Two dimensional Yang–Mills measure

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Context

- General setting for EQFT:
  - Set of fields $\mathcal{A}$: functions, distributions,...
  - Action: $S : \mathcal{A} \to \mathbb{R}_+$, like an energy
  - Set of observables Φ, physical relevant information on the field
  - An observable $\phi \in \Phi$ is a function on $\mathcal{A}$
- Construction of EQFT
  - Measure on $\mathcal{A}$ with formal expression

$$\mathrm{d}\mu(A) = \frac{1}{Z} \exp\big(- S(A)\big)\mathrm{d}A$$

  - The averages $\mathbb{E}[\phi(A)], \phi \in \Phi$

Example

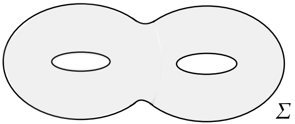
- $\mathcal{A} = \{f : [0,1] \to \mathbb{R} : f(0) = 0\}$
- $S(f) = \int_0^1 f'^2(x)\mathrm{d}x$
- Measure $\mathrm{d}\mu(f) = \frac{1}{Z}\exp\left(-S(f)\right)\mathrm{d}f$
- $\Phi = \{\mathrm{ev}_{s_1,\cdots,s_n} : \mathcal{A} \to \mathbb{R}^n \ : \ 0 \leq s_1 < \cdots < s_n \leq 1\}$ with

$$\mathrm{ev}_{s_1,\ldots,s_n}(f) = \big(f(s_1),\ldots,f(s_n)\big)$$

- Problems:
  - No Lebesgue measure $\mathrm{d}f$
  - $S(f)$ not well defined
  - Could be: $Z = \infty$
- μ is the Wiener measure
- A random function sampled under μ is a Brownian motion

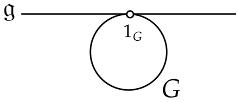
Setting

- Similar sort of question, different type of fields
- Hint on physical meaning
- Setting:
  - Σ: compact surface
  - σ: area measure
  - Space time



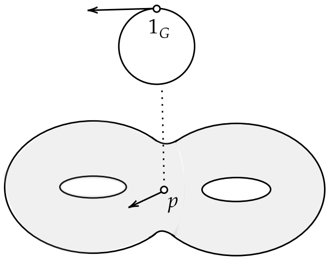
Set up

- Setting:
  - G: compact Lie group
  - Example:
  $$U(N) = \{X \in M_N(\mathbb{C}) : XX^* = I\}$$
  - Represents the 'state' of some particles, example: phase
  - $\mathfrak{g}$: its Lie algebra
  - Example:
  $$\mathfrak{u}(N) = \{X \in M_N(\mathbb{C}) : X = -X^*\}$$
  - Tangent space at 1_G
  - Infinitesimal change of state



Connection

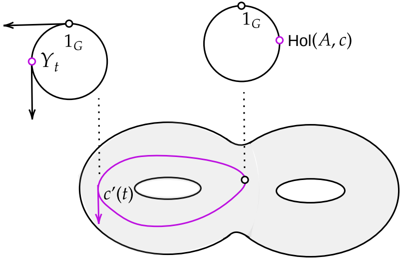
- Central object: a **connection** $A \in \Omega^1(\Sigma, \mathfrak{g})$
  - Represents a (physical) field
  - Given $p \in \Sigma$, $A_p : T_p\Sigma \to \mathfrak{g} = T_1G$ linear
  - For an infinitesimal movement of $\epsilon v_p \in T_p\Sigma$, the field change the state of the particle by $\epsilon A_p(v_p)$, provided it was at 1_G



Holonomy

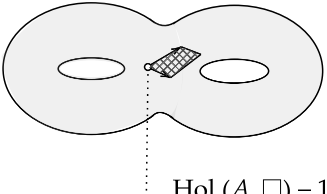
- A connection $A \in \Omega^1(\Sigma, \mathfrak{g})$ has a **holonomy**
  - A curve $c : [0,1] \to \Sigma$
  - Sums up all the infinitesimal changes of the state along c
  - $\mathrm{St}_0 \mathrm{Hol}(A, c) = \mathrm{St}_1$
  - Multiplicative version of $\int_c A$
  - Final value of the solution to

$$Y_t' = -Y_t A_{c(t)}(c'(t)) \quad Y_0 = 1_G$$



Curvature

- A connection $A \in \Omega^1(\Sigma, \mathfrak{g})$ has a **curvature**
  - Infinitesimal version of the holonomy
  - Curvature is when small (contractible) loops have non trivial holonomy
  - It is a $2-$form $\Omega = dA + \frac{1}{2}[A \wedge A]$, non-linear in A
  - For $X, Y \in T_p\Sigma$,



$$\Omega(X, Y) \approx \frac{\mathrm{Hol}\,(A, \square) - 1_G}{\sigma(\boxtimes)}$$

Yang–Mills action

- A connection $A \in \Omega^1(\Sigma, \mathfrak{g})$ has a **Yang–Mills action**
  - Global measure of the curvature strength of A
  - With the invariant inner product on $\mathfrak{g}$ we measure the curvature strength $\|\Omega_p\|$ at each point p
  - Not quadratic
  - Formula :

$$S_{YM}(A) = \int_\Sigma \|\Omega_p\|^2 \mathrm{d}\sigma(p)$$

Main question

- Main question: How to give a meaning to the probability measure

$$\mathrm{d}\mu(A) = \frac{1}{Z} \exp\big(- S_{YM}(A)\big)\, \mathrm{d}A$$

  with Z a normalization constant, and $\mathrm{d}A$ the Lebesgue measure
- When Σ is of dimension 3 or 4 this question is known to be very hard
- Citation from *Gauge Fields, Knots and Gravity*, Baez & Muniain:

> While many mathematicians have torn out their hair trying to provide a rigorous foundation for path integrals, with only partial success, physicists sail right along using them very effectively.

Observation

- Holonomies along sufficiently many curves determine connection
  - Analogy: a L^∞ function is determined by its integrals on Borel set
- A returned way of defining the measure: G-valued stochastic process indexed by paths
  - Analogy: a random L^∞ function is a real-valued stochastic process indexed by Borel sets
- No analytical object, no regularity property
  - Is there an L^∞ random function that produces these integrals ?
  - Is it distributional, differentiable, better regularity ?

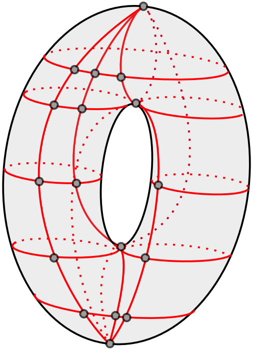
Literature

- In dimension 2 several constructions, two general approaches
  - A probability measure on $G^{\text{curves}(\Sigma)}$
    - Driver (1989), Semgupta (1992-1997), Lévy (2000-2010).
  - A probability measure on $\Omega^1(\Sigma, \mathfrak{g})$
    - Random distribution
    - Driver (on the plane 1989), Chevyrev (on the Torus 2018)
- Studied by several author: Cao, Chandra, Chatterjee, Dahlqvist, Gabriel, Hairer, Klose, Lemoine, Mohamed, Park, Pfeffer, Sheffield, Shen, Zhu, Zhu, and others

Main theorem (Prelim.)

- Graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ on Σ
- Discrete connection: $g : \mathbb{E} \to G$
- For c path in $\mathbb{G}$

$$\mathrm{Hol}(g, c) = \prod_{e \in c} g_e$$

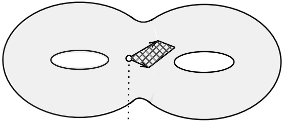


- Suppose we could sample A under $\mathrm{d}\mu(A) = \frac{1}{Z} \exp\left(- S_{YM}(A)\right) \mathrm{d}A$
  What would be the law of $(\mathrm{Hol}(A, e))_{e \in E}$?
- Need to approximate $S_{YM}(A)$ accessing only $\{\mathrm{Hol}(\mathrm{A}, \mathrm{e}) : e \in \mathbb{E}\}$

Main theorem (Prelim.)

- For a face $F \in \mathbb{F}$,

$$\int_F \|\Omega_p\|^2 \mathrm{d}\sigma(p) \approx \frac{d\left(\prod_{e\in\partial F} g_e, 1_G\right)^2}{\sigma(F)}$$



- Action of a discrete connection:

$$S^{\mathrm{disc}}_{YM}\left((g_e)_{e\in E}\right) = \sum_{F\in\mathbb{F}} \frac{d\left(1_G, \mathrm{Hol}(g,\partial F)\right)^2}{\sigma(F)}$$

- Pick edge holonomies under

$$\mathrm{d}\mu\left((g_e)_{e\in\mathbb{E}}\right) = \frac{1}{Z}\exp\left(-S^{\mathrm{disc}}_{YM}\left((g_e)_{e\in\mathbb{E}}\right)\right)\mathrm{d}g$$

Main theorem (Statement)

Theorem (With N. V. Dang)

There exists a sequence of graphs $\big(\mathbb{G}_N = (\mathbb{V}_N, \mathbb{E}_N, \mathbb{F}_N)\big)_{N\geq 0}$ on Σ and a sequence of random 1-forms $(A_N)_{N\geq 0}$ such that the following holds

- The holonomies of A_N on the edges of $\mathbb{G}_N$ induce the desired law
- There exists a random distributional 1-form A such that in distribution

$$A_N \xrightarrow{N\to\infty} A$$

- Gives a construction of A
- A is a universal limit
- Independent of the choice of the approximation of the curvature

Merci !