



The spatio-temporal statistical structure of turbulent dissipation and its stochastic modeling

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Navier-Stokes equations and energy budget

Navier-Stokes equations on \mathbb{R}^3 or \mathbb{T}^3

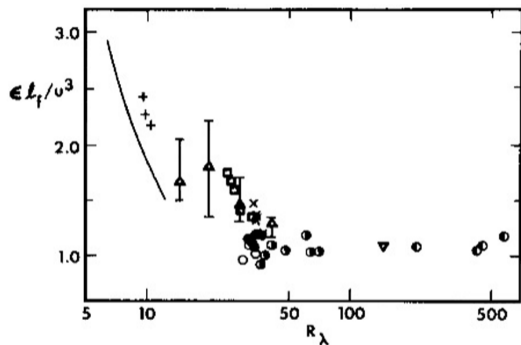
$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p^\nu + \nu \Delta u^\nu + f, \\ \frac{1}{2} \partial_t (u^\nu)^2 + \nabla \cdot J^\nu(t, x) = -\nu \|\nabla u^\nu\|^2 + u^\nu \cdot f \end{cases} \quad (\text{NSE})$$

Locally, kinetic energy is dissipated at a rate

$$\varepsilon^\nu(t, x) = \nu \|\nabla u^\nu\|^2 = \frac{\nu}{2} \sum_{i=1}^3 \left(\frac{\partial u_i^\nu}{\partial x_j} + \frac{\partial u_j^\nu}{\partial x_i} \right)^2$$

Why is the dissipation rate central to turbulence phenomenology? How to model its statistical structure?

Some experimental observations on $\varepsilon^\nu(t, x) = \nu \|\nabla u^\nu\|^2$



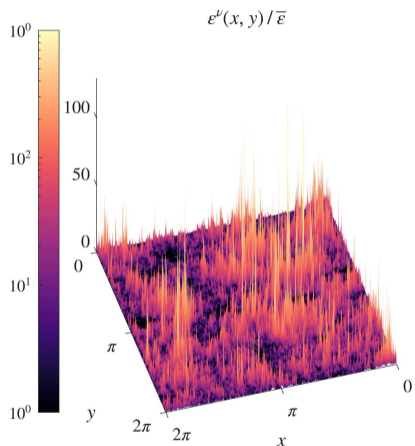
K. R. Sreenivasan, On the scaling of the turbulence energy dissipation rate, 1984

Remarkably

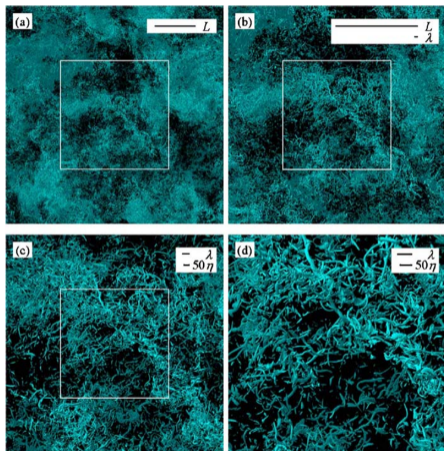
- $\lim_{\nu \rightarrow 0} \mathbb{E} [\varepsilon^\nu(t, x)] = \bar{\varepsilon} > 0.$
- $\lim_{\nu \rightarrow 0} \mathbb{E} [(u^\nu(t, x))^2] = \sigma^2 < \infty.$
- These two observations + dimension argument = Kolmogorov's phenomenology (1941)

What about higher order moments and correlations?

In the inviscid limit, $\varepsilon^\nu(t, x)$ is a positive measure

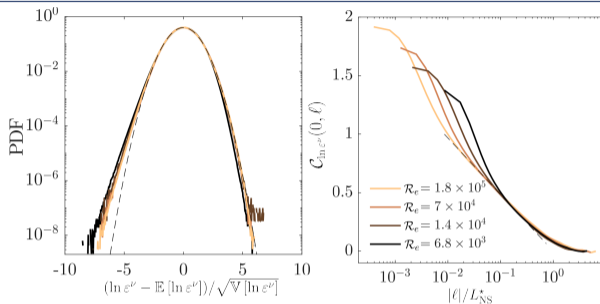


JHU Turbulence database, 4096^3 DNS.



Y. Kaneda and T. Ishihara, High-resolution direct numerical simulation of turbulence, 2006

Spatial structure of $\ln \varepsilon^\nu$



JHU Database: 1024^3 , 4096^3 , 8192^3 and $32,768^3$ DNS

- $\lim_{\nu \rightarrow 0} \mathcal{C}_{\ln \varepsilon^\nu}(\ell) = \mathbb{E}[\ln \varepsilon^\nu(x + \ell) \ln \varepsilon^\nu(x)]_c = \mu \ln \frac{L}{|\ell|} + f(\ell), \mu \simeq 0.2.$

Is it possible to give a rigorous meaning to the exponential of a log-correlated field?

Gaussian Multiplicative Chaos (Kahane, 1985)

Given X a Gaussian field (on \mathbb{R}^d or \mathbb{T}^d) and f continuous and bounded such that

$$\mathbb{E}[X(x)X(x + \ell)] = \ln\left(\frac{L}{|\ell|}\right) + f(|\ell|), \quad X \text{ is not defined pointwise}$$

Kahane (1985) make sense of " $e^{\gamma X(x)} dx$ " as a singular positive measure. Consider X^ϵ a mollification of X , then

$$M_\gamma^\epsilon(x) dx = e^{\gamma X^\epsilon(x) - \frac{\gamma^2}{2} \mathbb{E}(X^\epsilon)^2} dx \xrightarrow{\epsilon \rightarrow 0} M_\gamma(x) dx.$$

$M_\gamma(x) dx$ is a positive singular measure, non-degenerate if $0 < \gamma^2 < 2d$



Summary: GMC versus dissipation field

Dissipation

$$M_\gamma^\epsilon(x) = \bar{\epsilon} e^{\gamma X^\epsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[(X^\epsilon)^2]}$$

$\ln \varepsilon^\nu \simeq$ Gaussian

$X^\epsilon =$ Gaussian

$$\mathcal{C}_{\ln \varepsilon^\nu}(\ell) \sim \mu \ln \frac{L}{|\ell|}$$

$$\mathcal{C}_{\ln M_\gamma^\epsilon}(\ell) = \gamma^2 \mathbb{E}[X^\epsilon(x) X^\epsilon(x + \ell)] \sim \gamma^2 \ln \frac{L}{|\ell|}$$

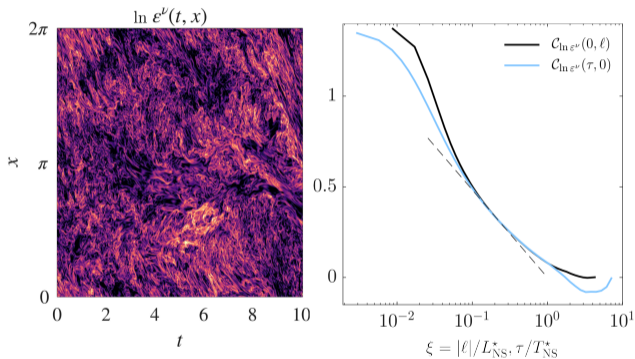
$$\mathbb{E}[\varepsilon^\nu] = \bar{\epsilon}$$

$$\mathbb{E}[M_\gamma^\epsilon] = \bar{\epsilon}$$

$$\text{Var}[\ln \varepsilon^\nu] \propto -\ln \eta_K$$

$$\text{Var}[\ln M_\gamma^\epsilon] \propto -\ln \epsilon$$

What about the temporal structure?



(Ruffenach & Chevillard 2026)

- $\ln \varepsilon^\nu$ is also correlated over large time scales.
- $C_{\ln \varepsilon^\nu}(\tau) = \mu \ln \frac{T}{|\tau|} + g(\tau)$,
 $\mu \simeq 0.2$.

Can we model this using a spatio-temporal GMC?

A spatio-temporal GMC

Goal is to build a spatio-temporal GMC:

$$M_{\gamma,\beta}^\epsilon(t, x) = \bar{\epsilon} e^{\gamma X_\beta^\epsilon(t, x) - \frac{\gamma^2}{2} \mathbb{E}(X_\beta^\epsilon)^2}$$

Where X_β^ϵ is a Gaussian field such that:

$$\mathcal{C}_{\ln M_{\gamma,\beta}}(0, \ell) \simeq \gamma^2 \ln \frac{L}{|\ell|} \quad \text{and} \quad \mathcal{C}_{\ln M_{\gamma,\beta}}(\tau, 0) \simeq \frac{\gamma^2}{2\beta} \ln \frac{T}{|\tau|}$$

In (Ruffenach & Chevillard, 2026), we propose a Fourier space dynamics generating such field X_β^ϵ .

The Fourier space dynamics

Ornstein-Uhlenbeck process on Fourier modes

$$d\widehat{X}_\beta^\epsilon(t, k) = -\frac{\widehat{X}_\beta^\epsilon(t, k)}{T_{k,\beta}} dt + G_k^\epsilon \sqrt{\frac{2}{T_{k,\beta}}} d\widehat{W}(t, k)$$

With scale dependent time-scale $T_{k,\beta}$ and kernel G_k^ϵ defined as

$$T_{k,\beta}^{-1} = D_3 |k|^{2\beta} \quad \text{and} \quad G_k^\epsilon = \frac{1}{\sqrt{A_{d-1}} |k|^{\frac{d}{2}}} \mathbb{1}_{\frac{1}{L} \leq |k| \leq \frac{1}{\epsilon}}(k), \quad A_{d-1} = |\mathbb{S}_{d-1}|$$

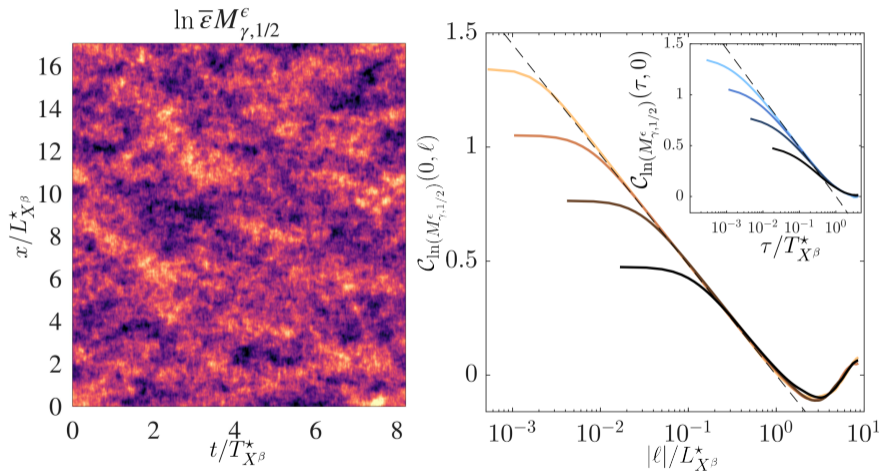
The field X_β^ϵ is correlated as

$$\mathbb{E}[X_\beta^\epsilon(t + \tau, x + \ell) X_\beta^\epsilon(t, x)] = \int_{\mathbb{R}^d} |G_k^\epsilon|^2 e^{-\frac{|\tau|}{T_{k,\beta}}} e^{2i\pi k \cdot \ell} dk$$

Or equivalently (in the limit $\epsilon \rightarrow 0$),

$$\mathcal{C}_{\ln M_{\gamma,\beta}}(0, \ell) = \gamma^2 \ln \frac{L}{|\ell|} + \gamma^2 f(\ell) \quad \text{and} \quad \mathcal{C}_{\ln M_{\gamma,\beta}}(\tau, 0) = \frac{\gamma^2}{2\beta} \ln \frac{T}{|\tau|} + \gamma^2 g(\tau)$$

Some simulations



(Ruffenach & Chevillard 2026)

Conclusion

Summary

- In good approximation, ε^ν can be treated as lognormal.
- $\ln \varepsilon^\nu$ is correlated logarithmically in both space and time and with the same intermittency coefficient μ .
- One can define a time dynamics (causal and Fourier-Markovian) generating a spatio-temporal GMC modeling space-time statistics of turbulent dissipation as a positive singular measure.

Perspectives

- Generalize to log-infinitely divisible processes (Bacry & Muzy, 2003)
- Define a spatio-temporal matrix valued GMC to add a time evolution to the very realistic random vector field of Peirera, Garban, Chevillard 2016.

Thank you!

Breakdown of K41

With the coarse grained dissipation field $\varepsilon_\ell = \lim_{\nu \rightarrow 0} \varphi_\ell * \varepsilon^\nu$, refined similarity hypothesis states

$$\mathbb{E} [(\delta_\ell^\parallel u)^p] \sim C_p (\mathbb{E} [(\varepsilon_\ell)^p])^{1/3} |\ell|^{p/3}.$$

K41 holds if $\mathbb{E} [(\varepsilon_\ell)^p] = \bar{\varepsilon}^p$. Now, assuming ε^ν is log-normal, one has

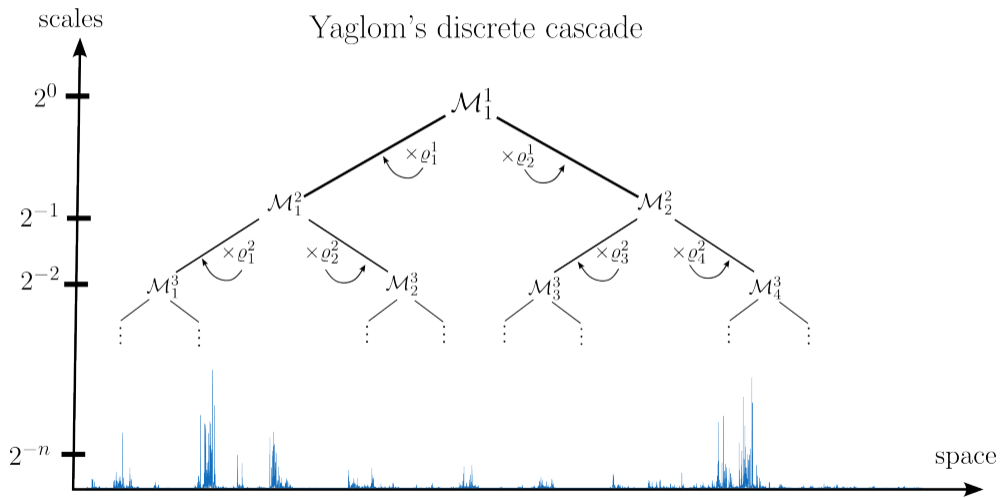
$$\mathbb{E} [(\varepsilon_\ell^\nu)^q] = \bar{\varepsilon}^q \int_{(\mathbb{R}^3)^q} \left(\prod_{i=1}^q \varphi(x - y_i) \right) e^{\sum_{i < j=1}^q C_{\ln \varepsilon^\nu}(|\ell|(y_i - y_j))} d^{3p} y.$$

Therefore, if

$$\lim_{|\ell| \rightarrow 0} \lim_{\nu \rightarrow 0} C_{\ln \varepsilon^\nu}(|\ell|(y_i - y_j)) = \lim_{\nu \rightarrow 0} \text{Var}[\ln \varepsilon^\nu] < +\infty$$

the dissipation rate is scale independent and K41 holds

Dyadic cascade



MF computation of $\text{Var}[\ln \varepsilon^\nu]$

The logarithmic correlations evaluated at Kolmogorov scale suggest:

$$\text{Var}[\ln \varepsilon^\nu] \underset{\mathcal{R}_e \rightarrow \infty}{\sim} \mu \ln \frac{L}{\eta_K} \underset{\mathcal{R}_e \rightarrow \infty}{\sim} \frac{3\mu}{4} \ln \mathcal{R}_e.$$

This is not consistent with the dissipative range behavior observed in DNS. Instead, a more involved MF computation yield

$$\text{Var}[\ln \varepsilon^\nu] \underset{\mathcal{R}_e \rightarrow \infty}{\sim} \frac{3\mu}{4} \frac{(1 + \mu/4)^2}{(1 + \mu/8)^3} \ln \mathcal{R}_e$$

MF is needed to understand the finite Reynolds number corrections.

Log or not log ?

