

Mixed 3D-2D asymptotics for the rotating magnetohydrodynamic system

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1 Introduction

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- Examples of penalized geophysical fluids models
- Hidden asymptotics

2 Asymptotics for the rotating MHD system

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- About the limit systems
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- Step 3: Energy estimates and bootstrap
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Incompressible Navier-Stokes system

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v = -\nabla p, & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases}$$

- Unknowns: **velocity** $v(t, x) \in \mathbb{R}^d$, and **pressure** $p(t, x) \in \mathbb{R}$.
- Kinematic viscosity $\nu > 0$.
- Pressure and velocity: $p = -\sum_{i,j=1}^d \partial_i \partial_j \Delta^{-1}(v^i v^j)$.
- Scaling invariance: for $\lambda > 0$, $(\lambda v, \lambda^2 p)(\lambda^2 t, \lambda x)$.
- **Fundamental results:** **Leray** (Weak global solution if $v_0 \in L^2(\mathbb{R}^d)$, **uniqueness when $d = 2$**) and **Fujita-Kato** (Unique strong **local** solution if $v_0 \in \dot{H}^{\frac{d}{2}-1}$, **global solution for small data**). **Energy methods**.

A few geophysical models

- Geophysical fluids: **Rotation of the Earth** (Coriolis and centrifugal forces), **Gravity** (vertical stratification of density).
- Scales, **Rossby** and **Froude** numbers.
- Small parameters $Ro = \varepsilon$, $Fr = \varepsilon F$ ($F > 0$)

Primitive system

- $U_\varepsilon(t, x) = (v_\varepsilon, \theta_\varepsilon) = (v_\varepsilon^1, v_\varepsilon^2, v_\varepsilon^3, \theta_\varepsilon)$,
- Velocity: $v_\varepsilon(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,
- Scalar potential temperature: $\theta_\varepsilon(t, x)$,
- Geopotential: $\phi_\varepsilon(t, x)$.

Primitive system

Primitive system

$$\begin{cases} \partial_t U_\varepsilon + U_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U_\varepsilon = \frac{1}{\varepsilon} (-\nabla \Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_\varepsilon|_{t=0} = U_{0,\varepsilon}. \end{cases} \quad (PE_\varepsilon)$$

where $U_\varepsilon \cdot \nabla U_\varepsilon \stackrel{\text{def}}{=} v_\varepsilon \cdot \nabla U_\varepsilon$ and

$$LU_\varepsilon \stackrel{\text{def}}{=} (\nu \Delta v_\varepsilon, \nu' \Delta \theta_\varepsilon), \quad \mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

Remarks

- The terms $\mathcal{A}U_\varepsilon$ and $(\nabla\Phi_\varepsilon, 0)$ are said to be **penalized** and lead the asymptotics together with the divergence-free condition.
- \mathcal{A} is skewsymmetric, **energy methods** easily adapted to obtain Leray and Fujita-Kato results in similar spaces ($s \in \mathbb{R}$, $T \in]0, \infty[$):

$$\begin{cases} \dot{E}_T^s = \mathcal{C}([0, T], \dot{H}^s) \cap L^2([0, T], \dot{H}^{s+1}), \\ \|f\|_{\dot{E}_T^s}^2 \stackrel{\text{def}}{=} \|f\|_{L_T^\infty \dot{H}^s}^2 + \min(\nu, \nu') \|f\|_{L_T^2 \dot{H}^{s+1}}^2. \end{cases}$$

Homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ endowed with the norm

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Weak and strong solutions

For any **fixed** $\varepsilon > 0$

Theorem (J. Leray, 1933)

If $U_{0,\varepsilon} \in L^2(\mathbb{R}^3)$, then there exists a Leray solution
 $U_\varepsilon \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ (+energy).

No uniqueness ($d = 3$).

Theorem (H. Fujita and T. Kato, 1963, **scaling invariance**)

If $U_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then there exists a unique maximal lifespan
 $T_\varepsilon^* > 0$ and a unique solution $U_\varepsilon \in C_T \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap L_T^2 \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ for
all $T < T_\varepsilon^*$.
+ blow-up criteria and weak-strong uniqueness.

Study of the asymptotics when $\varepsilon \rightarrow 0$

Procedure

- The penalized terms impose a **limit system** and a **special structure/decomposition** linked with it ($U = U_{lim} + U_{osc}$),
- Notion of "well-/ill-prepared" **initial data**,
- **Globally well-posed limit system** (strong solutions),
- **Better results** for the lifespan of strong solutions are transmitted (for strong enough rotation/stratification i.e. $\varepsilon \rightarrow 0$),
- Convergence rates.
- Sometimes: **Partial limits, hidden asymptotics** if we do not consider adapted initial data.

Hidden asymptotics for the Rotating fluids system

Considering only the first 3 components:

$$LU_\varepsilon \stackrel{\text{def}}{=} (\nu \Delta v_\varepsilon, \nu' \Delta \theta_\varepsilon), \quad \mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix},$$

we obtain the:

Rotating fluids system

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon - \nu \Delta v_\varepsilon + \frac{e_3 \wedge v_\varepsilon}{\varepsilon} = -\frac{1}{\varepsilon} \nabla p_\varepsilon, \\ \operatorname{div} v_\varepsilon = 0, \\ v_\varepsilon|_{t=0} = v_0. \end{cases} \quad (RF_\varepsilon)$$

First approach: well-prepared initial data, resonances for (RF_ε) , (PE_ε) ...

- T. Beale, A. Bourgeois ('94),
- S. Schochet ('94)
- P. Embid, A. Majda ('96, '98),
- E. Grenier ('97)
- B. Desjardins, E. Grenier ('98),
- I. Gallagher ('98),
- A. Babin, A. Mahalov, B. Nicolaenko ('96, '99, '01).
- J.-Y. Chemin ('97, $\nu \equiv \nu$),
- D. Iftimie ($F=1$, $\nu = \nu' = 0$) ('99)
- FC ('18, general case)

Dispersive approach: ill-prepared initial data for (RF_ε) , (PE_ε) ...

- J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier ('00, '02, '02 (Ekman), '06),
- A. Dutrifoy ('05),
- V.-S. Ngo ($\nu \rightarrow 0$) ('09),
- M. Hieber, Y. Shibata ('10),
- T. Iwabuchi, R. Takada ('15, '13, '14),
- Y. Koh, S. Lee, R. Takada (Littman) ('14)
- FC ('04-'23, '26)

see also:

- I. Gallagher, L. Saint Raymond ('06, '06),
- I. Gallagher ('08)

Asymptotics for the Rotating fluids

Limit system: 2D-NS with 3 components (Global strong solutions)

$$\begin{cases} \partial_t \bar{u}_h + \bar{u}_h \cdot \nabla_h \bar{u}_h - \nu \Delta_h \bar{u}_h = -\nabla_h \bar{p}, \\ \partial_t \bar{u}_3 + \bar{u}_h \cdot \nabla_h \bar{u}_3 - \nu \Delta_h \bar{u}_3 = 0, \\ \operatorname{div}_h \bar{u}_h = 0, \\ \bar{u}|_{t=0} = \bar{u}_0. \end{cases} \quad (2D - NS^3)$$

Asymptotics (Chemin, Desjardins, Gallagher, Grenier, 2002)

- $v_\varepsilon|_{t=0} = v_0(x) + \bar{u}_0(x_h)$.
- Direct study of $v_\varepsilon - \bar{u} - W_\varepsilon$, where W_ε solves

$$\begin{cases} \partial_t W_\varepsilon - \nu \Delta W_\varepsilon + \frac{1}{\varepsilon} \mathbb{P}(e_3 \wedge W_\varepsilon) = 0, \\ W_\varepsilon|_{t=0} = v_0. \end{cases}$$

Asymptotics for the Rotating fluids

- Mathematical proof of the **Taylor-Proudman theorem (Physics)** which states for strong rotation a column structure (that is a limit velocity independent of x_3).
- We have to consider **non-conventional initial data** to reach such limit (for "Navier-Stokes"-classical/conventional initial data $v_0 \in L^2(\mathbb{R}^3)$ or $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, **the limit is zero**).
- Generalization and convergence rates for large ill-prepared initial data (FC TJM '23, Preprint '26).

Hidden asymptotics for the Strongly stratified Boussinesq model without rotation

Neglecting the rotational effect we obtain the:

Strongly stratified Boussinesq system (also NS-Boussinesq)

$$\begin{cases} \partial_t U_\varepsilon + U_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon} \mathcal{B}U_\varepsilon = \frac{1}{\varepsilon} (-\nabla \Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_\varepsilon|_{t=0} = U_{0,\varepsilon}. \end{cases} \quad (S_\varepsilon)$$

$$LU_\varepsilon \stackrel{\text{def}}{=} (\nu \Delta v_\varepsilon, \nu' \Delta \theta_\varepsilon), \quad \mathcal{B} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Previous results, inviscid case $\nu = \nu' = 0$

- K. Widmayer (CMS '18): convergence towards $(\bar{u}(x_1, x_2, x_3), 0, 0)$

$$\begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla_h \bar{u} = -\nabla_h \bar{p}, \\ \operatorname{div}_h \bar{u} = 0, \\ \bar{u}|_{t=0} = (\mathbb{P}_2 U_0)^h, \end{cases} \quad (1)$$

\mathbb{P}_2 orthogonal projector onto horizontal-divergence-free vectorfields

- R. Takada (ARMA '19): Existence result and convergence rate:

$$\|U_\varepsilon - (\bar{u}, 0, 0)\|_{L_T^q W^{1,\infty}} \leq C \varepsilon^{\frac{1}{q}}.$$

- S. Lee, R. Takada (IUMJ '17): ($\nu = \nu'$) For any $\varepsilon > 0$, if $U_0 = \mathbb{P}_2 U_0 + U_{0,osc}$ where $\mathbb{P}_2 U_0 \in \dot{H}^{\frac{1}{2}}$, $U_{0,osc} \stackrel{def}{=} (I_d - \mathbb{P}_2) U_0 \in \dot{H}^{\frac{1}{2}+\delta}$ ($\delta \in]0, \frac{1}{8}[$) are small:

$$\|\mathbb{P}_2 U_0\|_{\dot{H}^{\frac{1}{2}}} \leq \delta_2, \text{ and } \|U_{0,osc}\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq \delta_1 \varepsilon^{-\frac{\delta}{2}},$$

there exists a unique global **mild solution** $U_\varepsilon \in L^4(\dot{W}^{\frac{1}{2},3})$.

if $\|\mathbb{P}_2 U_0\|_{\dot{H}^{\frac{1}{2}}}$ is sufficiently small, there exists a global solution for small enough ε .

- S. Scrobogna (DCDS '20): Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ with $U_{0,S} = \mathbb{P}_2 U_0 \in H^1(\mathbb{R}^3)$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, there exists a unique global solution $U_\varepsilon \in \dot{E}^{\frac{1}{2}}$. Moreover, U_ε converges to $(\tilde{v}^h, 0, 0)$, where \tilde{v}^h is the unique global solution of the two-component 3D-Navier-Stokes system:

$$\begin{cases} \partial_t \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \tilde{v}^h - \nu \Delta \tilde{v}^h &= -\nabla_h \tilde{\pi}^0, \\ \operatorname{div}_h \tilde{v}^h &= 0, \\ \tilde{v}^h|_{t=0} &= \mathbb{P}_2 U_0, \end{cases} \quad (3D - NS^2)$$

- S. Scrobogna (DCDS '20): Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ with $U_{0,S} = \mathbb{P}_2 U_0 \in H^1(\mathbb{R}^3)$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, there exists a unique global solution $U_\varepsilon \in \dot{E}^{\frac{1}{2}}$. Moreover, U_ε converges to $(\tilde{v}^h, 0, 0)$, where \tilde{v}^h is the unique global solution of the two-component 3D-Navier-Stokes system:

$$\begin{cases} \partial_t \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \tilde{v}^h - \nu \Delta \tilde{v}^h &= -\nabla_h \tilde{\pi}^0, \\ \operatorname{div}_h \tilde{v}^h &= 0, \\ \tilde{v}^h|_{t=0} &= \mathbb{P}_2 U_0, \end{cases} \quad (3D - NS^2)$$

Something surprising...

Extension and question

Question: why does the previous limit not depend on ν' ?

Thanks to a formal approach we can clearly identify the complete limit system.

Extension and question

$$\begin{cases} \partial_t \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \tilde{v}^h - \nu \Delta \tilde{v}^h &= -\nabla_h \tilde{\pi}^0, \\ \operatorname{div}_h \tilde{v}^h &= 0, \\ \tilde{v}^h|_{t=0} &= \tilde{v}_0^h, \end{cases} \quad (2)$$

and

$$\begin{cases} \partial_t \tilde{\theta} - \nu' \partial_3^2 \tilde{\theta} &= 0, \\ \tilde{\theta}|_{t=0} &= \tilde{\theta}_0. \end{cases} \quad (3)$$

- Both systems have global unique solutions.
- **Special structure** linked with the velocity part:

$$U = \mathbb{P}_2 U + U_{osc}.$$

Results: FC CPDE '24, JMPA '25.

We prove global existence (when ε is small enough) of strong solutions for the following initial data ($U_{0,\varepsilon,osc}$ can be very large):

$$U_\varepsilon|_{t=0}(x) = (\tilde{v}^h(x), 0, 0) + U_{0,\varepsilon,osc}(x) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\theta}_0(x_3) \end{pmatrix}$$

Moreover, there exist $\mathbb{D}_0, K > 0$ such that:

$$\|U_\varepsilon - (\tilde{v}^h, 0, \tilde{\theta})\|_{L^2L^\infty} \leq \mathbb{D}_0 \varepsilon^K.$$

Asymptotics for the rotating MHD system

We consider the following penalized system:

Rotating MHD system

$$\begin{cases} \partial_t u_\varepsilon - \nu \Delta u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon \wedge e_3 + u_\varepsilon \cdot \nabla u_\varepsilon - b_\varepsilon \cdot \nabla b_\varepsilon = -\nabla p_\varepsilon, \\ \partial_t b_\varepsilon - \nu' \Delta b_\varepsilon + u_\varepsilon \cdot \nabla b_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon = 0, \\ \operatorname{div} u_\varepsilon = 0, \\ (u_\varepsilon, b_\varepsilon)|_{t=0} = (u_{0,\varepsilon}, b_{0,\varepsilon}). \end{cases}$$

(MHD_ε)

u_ε : velocity of the flow of charged particles (p_ε pressure),

b_ε : magnetic field induced by its motion,

ε : Rossby number.

Remark: $\operatorname{div} b_\varepsilon = 0$.

Some references for the rotating MHD system

- B. Desjardins, E. Dormy and E. Grenier (Nonlinearity '99), \mathbb{R}^3 , $\nu = \varepsilon$, $\nu' = \varepsilon^{-1}$, $(u_{0,\varepsilon}, b_{0,\varepsilon}) \rightarrow (\bar{u}_0(x_h), 0)$. **Limit:** $(\bar{u}_E(x_h), 0)$, where \bar{u}_E solves 2D-Euler with 3 components.
- J. Benameur, M. Ghazel and M. Majdoub (AA '05), \mathbb{R}^3 or \mathbb{T}^3 , $\nu = \varepsilon$, $\nu' = \varepsilon^{-1}$, $(u_0, b_0)(x_h, x_3) \in H^s$ ($s > 7/2$), local existence, **limit:** $(\bar{u}_E(x_h), 0)$ in \mathbb{T}^3 , $(0, 0)$ in \mathbb{R}^3 .
- J. Benameur, S. Ibrahim and M. Majdoub (DIE '05), $\nu = \varepsilon$, $\nu' = 1$, $(u_0, b_0)(x_h, x_3)$, local existence, **limit:** (\bar{u}, \bar{b}) in \mathbb{T}^3 , $(0, e^{t\Delta} b_0)$ in \mathbb{R}^3 .

Some references for the rotating MHD system

- V.-S. Ngo (AAM, '17), $\nu_h = \varepsilon^\alpha = \nu'_h$, **global existence**.
- J. Ahn, J. Kim, J. Lee (JEE '21), \mathbb{R}^3 , $\nu = \nu' = 1$, **global existence** (smallness assumptions on the initial data).
- R. Takada, K. Yoneda (FE '24), $\nu, \nu' > 0$, **global existence** (more general assumptions).
- H. Ohyama, K. Yoneda (ADE '25), $\nu, \nu' > 0$, **global existence**, **limit:** (\bar{u}, \bar{b}) in $\mathbb{R}^2 \times \mathbb{T}$ (data independent of ε).
- H. Ohyama (Math. Nach. '26), $\nu = \varepsilon, \nu' = 1$, **long-time existence**, **limit:** $(0, e^{t\Delta} b_0)$ in \mathbb{R}^3 .

Formal approach for the limit: rewriting the pressure

Taking the divergence of the velocity part of (MHD_ε) we can split the pressure into $p_\varepsilon = p_\varepsilon^0 + \frac{1}{\varepsilon}p_\varepsilon^1$, where:

$$\begin{cases} \Delta p_\varepsilon^0 = -\sum_{i,j=1}^3 \partial_i \partial_j (u_\varepsilon^i u_\varepsilon^j - b_\varepsilon^i b_\varepsilon^j), \\ \Delta p_\varepsilon^1 = \partial_2 u_\varepsilon^1 - \partial_1 u_\varepsilon^2. \end{cases} \quad (4)$$

and clearly identify the actual penalized terms, (MHD_ε) turns into:

$$\begin{cases} \partial_t u_\varepsilon^1 - \nu \Delta u_\varepsilon^1 + u_\varepsilon \cdot \nabla u_\varepsilon^1 - b_\varepsilon \cdot \nabla b_\varepsilon^1 = -\partial_1 p_\varepsilon^0 - \frac{1}{\varepsilon}(\partial_1 p_\varepsilon^1 + u_\varepsilon^2), \\ \partial_t u_\varepsilon^2 - \nu \Delta u_\varepsilon^2 + u_\varepsilon \cdot \nabla u_\varepsilon^2 - b_\varepsilon \cdot \nabla b_\varepsilon^2 = -\partial_2 p_\varepsilon^0 - \frac{1}{\varepsilon}(\partial_2 p_\varepsilon^1 - u_\varepsilon^1), \\ \partial_t u_\varepsilon^3 - \nu \Delta u_\varepsilon^3 + u_\varepsilon \cdot \nabla u_\varepsilon^3 - b_\varepsilon \cdot \nabla b_\varepsilon^3 = -\partial_3 p_\varepsilon^0 - \frac{1}{\varepsilon} \partial_3 p_\varepsilon^1, \\ \partial_t b_\varepsilon - \nu' \Delta b_\varepsilon + u_\varepsilon \cdot \nabla b_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon = 0. \end{cases}$$

Formal approach for the limit: dealing with the penalized terms

Assuming that $(u_\varepsilon, b_\varepsilon, p_\varepsilon^0, p_\varepsilon^1) \xrightarrow{\varepsilon \rightarrow 0} (\tilde{u}, \tilde{b}, \tilde{p}^0, \tilde{p}^1)$ in a sufficiently strong way (for derivatives and nonlinear terms...) if we assume in addition that:

$$\begin{cases} -\frac{1}{\varepsilon}(\partial_1 p_\varepsilon^1 + u_\varepsilon^2) \xrightarrow{\varepsilon \rightarrow 0} \tilde{A}, \\ -\frac{1}{\varepsilon}(\partial_2 p_\varepsilon^1 - u_\varepsilon^1) \xrightarrow{\varepsilon \rightarrow 0} \tilde{B}, \\ -\frac{1}{\varepsilon} \partial_3 p_\varepsilon^1 \xrightarrow{\varepsilon \rightarrow 0} \tilde{C}, \end{cases}$$

we immediately get that:

Formal approach for the limit: the limit system

the formal limits not only solve:

$$\begin{cases} \partial_t \tilde{u} - \nu \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} - \tilde{b} \cdot \nabla \tilde{b} = -\nabla \tilde{p}^0 + (\tilde{A}, \tilde{B}, \tilde{C}), \\ \partial_t \tilde{b} - \nu' \Delta \tilde{b} + \tilde{u} \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla \tilde{u} = 0, \end{cases}$$

but also satisfy:

$$\begin{cases} \partial_1 \tilde{p}^1 + \tilde{u}^2 = 0, \\ \partial_2 \tilde{p}^1 - \tilde{u}^1 = 0, \\ \partial_3 \tilde{p}^1 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{div} \tilde{u} = 0 = \operatorname{div} \tilde{b}, \\ \Delta \tilde{p}^0 = -\sum_{i,j=1}^3 \partial_i \partial_j (\tilde{u}^i \tilde{u}^j - \tilde{b}^i \tilde{b}^j), \\ \Delta \tilde{p}^1 = \partial_2 \tilde{u}^1 - \partial_1 \tilde{u}^2. \end{cases}$$

It follows that \tilde{p}^1 and $(\tilde{u}^1, \tilde{u}^2) = (\partial_2 \tilde{p}^1, -\partial_1 \tilde{p}^1)$ are independent of x_3 . So does \tilde{u}^3 (as $\operatorname{div} \tilde{u} = 0$) and $\partial_3 \tilde{u} = 0$.

Formal approach for the limit: How to get rid of the parameters ?

We can also get that:

$$\begin{cases} \partial_1 \tilde{A} + \partial_2 \tilde{B} + \partial_3 \tilde{C} = 0, \\ -\frac{1}{\varepsilon} \partial_3 u_\varepsilon^3 \xrightarrow{\varepsilon \rightarrow 0} \partial_1 \tilde{B} - \partial_2 \tilde{A} \stackrel{\text{def}}{=} \tilde{G}, \end{cases}$$

which leads to

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = -\nabla_h \Delta_h^{-1} \partial_3 \tilde{C} + \nabla_h^\perp \Delta_h^{-1} \tilde{G},$$

and allows us to rewrite the velocity part as follows:

Formal approach for the limit: How to get rid of the parameters ?

$$\begin{cases} \partial_t \tilde{u}^1 - \nu \Delta_h \tilde{u}^1 + \tilde{u}^h \cdot \nabla_h \tilde{u}^1 = -\partial_1 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^1 - \partial_1 \Delta_h^{-1} \partial_3 \tilde{C} - \partial_2 \Delta_h^{-1} \tilde{G}, \\ \partial_t \tilde{u}^2 - \nu \Delta_h \tilde{u}^2 + \tilde{u}^h \cdot \nabla_h \tilde{u}^2 = -\partial_2 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^2 - \partial_2 \Delta_h^{-1} \partial_3 \tilde{C} + \partial_1 \Delta_h^{-1} \tilde{G} \\ \partial_t \tilde{u}^3 - \nu \Delta_h \tilde{u}^3 + \tilde{u}^h \cdot \nabla_h \tilde{u}^3 = -\partial_3 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^3 + \tilde{C}. \end{cases}$$

Using that $\partial_3 \tilde{u} = 0$, we can write

$$\partial_3(-\partial_3 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^3 + \tilde{C}) = 0.$$

Applying ∂_3 to the equation on \tilde{b} , we get the linear system:

$$\begin{cases} \partial_t \partial_3 \tilde{b} - \nu' \Delta \partial_3 \tilde{b} + \tilde{u} \cdot \nabla \partial_3 \tilde{b} - \partial_3 \tilde{b} \cdot \nabla \tilde{u} = 0, \\ \partial_3 \tilde{b}|_{t=0} = \partial_3 \tilde{b}_0, \end{cases}$$

so $\partial_3 \tilde{b}_0 = 0 \implies \partial_3 \tilde{b} = 0$ for all times.

Formal approach for the limit: How to get rid of the parameters ?

If we impose $\partial_3 \tilde{b}_0 = 0$ and $(\tilde{C}, \tilde{G}) = (\partial_3 \tilde{\rho}^0, 0)$ we end-up with:

Limit system

$$\begin{cases} \partial_t \tilde{u} - \nu \Delta_h \tilde{u} + \tilde{u}^h \cdot \nabla_h \tilde{u} - \tilde{b}^h \cdot \nabla_h \tilde{b} = - \begin{pmatrix} \nabla_h \tilde{q}^0 \\ 0 \end{pmatrix}, \\ \partial_t \tilde{b} - \nu' \Delta_h \tilde{b} + \tilde{u}^h \cdot \nabla_h \tilde{b} - \tilde{b}_h \cdot \nabla_h \tilde{u} = 0, \\ \operatorname{div}_h \tilde{u} = 0, \\ (\tilde{u}, \tilde{b})|_{t=0} = (\tilde{u}_0, \tilde{b}_0)(x_h). \end{cases} \quad (2D - MHD^3)$$

Nice 2D-system (close to 2D-Navier-Stokes).

Choice of initial data

This motivates us to consider as initial data:

$$(u_\varepsilon, b_\varepsilon)|_{t=0} = (\tilde{u}_0(x_h) + v_{0,\varepsilon}(x), \tilde{b}_0(x_h) + c_{0,\varepsilon}(x)). \quad (5)$$

Problem: classical results (Leray, Fujita Kato) only apply to conventional initial data (i.-e. in some functional space taking into consideration all space variables (x_1, x_2, x_3)).

Rewriting the limit system

In order to study $(u_\varepsilon - \tilde{u}, b_\varepsilon - \tilde{b})$ we first need to rewrite this system as follows:

$$\begin{cases} \partial_t \tilde{u} - \nu \Delta \tilde{u} + \frac{1}{\varepsilon} \tilde{u} \wedge \mathbf{e}_3 + \tilde{u} \cdot \nabla \tilde{u} - \tilde{b} \cdot \nabla \tilde{b} = -\nabla \tilde{q}^0 - \frac{1}{\varepsilon} \nabla \tilde{p}^1, \\ \partial_t \tilde{b} - \nu' \Delta \tilde{b} + \tilde{u} \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla \tilde{u} = 0, \\ \operatorname{div} \tilde{u} = 0, \\ (\tilde{u}, \tilde{b})|_{t=0} = (\tilde{u}_0, \tilde{b}_0)(x_h). \end{cases}$$

(2D – MHD³)

What system to study ?

The difference $(v_\varepsilon, B_\varepsilon) = (u_\varepsilon - \tilde{u}, b_\varepsilon - \tilde{b})$ satisfies

$$\left\{ \begin{array}{l} \partial_t v_\varepsilon - \nu \Delta v_\varepsilon + \frac{1}{\varepsilon} v_\varepsilon \wedge e_3 + v_\varepsilon \cdot \nabla v_\varepsilon + v_\varepsilon \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla v_\varepsilon \\ \quad - B_\varepsilon \cdot \nabla B_\varepsilon - B_\varepsilon \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla B_\varepsilon = -\nabla q_\varepsilon, \\ \partial_t B_\varepsilon - \nu' \Delta B_\varepsilon + v_\varepsilon \cdot \nabla B_\varepsilon - B_\varepsilon \cdot \nabla v_\varepsilon + \tilde{u} \cdot \nabla B_\varepsilon - B_\varepsilon \cdot \nabla \tilde{u} \\ \quad + v_\varepsilon \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla v_\varepsilon = 0, \\ \operatorname{div} v_\varepsilon = 0, \\ (v_\varepsilon, B_\varepsilon)|_{t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{array} \right.$$

As dispersion will only affect the velocity we expect

$$v_\varepsilon = u_\varepsilon - \tilde{u} \rightarrow 0 \text{ but nothing forces } B_\varepsilon = b_\varepsilon - \tilde{b} \rightarrow 0,$$

What system to study ?

The difference $(v_\varepsilon, B_\varepsilon) = (u_\varepsilon - \tilde{u}, b_\varepsilon - \tilde{b})$ satisfies

$$\left\{ \begin{array}{l} \partial_t v_\varepsilon - \nu \Delta v_\varepsilon + \frac{1}{\varepsilon} v_\varepsilon \wedge e_3 + v_\varepsilon \cdot \nabla v_\varepsilon + v_\varepsilon \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla v_\varepsilon \\ \quad - B_\varepsilon \cdot \nabla B_\varepsilon - B_\varepsilon \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla B_\varepsilon = -\nabla q_\varepsilon, \\ \partial_t B_\varepsilon - \nu' \Delta B_\varepsilon + v_\varepsilon \cdot \nabla B_\varepsilon - B_\varepsilon \cdot \nabla v_\varepsilon + \tilde{u} \cdot \nabla B_\varepsilon - B_\varepsilon \cdot \nabla \tilde{u} \\ \quad + v_\varepsilon \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla v_\varepsilon = 0, \\ \operatorname{div} v_\varepsilon = 0, \\ (v_\varepsilon, B_\varepsilon)|_{t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{array} \right.$$

As dispersion will only affect the velocity we expect $v_\varepsilon = u_\varepsilon - \tilde{u} \rightarrow 0$ but nothing forces $B_\varepsilon = b_\varepsilon - \tilde{b} \rightarrow 0$,

What system to study ?

To filter this, we simply introduce:

$$\begin{cases} \partial_t c_\varepsilon - \nu' \Delta c_\varepsilon + \tilde{u} \cdot \nabla c_\varepsilon - c_\varepsilon \cdot \nabla \tilde{u} = 0, \\ c|_{t=0} = c_{0,\varepsilon}, \end{cases} \quad (3D - M)$$

so that we will finally study:

$$D_\varepsilon = \begin{pmatrix} v_\varepsilon \\ d_\varepsilon \end{pmatrix} \stackrel{\text{def}}{=} U_\varepsilon - \begin{pmatrix} \tilde{u} \\ \tilde{b} + c_\varepsilon \end{pmatrix} = \begin{pmatrix} u_\varepsilon - \tilde{u} \\ b_\varepsilon - \tilde{b} - c_\varepsilon \end{pmatrix},$$

What system to study ?

which satisfies the following MHD system:

$$\left\{ \begin{array}{l} \partial_t v_\varepsilon - \nu \Delta v_\varepsilon + \frac{1}{\varepsilon} v_\varepsilon \wedge e_3 + v_\varepsilon \cdot \nabla v_\varepsilon + v_\varepsilon \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla v_\varepsilon \\ \quad - d_\varepsilon \cdot \nabla d_\varepsilon - d_\varepsilon \cdot \nabla (\tilde{b} + c_\varepsilon) - (\tilde{b} + c_\varepsilon) \cdot \nabla d_\varepsilon \\ \quad = -\nabla q_\varepsilon + \tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b} + c_\varepsilon \cdot \nabla c_\varepsilon, \\ \partial_t d_\varepsilon - \nu' \Delta d_\varepsilon + v_\varepsilon \cdot \nabla d_\varepsilon - d_\varepsilon \cdot \nabla v_\varepsilon + \tilde{u} \cdot \nabla d_\varepsilon - d_\varepsilon \cdot \nabla \tilde{u} \\ \quad + v_\varepsilon \cdot \nabla (\tilde{b} + c_\varepsilon) - (\tilde{b} + c_\varepsilon) \cdot \nabla v_\varepsilon = 0, \\ \operatorname{div} v_\varepsilon = 0, \\ (v_\varepsilon, d_\varepsilon)|_{t=0} = (v_{0,\varepsilon}, 0). \end{array} \right. \quad (6)$$

Now we expect both components to go to zero.

Global existence for $(2D - MHD^3)$

For any $\tilde{u}_0, \tilde{b}_0 \in L^2(\mathbb{R}^2)^3$ there exists a unique global solution $(\tilde{u}, \tilde{b}) \in \dot{F}^0(\mathbb{R}^2)$. Moreover we have energy estimates and propagation of the \dot{H}^s -norm ($s \in]-1, 1[$).

The second system is a simple linear equation:

Global existence for $(3D-M)$

For any $s \in]-1, 1[$ and $c_{0,\varepsilon} \in \dot{H}^s(\mathbb{R}^3)$, there exists a unique global solution $c_\varepsilon \in \dot{E}^s(\mathbb{R}^3)$ of $(3D - M)$ (+ Energy estimates).

Theorem (FC, V.-S. Ngo, '25 in revision, FC preprint '26)

For any $\mathbb{C}_0 \geq 1$ (size), $\delta \in]0, \frac{1}{4}[$ (extra regularity), any $\mu, k \in [0, 1[$
 any $\gamma \in [0, \frac{\delta}{2}[$ (initial blow-up), there exists $\varepsilon_0, \mathbb{K}_0, \mathbb{F}_0 > 0$ such
 that for any $\varepsilon \in]0, \varepsilon_0]$ and any initial data as in (5) with:

$$\left\{ \begin{array}{l} \tilde{u}_0, \tilde{b}_0 \in H^\delta(\mathbb{R}^2), \quad \text{with } \max(\|\tilde{u}_0\|_{H^\delta}, \|\tilde{b}_0\|_{H^\delta}) \leq \mathbb{C}_0, \\ (v_{0,\varepsilon}, c_{0,\varepsilon}) \in \left(\dot{H}^{\frac{1}{2}-\delta}(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3) \right) \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3), \\ \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\mu\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma}, \\ \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}} \leq \mathbb{K}_0 |\ln \varepsilon|^{\frac{1}{2}}, \end{array} \right.$$

then there exists a **unique global strong solution** U_ε with

$$U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon) \in \dot{F}^{\frac{1}{2}}.$$

Moreover, introducing W_ε (solving some linear dispersive system precised later), we have:

$$\|U_\varepsilon - (\tilde{u}, \tilde{b}) - (W_\varepsilon, c_\varepsilon)\|_{\dot{F}^{\frac{1}{2}}} \leq \mathbb{F}_0 \varepsilon^{k(\frac{\delta}{2} - \gamma)}.$$

Finally, we can get rid of the fast oscillations W_ε as follows: for any $p \in [2, \frac{2}{\delta}[$ there exists \mathbb{F}_p such that,

$$\|U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon)\|_{L^p L^{\frac{3}{1-\frac{2}{p}}}} \leq \mathbb{F}_p \varepsilon^{k(\frac{\delta}{2} - \gamma)}.$$

Maximal reachable size

Theorem (FC preprint '26)

For any $\mathbb{C}_0 \geq 1$ (size), $\delta \in]0, \frac{1}{6}]$ (extra regularity), any $\mu \in [0, 1[$, there exists $m_0, \varepsilon_0, \mathbb{F}_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0]$ and any initial data as previously with:

$$\left\{ \begin{array}{l} \tilde{u}_0, \tilde{b}_0 \in H^\delta(\mathbb{R}^2), \quad \text{with } \max(\|\tilde{u}_0\|_{H^\delta}, \|\tilde{b}_0\|_{H^\delta}) \leq \mathbb{C}_0, \\ (v_{0,\varepsilon}, c_{0,\varepsilon}) \in \left(\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3) \right) \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3), \\ \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\mu\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq m_0 \varepsilon^{-\frac{\delta}{2}}, \\ \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}} \leq \mathbb{C}_0, \end{array} \right.$$

then there exists a **unique global strong solution** U_ε with

$$U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon) \in \dot{F}^{\frac{1}{2}}.$$

Moreover, we have:

$$\|U_\varepsilon - (\tilde{u}, \tilde{b}) - (W_\varepsilon, c_\varepsilon)\|_{\dot{F}^{\frac{1}{2}}} \leq \mathbb{F}_0 \max(m_0, \varepsilon^{\frac{\delta}{2}}).$$

Finally, we can get rid of the fast oscillations W_ε as follows: for any $p \in]2, \frac{2}{\delta}[$ there exists \mathbb{F}_p such that,

$$\|U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon)\|_{L^p L^{\frac{3}{1-\frac{2}{p}}}} \leq \mathbb{F}_p \max(m_0, \varepsilon^{\frac{\delta}{2}}).$$

Replacing m_0 by some $m(\varepsilon)$ gives convergence rates.

Step 1: Existence of local Fujita-Kato strong solutions

Local existence (FC, V.-S. Ngo, preprint '25)

Let $\varepsilon > 0$ fixed. For any $\tilde{u}, \tilde{b} \in L^2(\mathbb{R}^3)$, $v_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}$, $c_{0,\varepsilon} \in H^{\frac{1}{2}}$ and any initial data as in (5), there exists a unique local solution D_ε of (6) with lifespan $T_\varepsilon^* > 0$ such that for any $T < T_\varepsilon^*$, $D_\varepsilon \in \dot{F}_T^{1/2}$.

Moreover we have:

- **Regularity propagation** of the \dot{H}^s -norm ($s \in]-1, 1[$) + estimates.

- **Continuation criterion:**

$$\int_0^{T_\varepsilon^*} \|\nabla D_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau < \infty \implies T_\varepsilon^* = \infty.$$

- **Global existence for small data:** if $\|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} \leq c_0 \nu$ then $T_\varepsilon^* = +\infty$

Global existence and convergence: fast oscillations

Proving global existence and convergence requires us to introduce the following waves/fast oscillations (denoting as \mathbb{P} the Leray projector):

$$\begin{cases} \partial_t W_\varepsilon - \nu \Delta W_\varepsilon + \frac{1}{\varepsilon} \mathbb{P}(W_\varepsilon \wedge e_3) = \mathbb{P}(\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b} + c_\varepsilon \cdot \nabla c_\varepsilon), \\ W_\varepsilon|_{t=0} = v_{0,\varepsilon}. \end{cases} \quad (7)$$

which

- features dispersive properties,
- allow to "eat" this **non-vanishing external force terms** (by making them oscillate/disperse at infinity).

BUT these external force terms **do not share the same regularity** and are **not bounded w.r.t ε** .

Energy estimates and bootstrap

We study the following quantity

$$\tilde{D}_\varepsilon \stackrel{\text{def}}{=} U_\varepsilon - \begin{pmatrix} \tilde{u} + W_\varepsilon \\ \tilde{b} + c_\varepsilon \end{pmatrix} = D_\varepsilon - \begin{pmatrix} W_\varepsilon \\ 0 \end{pmatrix} = \begin{pmatrix} v_\varepsilon - W_\varepsilon \\ d_\varepsilon \end{pmatrix} = \begin{pmatrix} \delta_\varepsilon \\ d_\varepsilon \end{pmatrix},$$

which satisfies:

$$\begin{cases} \partial_t \delta_\varepsilon - \nu \Delta \delta_\varepsilon + \frac{1}{\varepsilon} \mathbb{P}(\delta_\varepsilon \wedge \mathbf{e}_3) = \sum_{i=1}^{13} F_i, \\ \partial_t d_\varepsilon - \nu' \Delta d_\varepsilon = \sum_{i=1}^{14} G_i, \\ (\delta_\varepsilon, d_\varepsilon)|_{t=0} = (0, 0), \end{cases} \quad (8)$$

where we define:

$$\left\{ \begin{array}{lll} F_1 \stackrel{\text{def}}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla \delta_\varepsilon), & F_2 \stackrel{\text{def}}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla W_\varepsilon), & F_3 \stackrel{\text{def}}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla \delta_\varepsilon), \\ F_4 \stackrel{\text{def}}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla W_\varepsilon), & F_5 \stackrel{\text{def}}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla_h \tilde{u}), & F_6 \stackrel{\text{def}}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla_h \tilde{u}), \\ F_7 \stackrel{\text{def}}{=} -\mathbb{P}(\tilde{u} \cdot \nabla \delta_\varepsilon), & F_8 \stackrel{\text{def}}{=} -\mathbb{P}(\tilde{u} \cdot \nabla W_\varepsilon), & F_9 \stackrel{\text{def}}{=} \mathbb{P}(d_\varepsilon \cdot \nabla d_\varepsilon), \\ F_{10} \stackrel{\text{def}}{=} \mathbb{P}(d_\varepsilon \cdot \nabla_h \tilde{b}), & F_{11} \stackrel{\text{def}}{=} \mathbb{P}(d_\varepsilon \cdot \nabla c_\varepsilon), & F_{12} \stackrel{\text{def}}{=} \mathbb{P}(\tilde{b} \cdot \nabla d_\varepsilon), \\ F_{13} \stackrel{\text{def}}{=} \mathbb{P}(c_\varepsilon \cdot \nabla d_\varepsilon), & G_1 \stackrel{\text{def}}{=} -\delta_\varepsilon \cdot \nabla d_\varepsilon, & G_2 \stackrel{\text{def}}{=} -W_\varepsilon \cdot \nabla d_\varepsilon, \\ G_3 \stackrel{\text{def}}{=} d_\varepsilon \cdot \nabla \delta_\varepsilon, & G_4 \stackrel{\text{def}}{=} d_\varepsilon \cdot \nabla W_\varepsilon, & G_5 \stackrel{\text{def}}{=} -\tilde{u} \cdot \nabla d_\varepsilon, \\ G_6 \stackrel{\text{def}}{=} d_\varepsilon \cdot \nabla_h \tilde{u}, & G_7 \stackrel{\text{def}}{=} -\delta_\varepsilon \cdot \nabla_h \tilde{b}, & G_8 \stackrel{\text{def}}{=} -\delta_\varepsilon \cdot \nabla c_\varepsilon, \\ G_9 \stackrel{\text{def}}{=} -W_\varepsilon \cdot \nabla_h \tilde{b}, & G_{10} \stackrel{\text{def}}{=} -W_\varepsilon \cdot \nabla c_\varepsilon, & G_{11} \stackrel{\text{def}}{=} \tilde{b} \cdot \nabla \delta_\varepsilon, \\ G_{12} \stackrel{\text{def}}{=} c_\varepsilon \cdot \nabla \delta_\varepsilon, & G_{13} \stackrel{\text{def}}{=} \tilde{b} \cdot \nabla W_\varepsilon, & G_{14} \stackrel{\text{def}}{=} c_\varepsilon \cdot \nabla W_\varepsilon. \end{array} \right. \quad (9)$$

Bootstrap method

Consider a local solution (with lifespan T_ε^*) and define

$$T_\varepsilon = \sup\{t \in [0, T_\varepsilon^*[, \forall t' \leq t, \\ \|\delta_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}} + \|d_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{8C} \min(\nu, \nu')\}. \quad (10)$$

- Perform energy estimates in $\dot{H}^{\frac{1}{2}}$ and obtain a contradiction and prove $T_\varepsilon = T_\varepsilon^* = \infty$.
- Continue with energy estimates in \dot{H}^s .

Energy estimates: difficulties

- Products of the form $f(x_1, x_2, x_3) \times g(x_1, x_2)$.
- W_ε has **very large energy** but is **small** when viewed through special norms thanks to dispersion/Strichartz estimates ($L_t^p L_x^r$ or $\tilde{L}_t^p \dot{B}_{r,q}^0$).
- Most of the 27 external force terms are classical, a few of them are tricky (we have to calibrate estimates).

Energy estimates

We obtain:

A priori estimates, FC Preprint '26

$$\begin{aligned}
 & \|\delta_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|d_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \\
 & \leq \mathbb{D}_0 \left[\|\nabla W_\varepsilon\|_{L^2 L^3}^2 + \|W_\varepsilon\|_{L^4 L^6}^2 + \|W_\varepsilon\|_{L^{\frac{4}{1+(1-\mu)\delta}} L_{h,v}^{\infty,2}}^2 + \|W_\varepsilon\|_{L^{\frac{4}{1+(5-\mu)\delta}} L_{h,v}^{\frac{2}{1-2\delta},2}}^2 \right. \\
 & \left. + \||D|^{\frac{1}{2}} W_\varepsilon\|_{L^4 L^3}^2 + \|W_\varepsilon\|_{L^4 \dot{B}_{4,2}^{\frac{1}{4}}}^2 \right] \exp \mathbb{D}_0 \left\{ 1 + \|\nabla W_\varepsilon\|_{L^2 L^3}^2 + \|W_\varepsilon\|_{L^4 L^6}^4 \right\}. \quad (11)
 \end{aligned}$$

Smallness of W_ε : use of dispersive effects

We show W_ε is small thanks to Strichartz estimates: there exists a positive constant \mathbb{E}_0 (depending on $\nu, \nu', \delta, \mathbb{C}_0, \mu$) such that if $\delta \leq \frac{1}{6}$, for any $\varepsilon \in]0, 1]$, we have:

$$\begin{aligned} & \|\nabla W_\varepsilon\|_{L^2 L^3} + \|W_\varepsilon\|_{L^4 L^6} + \||D|^{\frac{1}{2}} W_\varepsilon\|_{L^4 L^3} + \|W_\varepsilon\|_{L^4 \dot{B}_{4,2}^{\frac{1}{4}}} \\ & + \|W_\varepsilon\|_{L^{\frac{4}{1+(1-\mu)\delta}} L_{h,v}^{\infty,2}}^2 + \|W_\varepsilon\|_{L^{\frac{4}{1+(5-\mu)\delta}} L_{h,v}^{\frac{1}{1-2\delta},2}} \leq \mathbb{E}_0 \max(m_0, \varepsilon^{\frac{\delta}{2}}). \end{aligned} \quad (12)$$

Moreover, for any $p \in]2, \frac{2}{\delta}[$, there exists $\mathbb{E}'_0 > 0$ (depending on $\nu, \nu', \delta, \mathbb{C}_0, \mu, p$) such that:

$$\|W_\varepsilon\|_{L^p L^{\frac{3}{1-\frac{2}{p}}}} \leq \mathbb{E}'_0 \max(m_0, \varepsilon^{\frac{\delta}{2}}). \quad (13)$$

End of the proof

- Injecting the previous estimates in the a priori estimates allows to close the bootstrap and prove that $T_\varepsilon = T_\varepsilon^* = \infty$.
- Convergence rates **without fast oscillations**: we simply write

$$D_\varepsilon = U_\varepsilon - \begin{pmatrix} \tilde{u} \\ \tilde{b} + c_\varepsilon \end{pmatrix} = \tilde{D}_\varepsilon + \begin{pmatrix} W_\varepsilon \\ 0 \end{pmatrix},$$

and from the energy estimates, \tilde{D}_ε is bounded in $L_t^p L^{1-\frac{3}{p}}$ thanks to interpolation and Sobolev injections.

- For the first Theorem, we get estimates in $L_t^2 L^\infty$ through

$$\dot{H}^{\frac{3}{2}-\alpha} \cap \dot{H}^{\frac{3}{2}+\beta} \hookrightarrow \dot{B}_{2,1}^{\frac{3}{2}} \hookrightarrow \dot{B}_{\infty,1}^0 \hookrightarrow L^\infty,$$

- We use once more the Strichartz estimates to bound W_ε in $L_t^2 \dot{B}_{\infty,1}^0$.

Strichartz Estimates

Consider the following system (external force terms F_{ext} , G_{ext} depend on (t, x))

$$\begin{cases} \partial_t f - \nu \Delta f + \frac{1}{\varepsilon} \mathbb{P}(f \wedge e_3) = F_{ext} + G_{ext}, \\ f|_{t=0} = f_{0,\varepsilon}. \end{cases} \quad (14)$$

In our problem, two different regularities:

$$\begin{cases} F_{ext} = c_\varepsilon \cdot \nabla c_\varepsilon, \\ G_{ext} = \tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b}. \end{cases}$$

We need two types of Strichartz estimates:

Isotropic Strichartz estimates

For any $d \in \mathbb{R}$, $r \geq 2$, $q \geq 1$, $\theta \in [0, 1]$, and $p \in [1, \frac{2}{\theta(1-\frac{2}{r})}]$, there exists a constant $C = C_{p,\theta,r} > 0$ such that for any divergence-free vectorfield $f_{0,\varepsilon}$, F_{ext} and G_{ext} , the solution f of (14) satisfies:

$$\| |D|^d f \|_{\tilde{L}^p \dot{B}_{r,q}^0} \leq \frac{\varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})}}{\nu^{\frac{1}{p}-\frac{\theta}{2}(1-\frac{2}{r})}} \left(C_{p,\theta,r} \left(\|f_{0,\varepsilon}\|_{\dot{B}_{2,q}^\sigma} + \|F_{ext}\|_{L^1 \dot{B}_{2,q}^\sigma} \right) + \frac{C_{p,\theta,r,k}}{\nu^{1-\frac{1}{k}}} \|G_{ext}\|_{L^k \dot{B}_{2,q}^{\sigma+\frac{2}{k}-2}} \right). \quad (15)$$

with $\sigma = d + \frac{3}{2} - \frac{3}{r} - \frac{2}{p} + \theta(1 - \frac{2}{r})$.

Anisotropic Strichartz estimates

For any $d \in \mathbb{R}$, $m > 2$, $\theta \in]0, 1]$, $p \in [1, \frac{4}{\theta(1-\frac{2}{m})}]$ there exists a constant $C = C_{p,\theta,m}$ such that for any divergence-free vectorfields $f_{0,\varepsilon}$, F_{ext} and G_{ext} , the solution f of (14) satisfies:

$$\| |D|^d f \|_{L^p L^m_{h,v}} \leq \frac{\varepsilon^{\frac{\theta}{4}(1-\frac{2}{m})}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{2}{m})}} \left(C_{p,\theta,m} \left(\|f_{0,\varepsilon}\|_{\dot{B}_{2,1}^\sigma} + \|F_{ext}\|_{L^1 \dot{B}_{2,1}^\sigma} \right) + \frac{C_{p,\theta,m,k}}{\nu^{1-\frac{1}{k}}} \|G_{ext}\|_{L^k_t \dot{B}_{2,1}^{\sigma+\frac{2}{k}-2}} \right). \quad (16)$$

with $\sigma = d + 1 - \frac{2}{m} - \frac{2}{p} + \frac{\theta}{2}(1 - \frac{2}{m})$.

Thank you for your attention !

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Link with the classical Boussinesq system

The classical Boussinesq system

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v + \kappa^2 \rho e_3 = -\nabla P, \\ \partial_t \rho + v \cdot \nabla \rho - \nu' \Delta \rho = 0, \\ \operatorname{div} v = 0. \end{cases} \quad (17)$$

Related with (17) through the study of solutions **near the following explicit stationary solution**: $(\bar{V}_\varepsilon, \bar{P}_\varepsilon)$ with

$$\bar{P}_\varepsilon(x_3) = \bar{P}_{0,\varepsilon} - \kappa^2 \bar{\rho}_{0,\varepsilon} x_3 + \frac{x_3^2}{2\varepsilon^2},$$

$$\bar{V}_\varepsilon(x_3) = \begin{pmatrix} 0 \\ \bar{\rho}_\varepsilon(x_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\rho}_{0,\varepsilon} - \frac{x_3}{\varepsilon^2 \kappa^2} \end{pmatrix},$$

The theorem for the stratified Boussinesq system gives global solutions for the classical Boussinesq system for initial data as follows:

$$V_\varepsilon|_{t=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \rho_{0,\varepsilon} - \frac{x_3}{\varepsilon^2 \kappa^2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\tilde{\theta}_0(x_3)}{\varepsilon \kappa^2} \end{pmatrix} + \begin{pmatrix} \tilde{v}_0^h(x) + v_{0,osc,\varepsilon}^h(x) \\ v_{0,osc,\varepsilon}^3(x) \\ \frac{\theta_{0,osc,\varepsilon}(x)}{\varepsilon \kappa^2} \end{pmatrix}.$$

Plus asymptotic expansion:

$$V_\varepsilon(t, x) \underset{\varepsilon \rightarrow 0}{\sim} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\rho}_{0,\varepsilon} - \frac{x_3}{\varepsilon^2 \kappa^2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\tilde{\theta}(t, x_3)}{\varepsilon \kappa^2} \end{pmatrix} + \begin{pmatrix} \tilde{v}^h(t, x) + \mathcal{O}(\varepsilon^K) \\ \mathcal{O}(\varepsilon^K) \\ \mathcal{O}(\varepsilon^{K-1}) \end{pmatrix}.$$