Liouville Theorems for the Stationary Navier-Stokes System for Incompressible Fluids

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entailing that f is a polynomial. Since f is assumed to be bounded, this implies that f is a constant, which is the sought result.

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It is important to note at this stage that the assumptions of v vanishing at infinity, $\operatorname{curl} v \in L^2, \operatorname{div} v = 0$ imply

$$v \in L^6 = L^{\frac{2d}{d-2}}, \quad d = 3.$$

Summing up, we assume that v is a solution of the Stationary Navier-Stokes System for Incompressible Fluids, vanishes at infinity with $\operatorname{curl} v \in L^2(\mathbb{R}^3)$. In particular, we know that v belongs to $\bigcap_{6 \le p \le \infty} L^p$. These are the standard hypotheses.

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Theorem (G.P. Galdi, 2011)

Let v be a solution of the Stationary Navier-Stokes System for Incompressible Fluids, vanishing at infinity, with $\operatorname{curl} v \in L^2(\mathbb{R}^3)$. Assuming moreover that v belongs to $L^{9/2}(\mathbb{R}^3)$, we obtain $v \equiv 0$.

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Note that

$$\underbrace{\bigcap_{\frac{9}{2} \le p \le \infty} L^p}_{\text{Galdi's assumption}} \subset \underbrace{\bigcap_{6 \le p \le \infty} L^p}_{\text{standard set}},$$

with a strict inclusion so the above theorem is assuming much more than the standard hypotheses.

The L^p condition is related to the behaviour at infinity and in some sense to the decay at infinity. Defining

$$L^p_\infty = \{u \in L^1_{\mathrm{loc}},\ \exists K \text{ compact such that } u \in L^\infty(K^c) \cap L^p(K^c)\},$$

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$$\int_{K_1^c} |f|^{p_2} dx \le \|f\|^{p_2-p_1}_{L^\infty(K_1^c)} \int_{K_1^c} |f|^{p_1} dx < +\infty.$$

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As a consequence, for compact intervals of $[1, +\infty]$, $J_1 \supset J_2$, we have $\min J_1 \le \min J_2$ and

$$L^{\min J_1}_\infty = \bigcap_{p \in J_1} L^p_\infty \subset \bigcap_{p \in J_2} L^p_\infty = L^{\min J_2}_\infty.$$

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Note that the Galdi assumption $v \in L^{9/2}(\mathbb{R}^3)$ is quite good, since it beats in particular the scaling of the equation at $L^3(\mathbb{R}^3)$: we have

$$\underbrace{\bigcap_{3\le p\le\infty}}_{\text{scaling}} L^p \subset \underbrace{\bigcap_{\frac{9}{2}\le p\le\infty}}_{\text{Galdi's assumption}} L^p \subset \underbrace{\bigcap_{6\le p\le\infty}}_{\text{standard set}} L^p.$$

Another very interesting result is due to D. Chae.

Theorem (D. Chae, 2014)

Let v be a solution of the Stationary Navier-Stokes System for Incompressible Fluids, vanishing at infinity, with $\operatorname{curl}^2 v \in L^{6/5}(\mathbb{R}^3)$. *Then we obtain* $v \equiv 0$.

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Note that here, there is no assumption on the curl itself, but that $\operatorname{curl} v \in L^2(\mathbb{R}^3)$ actually follows from the assumption since

$$\operatorname{curl} v \in \dot{W}^{1,\frac{6}{5}}(\mathbb{R}^3) \subset W^{0,2}(\mathbb{R}^3), \qquad \frac{1-0}{3} = \frac{5}{6} - \frac{1}{2}.$$

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Clearly, Chae is assuming more than the standard hypotheses, but it respects the scaling of the problem, thanks to the above Sobolev inclusion, which is sharp.

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As a consequence, the stationary Navier-Stokes system can be written as (we use the notation $\mathbf{C}$ for the curl operator)

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where R_0 is a standard singular integral, which is a Fourier multiplier bounded on every L^p, $1 < p < +\infty$, so that we can take

$$Q = R_0(v \otimes v) + \frac{1}{2}|v|^2 = S_0(v \otimes v) \in L^3.$$

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without an integration by parts.

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We may always multiply pointwisely to get

$$\nu < \mathbf{C}^2 v, v >_{\mathbb{R}^3} + \underbrace{\det\big(\mathbf{C} v, v, v\big)}_{=0} + \operatorname{div}(Q v) = 0,$$

a very helpful identity, essentially sufficient to get Galdi's $L^{9/2}$ result.

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proving the following

Lemma 1. Let v be a vector field on $\mathbb{R}^3$ such that such that the standard hypotheses are satisfied. Then we have

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typically a non-linear estimate, true for the solutions of the non-linear equation.

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All of the above is plausible but still quite vague and we would like to provide a couple of clearcut results supporting that perspective.

Standard assumptions: $\begin{cases} \nu \mathbf{C}^2 v + \mathbb{P}(\mathbf{C}v \times v) = 0, \operatorname{div} v = 0, \\ \lim_{|x| \to +\infty} v(x) = 0, \\ \mathbf{C}v \in L^2. \end{cases}$

Theorem 1. [An improvement of Galdi's result] Let us assume that v satisfies the standard assumptions. Moreover we assume that there exists $\alpha_0 \in C_c^\infty(\mathbb{R}^3_\xi)$, equal to 1 in a neighborhood of 0, such that

$$\alpha_0(D)v \in L^{9/2}(\mathbb{R}^3).$$

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This result means that the assumption $v \in L^{9/2}$ in Galdi's Theorem is too strong and that it is enough to assume that $v_{[0]} = \alpha_0(D)v$, the projection of v onto the space of vectors with spectrum in a given neighborhood of 0 belongs to $L^{9/2}$.

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A key point in our proof is that we are able to prove that the part of v with large spectrum belongs to L^2, that is with $\alpha_1(\xi) = 1 - \alpha_0(\xi)$ (which vanishes near 0)

$$\alpha_1(D)v \in L^2.$$

Theorem 2. [An improvement of Chae's result] Let us assume that v satisfies the standard assumptions. Moreover we assume that there exists $\alpha_0 \in C_c^\infty(\mathbb{R}^3_\xi)$, equal to 1 in a neighborhood of 0, such that

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The Wiener algebra is nice, much nicer than $C^0_{(0)}$ and L^∞: although these three algebras are stable by product (i.e. are algebras . . .), the Wiener algebra is also stable under the action of standard singular integrals, which is not the case of the two others.

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Enhanced Regularity Results, Wiener Algebras

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We define, for $k \in \mathbb{N}$, the multiplicative Banach algebra

$$\mathcal{V}^{k,\omega} = \{v \in \mathcal{W}, \forall \alpha \in \mathbb{N}^3, |\alpha| \le k, \partial_x^\alpha v \in \mathcal{W}\},$$

and the Fréchet algebra $\mathcal{V}^{\infty,\omega} = \cap_{k\ge 0}\mathcal{V}^{k,\omega}$,

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Then $\mathcal{A}$ is a Fréchet algebra, stable under differentiation and under the action of singular integrals.

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In particular v, Q are C^∞, with limit 0 at infinity as well as all their derivatives: in fact we have $\mathcal{V}^{k,\omega} \subset C^k_{(0)}$ (C^k functions with limit 0 at infinity).

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The smoothness of v was already proven by Galdi, but the fact that v, Q belong to $\mathcal{A}$ seems to be new and is quite helpful for our purpose.

Indeed the Stationary Navier-Stokes System reads,

$$\nu \mathbf{C}^2 v + \mathbb{P}\big(\mathbf{C} v \times v\big) = 0, \quad \operatorname{div} v = 0,$$

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Many important phenomena could occur near $\xi = 0$, and, in fact, it is particularly conspicuous for the Liouville problem.

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are not onto, but the *homogeneity* of $\mathcal{W}$ is the same as the homogeneity of L^∞, since defining for $\Lambda \in Gl(d, \mathbb{R})$, $u_\Lambda(x) = u(\Lambda x)$, we have, say for $u \in \mathcal{W}$,

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Although replacing $v \in C^k_{(0)}$ by $v \in \mathcal{V}^{k,\omega}$, as replacing $v \in C^0_{(0)}$ by $v \in \mathcal{W}$, seem minute improvements, they are both needed for our spectral arguments to work.

3. Elements for the proof

On the Wiener algebras

For $s \geq 0$, we have defined $\mathcal{V}^{s,\omega} = \{v \in \mathcal{W}, \langle D\rangle^s v \in \mathcal{W}\}, \quad \langle \xi\rangle^s = (1 + |\xi|^2)^{s/2}$.

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Proof. It is obvious that $\mathcal{V}^{s,\omega}$ is a Banach space isomorphic to $\mathcal{V}^{0,\omega} = \mathcal{W}$. Let us check the algebra property: let v_1, v_2 be in $\mathcal{V}^{s,\omega}$ for some $s \geq 0$.

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$$(\sharp) \qquad \langle \xi \rangle^s \widehat{v_1 v_2}(\xi) = \int \frac{\langle \xi \rangle^s}{\langle \xi - \eta \rangle^s \langle \eta \rangle^s} \left[\hat{v}_1(\eta)\langle \eta \rangle^s\right] \left[\hat{v}_2(\xi - \eta)\langle \xi - \eta \rangle^s\right] d\eta.$$

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Notice that for $\xi_1, \xi_2 \in \mathbb{R}^d$, we have $\frac{2+|\xi_1+\xi_2|^2}{(2+|\xi_1|^2)(2+|\xi_2|^2)} \leq 1$. We get by integration of the absolute value of the lhs of $(\sharp)$, the Banach algebra property

$$(\natural) \qquad \|v_1 v_2\|_{\mathcal{W}^{s,\omega}} \leq \|v_1\|_{\mathcal{W}^{s,\omega}} \|v_2\|_{\mathcal{W}^{s,\omega}}.$$

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The listener may have realized that we have surreptitiously changed the $1+|\xi|^2$ into a $2+|\xi|^2$, a rather harmless modification needed to have $(\natural)$ with the constant 1. By the way, the optimal change would have been $\frac{4}{3}+|\xi|^2$. Obviously not the point.

For $s \ge 0$, we have defined $\mathcal{V}^{s,\omega} = \{v \in \mathcal{W}, \langle D\rangle^s v \in \mathcal{W}\}, \quad \langle\xi\rangle^s = (1+|\xi|^2)^{s/2}.$

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Let $m(D)$ be a Fourier multiplier with $m(\xi)$ bounded and let $v \in \mathcal{W}^{s,\omega}$. We have

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and thus $\|m(D)v\|_{\mathcal{V}^{s,\omega}} = \int |m(\xi)\hat{v}(\xi)| \langle \xi \rangle^s d\xi \leq \|m\|_{L^\infty} \|v\|_{\mathcal{V}^{s,\omega}}$, concluding the proof of the lemma.

Lemma 4. *Let v be a vector field satisfying the standard assumptions. Then $v \in \mathcal{W}$ and we have for all $r > 0$,*

$$\nu^2 r^4 \int_{|\xi|\geq r} |\hat{v}(\xi)|^2 d\xi \leq \|\mathbf{C}v\|_{L^2}^2 \|v\|_{L^\infty}^2 < +\infty.$$

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Proof. We have

$$\mathbf{1}_{\{|\xi|\geq r\}} \hat{v}(\xi) = \underbrace{\left[|\xi|^{-2}\mathbf{1}_{\{|\xi|\geq r\}}\right]}_{\in L^\infty(\mathbb{R}^3)} \underbrace{\left[|\xi|^2|\hat{v}(\xi)|\right]}_{\substack{\in L^2(\mathbb{R}^3) \\ \text{since } \mathbf{C}^2 v \in L^2}}, \quad \text{which belongs to } L^2,$$

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`Proof.` We have

$$\mathbf{1}_{\{|\xi|\geq r\}} \hat{v}(\xi) = \underbrace{\left[|\xi|^{-2} \mathbf{1}_{\{|\xi|\geq r\}}\right]}_{\in L^\infty(\mathbb{R}^3)} \underbrace{\left[|\xi|^2 |\hat{v}(\xi)|\right]}_{\substack{\in L^2(\mathbb{R}^3) \\ \text{since } \mathbf{C}^2 v \in L^2}}, \quad \text{which belongs to } L^2,$$

and

$$\begin{aligned}
\int_{|\xi|\geq r} |\hat{v}(\xi)|^2 d\xi \leq r^{-4} \int |\xi|^4 |\hat{v}(\xi)|^2 d\xi \\
= r^{-4} \|\mathbf{C}^2 v\|_{L^2}^2 = r^{-4} \nu^{-2} \|\mathbb{P}(\mathbf{C}v \times v)\|_{L^2}^2 \\
\leq r^{-4} \nu^{-2} \|\mathbf{C}v \times v\|_{L^2}^2 \\
\leq r^{-4} \nu^{-2} \|\mathbf{C}v\|_{L^2}^2 \|v\|_{L^\infty}^2, \qquad \text{qed.}
\end{aligned}$$

Some properties of the Bernoulli head pressure.

We recall that the SNSI reads

$$\nu \mathbf{C}^2 v + \mathbb{P}(\mathbf{C} v \times v) = 0, \quad \operatorname{div} v = 0,$$

or as

$$\begin{cases} \nu \mathbf{C}^2 v + (\mathbf{C} v \times v) + \nabla Q = 0, \quad \operatorname{div} v = 0, \\ Q = R_0(v \otimes v) + \frac{1}{2}|v|^2 = S_0(v \otimes v). \end{cases}$$

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We get immediately

$$-\Delta Q = \operatorname{div}(\mathbf{C}v \times v) = <\mathbf{C}^2 v, v>_{\mathbb{R}^3} - |\mathbf{C}v|^2_{\mathbb{R}^3} = -\nu^{-1} <v, \nabla Q>_{\mathbb{R}^3} - |\mathbf{C}v|^2_{\mathbb{R}^3}.$$

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Lemma (Chae's key Lemma)

Let v be a vector field satisfying the standard assumptions. The Bernoulli head pressure Q takes negative values.

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Lemma (Chae's key Lemma)

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This result was proven by D. Chae, with a new multiplier method: instead of multiplying the equation by any χv (say with χ radial smooth compactly function), he chose to use a very particular function χ, appearing as a carefully chosen function of Q.

We have the following (technical) result:

Theorem 3. Let Q be the Bernoulli head pressure. If $(\Delta Q)_+$ *or* $(\Delta Q)_-$ belongs to $L^1(\mathbb{R}^3)$, then $v = 0$.

NB. Note that the assumptions of this theorem hold true for instance when Q is superharmonic or subharmonic.

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We use essentially the following fact: for any $\varepsilon_0 > 0$ which is a regular value of Q, we have

$$\int_{\{Q(x)<-\varepsilon_0\}} |\mathbf{C}v|^2 dx = \int_{\{Q(x)<-\varepsilon_0\}} \Delta Q \ dx,$$

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and using and the fact the set $\{x, Q(x) < -\varepsilon_k\}$ is a bounded regular open set for a sequence of positive ε_k with limit 0 (Sard's Theorem) which increases when ε_k decreases, we may apply the Beppo-Levi Theorem to get that

$$\int_{\{Q(x)<0\}} |\mathbf{C}v|^2 dx + \int_{\{Q(x)<0\}} (\Delta Q)_- dx = \int_{\{Q(x)<0\}} (\Delta Q)_+ dx.$$

The very first integral is finite; if one of the two following integrals is finite, then the other is also finite and ΔQ belongs to L^1. It is easy to prove in that case that $\int_{\mathbb{R}^3} (\Delta Q) \ dx = 0$.

An improvement of Galdi's result.

Theorem 1. Let us assume that the standard assumptions are fullfilled. Moreover we assume that there exists $\alpha_0 \in C_c^\infty(\mathbb{R}^3_\xi)$ whose support contains a neighborhood of 0 such that

$$\alpha_0(D)v \in L^{9/2}(\mathbb{R}^3).$$

Then $v \equiv 0$.

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This result means that the assumption $v \in L^{9/2}$ is too strong and that it is enough to assume that $v_{[0]}$, the projection of v onto the space of vectors with spectrum in a given neighborhood of 0 belongs to $L^{9/2}$.

Let ψ_0 be a given smooth compactly supported function defined on $\mathbb{R}^3$, valued in $[0, 1]$, equal to 1 on a neighborhood of 0. We define, for $\rho > 0$, $\chi_\rho(x) = \psi_0(x/\rho)$, and we note that $\|\nabla\chi_\rho\|_{L^3} = \|\nabla\psi_0\|_{L^3}$ (thus independent of ρ).

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$$\langle \nabla Q, \chi_\rho v\rangle = -\langle Q, v\cdot\nabla\chi_\rho\rangle = -\langle S_0(v\otimes v), v\cdot\nabla\chi_\rho\rangle.$$

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Let us check the second term: $\underbrace{\underbrace{v_0}_{L^6}\ \overbrace{S_0\Big[\underbrace{v_0}_{L^\infty} \otimes \underbrace{v_1}_{L^2}\Big]}^{L^2}}_{L^{3/2}\ \text{OK}}$. The two last terms are not very different.

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We start with
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\begin{aligned}
0 &= \langle \mathbf{C}^2 v + (\mathbf{C} v \times v) + \nabla Q, \chi_\rho v \rangle \\
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We check only two terms in $\langle Q, v \cdot \nabla\chi_\rho\rangle$, namely

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and we obtain $\mathbf{C}v = 0$, thus $\nabla v = 0$ so that $v = cst$ and $v = 0$ since it vanishes at infinity.

The discussion on the improvement of Chae's result ($\alpha_0(D)\mathbf{C}^2 v \in L^{6/5}$) is more technical. Since $-\Delta Q = \mathbf{C}^2 v \cdot v - |\mathbf{C} v|^2$, it is enough to check that $\mathbf{C}^2 v \cdot v \in L^1$. We have

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We have $\mathbf{C}^2 v_1 \cdot v_{00} = \underbrace{\beta_1(D)}_{|\xi| \ge 89/100} \left(\mathbf{C}^2 v_1 \cdot \alpha_{00}(D) v_0\right)$.

With $w_1 = \mathbf{C}^2 v_1$, we have since $\beta_1(D)\alpha_{00}(D) = 0$,

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$$\nu\mathbf{C}^2\omega + \overbrace{\underbrace{v}_{L^6}\cdot\underbrace{\nabla\omega}_{\sim\mathbf{C}^2 v\in L^{3/2}}}^{L^{6/5}} - \overbrace{\underbrace{\omega}_{L^2}\cdot\underbrace{\nabla v}_{L^2}}^{L^1} = 0,$$

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$$\frac{1}{i}\beta_1'(\xi) \cdot w_1'(x).$$

It is essentially enough to show that w_1' belongs to $L^{6/5}$, since $v_0 \in L^6$. On the other hand, we have

$$\nu \mathbf{C}^2\omega + \overbrace{\underbrace{v}_{L^6} \cdot \underbrace{\nabla\omega}_{\sim \mathbf{C}^2 v \in L^{3/2}}}^{L^{6/5}} - \overbrace{\underbrace{\omega}_{L^2} \cdot \underbrace{\nabla v}_{L^2}}^{L^1} = 0,$$

entailing that

$$\mathbf{C}^2\omega = \mathbf{C}(\mathbf{C}^2 v) \in L^1 + L^{6/5} \ni \nabla w_1.$$

We shall be left with

$$\nabla w_1 \cdot \alpha_{00}(D)v_0 \in L^1 \cdot \underbrace{\alpha_{00}(D)v_0}_{L^\infty} + L^{6/5} \cdot \underbrace{\alpha_{00}(D)v_0}_{L^6} \subset L^1, \qquad \texttt{qed.}$$

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Thanks for your attention