_\mu * \left(\text{V} + c_\varepsilon^1 \text{I}\right)\right) = \text{V} + c_\varepsilon^1 \text{I} + c_\varepsilon^1 \text{V} + (c_\varepsilon^1)^2 \text{I}. \quad (37)$$

When integrating, we get

$$\text{V} + c_\varepsilon^1 \text{I} + c_\varepsilon^1 \text{V} + (c_\varepsilon^1)^2 \text{I} + \tilde{c}_\varepsilon^2 \text{I}. \quad (38)$$

**Intermezzo:** Models in regularity structures.  
 On trees with polynomial decoration on nodes  $\rightarrow$  naturally extends polynomials. Example

$$X^k \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}_k, \quad X^k \bullet = \bullet_k. \quad (39)$$

Every tree  $\tau$  can be written inductively as

$$\tau = X^k \Xi \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i), \quad (40)$$

$a_i \in \mathbb{N}^d$ , or similarly without noise. The product joins the roots.

$$\mathcal{I}_a(\bullet) = \begin{array}{c} \bullet \\ | \\ a \end{array} \quad (41)$$

For such trees, we can naturally define a root extraction-contraction coproduct:

$$\Delta_r \mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \Delta_r \Xi = \mathbf{1} \otimes \Xi + \Xi \otimes \mathbf{1}, \quad \Delta_r X_i = \mathbf{1} \otimes X_i + X_i \otimes \mathbf{1},$$

$$\Delta_r \mathcal{I}_a(\tau) = (\mathcal{I}_a \otimes \text{id})\Delta_r \tau + \sum_{k \in \mathbb{N}^d} \frac{X^k}{k!} \otimes \mathcal{I}_{a+k}(\tau),$$

$$\Delta_r(\sigma\tau) = \Delta_r \sigma \Delta_r \tau.$$

Example (we keep only the terms with an even number of noises on the left):

$$\begin{aligned} \Delta_r \text{tree} &= \mathbf{1} \otimes \text{tree} + \text{tree} \otimes \mathbf{1} + \text{tree} \otimes \text{tree} + \text{tree} \otimes \text{tree} + \dots \\ &+ 2 \text{tree} \otimes \text{tree} + \dots \end{aligned}$$

Define the so-called **Preparation map**

$$R\tau = (\ell \otimes \text{id})\Delta_r\tau. \quad (42)$$

$\ell$  vanishes on trees with an even number of noises or positive degree.

$$R \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right) \mathbf{1} + \ell \left( \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right) \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ | \\ \mathbf{1} \end{array} \quad (43)$$

Define for every  $\tau$  the preparation-map-dependent pre-model.

$$\begin{aligned}
 (\mathbf{\Pi}_\varepsilon^{R\times} \Xi)(x) &= \xi_\varepsilon(x), \quad (\mathbf{\Pi}_\varepsilon^{R\times} \mathbf{1})(x) = 1, \quad (\mathbf{\Pi}_\varepsilon^{R\times} X_i)(x) = x_i \\
 (\mathbf{\Pi}_\varepsilon^{R\times} \mathcal{I}_a(\tau))(x) &= (\partial^a G * \mathbf{\Pi}_\varepsilon^R \tau)(x), \\
 (\mathbf{\Pi}_\varepsilon^{R\times} (\sigma\tau))(x) &= (\mathbf{\Pi}_\varepsilon^{R\times} \sigma)(x)(\mathbf{\Pi}_\varepsilon^{R\times} \tau)(x),
 \end{aligned} \tag{44}$$

and finally

$$(\mathbf{\Pi}_\varepsilon^R \tau)(x) = (\mathbf{\Pi}_\varepsilon^{R\times} (R\tau))(x). \tag{45}$$

We also denote

$$\mathbf{\Pi}_\varepsilon = \mathbf{\Pi}_\varepsilon^{\text{id}}. \tag{46}$$

$$\Pi_\varepsilon^R \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} = \Pi_\varepsilon \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) \Pi_\varepsilon \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right), \quad (47)$$

$$\begin{aligned} \Pi_\varepsilon^R \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} &= \Pi_\varepsilon \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) \Pi_\varepsilon \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) \Pi_\varepsilon \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \\ &\quad + \ell \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right)^2 \Pi_\varepsilon \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \ell \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}. \end{aligned}$$

Compare with the terms that appear in the flow approach

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + c_\varepsilon^1 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + c_\varepsilon^1 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + (c_\varepsilon^1)^2 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \tilde{c}_\varepsilon^2 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}. \quad (48)$$

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + c_\varepsilon^1 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + c_\varepsilon^1 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + (c_\varepsilon^1)^2 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \tilde{c}_\varepsilon^2 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}. \quad (49)$$

## Theorem (Bruned, M., 2025)

We show an ansatz of the form

$$F_{\varepsilon,\mu}[\varphi] = \sum_{\tau \in T_0} \frac{1}{S(\tau)} (\Pi_{\varepsilon,\mu}^R \tau)[\varphi], \quad (50)$$

and the subsequent results

- 1 The ansatz solves the Polchinski equation
- 2 We recover the usual renormalised equation
- 3 We identify the renormalisation as being the BPHZ renormalisation in regularity structures.

This ansatz is comparable *after localization* to the one in regularity structures!

$$\Pi_{\varepsilon,\mu}^R \tau = (\ell \Upsilon \otimes \Pi_{\varepsilon,\mu}^{R \times})(M_{\text{loc}} \otimes \text{id}) \Delta_r \tau. \quad (51)$$

- $\Delta_r$  is a modified root-extraction coproduct
- $M_{\text{loc}}$  is a localization map. It inserts polynomials everywhere in the trees  $\sim$  Taylor
- The  $\Upsilon$ 's are the elementary differentials
- $\ell$  is a character on forests (later identified with the Bruned-Hairer-Zambotti character)
- $\Pi_{\varepsilon,\mu}^{R \times}$  is multiplicative and satisfies

$$\Pi_{\varepsilon,\mu}^{R \times} \mathcal{I}_a(\tau) = \partial^a(G - G_\mu) * \Pi_{\varepsilon,\mu}^R \tau. \quad (52)$$

The renormalised equation reads formally

$$(\partial_t + (-\Delta)^{\frac{\lambda}{2}})\varphi = g(\varphi, \dots) + f(\varphi, \dots)\xi + \sum_{\sigma \in T_0^X} \frac{\ell(\sigma)}{S(\sigma)} \Upsilon[\sigma][\varphi]. \quad (53)$$

$\varphi^4$  equation:

$$(\partial_t - \Delta)\varphi_\varepsilon = -\varphi_\varepsilon^3 + C_\varepsilon \varphi_\varepsilon + \xi_\varepsilon, \quad (54)$$

KPZ equation:

$$(\partial_t - \Delta)u_\varepsilon = (\partial_x u_\varepsilon)^2 + \xi_\varepsilon - C_\varepsilon. \quad (55)$$

THANK YOU!