Non-local models with turbulent hydrodynamic flavor

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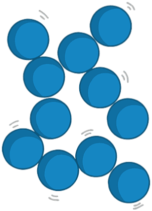
Part I

Introduction

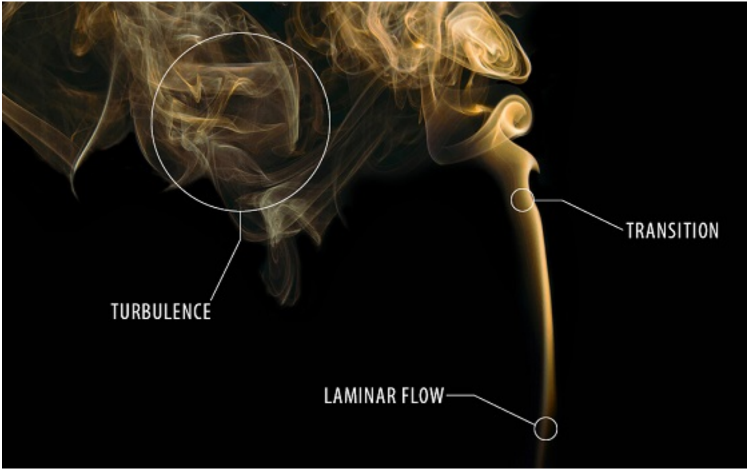
Physical scales (from human perspective)



kinetic

hydrodynamic

thermodynamic


TURBULENCE
TRANSITION
LAMINAR FLOW

Physcial quantities & fundamental laws

- Velocity : $v(t,x)$ $\qquad x \in \Omega \subset \mathbb{R}^d$ or $\mathbb{T}^d$
- Density : $\rho(t,x)$
- Flow : $\Phi_t : \Omega \to \Omega$

$$\Phi_0(x) = x \qquad \frac{d}{dt}\Phi_t(x) = v(t, \Phi_t(x))$$

- Fluid particle : $(\pi_t)_{t\in\mathbb{R}} \subset \Omega$ s.t. $\pi_t = \phi_t(\pi_0)$

$$\frac{d}{dt}\left(\int_{\pi_t} f(t,x)dx\right) = \int_{\pi_t} \partial_t f + \underbrace{\mathrm{div}(v \otimes f)}_{\partial_i(v_i f) = (v\cdot\nabla)f + f\,\mathrm{div}\,v}$$

Universal balance laws $\sim$ PDE in distributional sense

$\forall (\pi_t)_{t\in\mathbb{R}}, \ldots$

- Volume: $\displaystyle\int_{\pi_t} dx$ conserved iff $\operatorname{div} v = 0$.
- Mass:

$$\frac{d}{dt}\left(\int_{\pi_t}\rho(t,x)dx\right) = 0 \qquad \sim \qquad \partial_t\rho + \operatorname{div}(\rho v) = 0$$

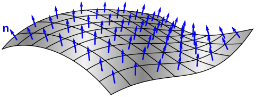
- Momentum (Newton's law):

$$\frac{d}{dt}\left(\int_{\pi_t}\rho(t,x)v(t,x)dx\right) = F(\pi_t) = \underbrace{\int_{\pi_t}\rho f}_{\text{volume}} + \underbrace{\int_{\partial\pi_t}\mathbf{T}\cdot\mathbf{n}}_{\text{surface}}$$

Cauchy equation

$$\partial_t v + v\cdot\nabla v = f + \frac{\operatorname{div}\mathbf{T}}{\rho}$$

$$\mathbf{T} = \underbrace{-p(t,x)\,\mathrm{id}}_{\text{Euler}} + \underbrace{\nu\rho\left(\frac{\nabla v + \nabla v^T}{2}\right)}_{\text{Navier-Stokes}} + \dots$$



Classical models in (viscous) hydrodynamics

- Incompressible Navier-Stokes ($u : \mathbb{T}^d \to \mathbb{R}^d$, $d \geq 2$) – 1822-1850:

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \end{cases}$$

- Burgers' model ($v : \mathbb{T} \to \mathbb{R}$) – 1948 :

$$\partial_t v - \nu \partial_x^2 v + v \partial_x v = 0$$

- Viscosity : $\nu > 0$
- Formal energy conservation:

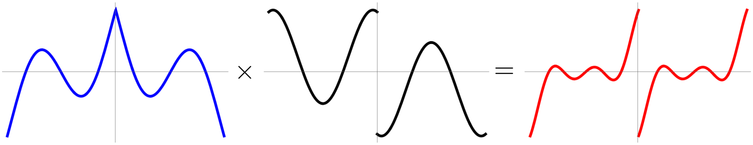
$$\int_{\mathbb{T}^d} (u \cdot \nabla) u \cdot u = 0 \qquad \int_{\mathbb{T}} |v|^{p-2} v \partial_x v = 0$$

$\Longrightarrow$ a priori bound in $L_t^\infty(L_x^2) \cap L_t^2(\dot{H}_x^1)$ (Leray space, 1934)
or $v \in L_t^\infty(L_x^p)$ and $v^{p/2} \in L_t^2(\dot{H}_x^1)$ (Burgers, $1 \leq p < \infty$).

- Conservative form:

$$(u\cdot\nabla)u = \mathrm{div}(u\otimes u) - \cancel{(\mathrm{div}\, u)}u, \qquad v\partial_x v = \frac{1}{2}\partial_x(v^2)$$

- High $\times$ low regularity inherits the lowest regularity:



unless the regularity of both factors is high enough.

(Kato, 1972 – Koch-Tataru, 2000)

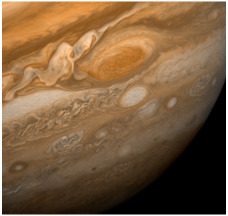
For Navier-Stokes equation

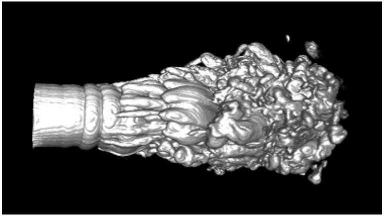
- Leray projector: $\mathbb{P} : L^2 \to L^2_{\text{div}} = \{u\,;\,\text{div}\,u = 0\}$

$$\mathbb{P} = \text{id} - \nabla\Delta^{-1}\,\text{div} = \mathcal{F}^{-1}\left[\left(\delta_{i,j} - \frac{\xi_i\xi_j}{|\xi|^2}\right)_{i,j}\begin{pmatrix} \cdot \\ \hat{u}_j(\xi) \\ \cdot \end{pmatrix}\right]$$

- Non-locality:

$$-\Delta p = \text{div}\,((u\cdot\nabla)u) = \sum_{i,j}(\partial_i u_j)(\partial_j u_i)$$



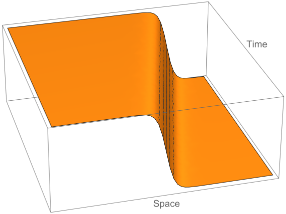


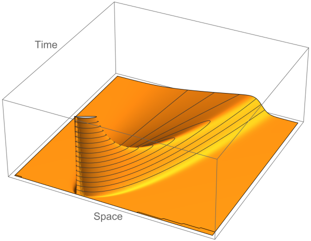
For Burgers equation

- Sign is preserved, max & min principle.
- Cole-Hopf transform:

$$u(t,x) = -2\nu\partial_x \ln\left(\phi(t,x)e^{\psi(t)}\right)$$

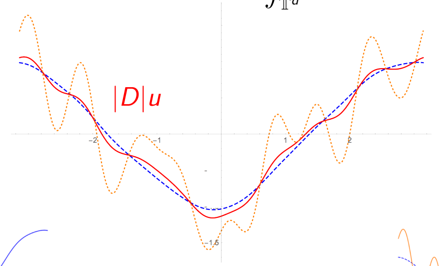
$$\phi = e^{\nu t\Delta} \exp\left(-\frac{1}{2\nu}\int^x u_0\right) \qquad \psi : \text{gauge (no effect on } u\text{)}$$

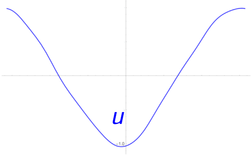


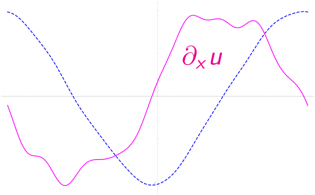


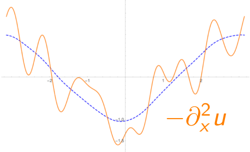
Pseudo-derivative

$$|D|u = (-\Delta)^{1/2}u = \mathcal{F}^{-1}\left(|\xi|\hat{u}(\xi)\right) = \text{p.v.}\int_{\mathbb{T}^d}(u(x)-u(y))\underbrace{K_d(x-y)}_{\sim|x-y|^{-d-1}}dy$$









Fractional heat flow on $\mathbb{T}^d$

The solution of the (linear) fractional heat flow

$$\begin{cases} \partial_t u + |D|u = f \\ u(0,\cdot) = u_0 \end{cases}$$

satisfies for $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ and $p \in [1,\infty]$:

$$\|u\|_{\widetilde{L}^{\rho_1}_T(B^{s+1/\rho_1}_{p,1})} \lesssim \|u_0\|_{B^s_{p,1}} + \|f\|_{\widetilde{L}^{\rho_2}_T(B^{s+1/\rho_2-1}_{p,1})}$$

with Chemin-Lerner norms:

$$\|u\|_{\widetilde{L}^{\rho}_T(B^s_{p,1})} = \left(\int_0^T |S_0 u(t)|^\rho \, dt\right)^{1/\rho} + \sum_q 2^{qs} \left(\int_0^T \|\Delta_q u(t)\|^\rho_{L^p} \, dt\right)^{1/\rho}$$

S_0 average operator, Δ_q Littlewood-Paley filters.

Classical inequalities between $u|D|u$ and $D(u^2)$

$$[u,|D|]u = u|D|u - |D|(u^2)$$

- Cordoba, 2004:

$$u|D|u = \frac{1}{2}|D|(u^2) + \int (u(x)-u(y))^2 K_d(x,y) \geq \frac{1}{2}|D|(u^2) \quad \text{pointwise.}$$

- Kenig-Ponce-Vega, 1993: $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $s_1 + s_2 = 1$, $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$

$$\| |D|(uv) - u|D|v - v|D|u \|_{L^p} \leq \|u\|_{\dot{B}^{s_1}_{p_1,q_1}} \|v\|_{\dot{B}^{s_2}_{p_2,q_2}}$$

$\rightsquigarrow$ pseudo-conservative form:

$$[u,|D|]u \quad = \quad -\frac{1}{2}|D|(u^2) \quad + \quad u|D|u - \frac{1}{2}|D|(u^2)$$

Part II

Non-local models

New non-local model(s)

Non-local Burgers equation: $u : [0, T) \times \mathbb{T}^d \to \mathbb{R}$

$$\partial_t u = [u, |D|]u \qquad \text{(NB)}$$

$$\partial_t u - \nu \Delta u = [u, |D|]u \qquad (\text{NB}_\nu)$$

- $[u, \partial_x]u = -u\partial_x u$
- Formal L^2 energy conservation: $\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$

$$\int_{\mathbb{T}^d} [u, |D|]u \cdot u = \int_{\mathbb{T}^d} u^2 |D| u - \int_{\mathbb{T}^d} u |D| (u^2) = 0$$

- Previous work: P.G. Lemarié, F. Lelièvre (2010); R. Shvydkoy, C. Imbert, F.V. (2016) + T. Jin, L. Silvestre (2018); J. Tan, F.V. (2024); R. Anton, K. Verdure, F.V. (2022-2025).

The surprising properties of $[u, |D|]u$

- "Self-weighted" pseudo-derivative:

$$|D| = J_1, \qquad [u, |D|]u = -J_u(u)$$

$$J_\omega u(x) = \int_{\mathbb{T}^d} (u(x) - u(y))\,\omega(y) K_d(x-y)dy$$

$$\partial_t u + J_u(u) = 0 \tag{NB}$$

- Sign $u = u_+ - u_-$ tied to regularization/singularization:

$$\underbrace{\partial_t u + J_{u_+}(u)}_{\text{frac. heat eq.}} - \underbrace{J_{u_-}(u)}_{\text{anti-heat}} = 0 \tag{NB}$$

- Momentum law for (NB):

$$\frac{d}{dt}\int_{\mathbb{T}^d} u = \int_{\mathbb{T}^d} u|D|u = \left\| |D|^{1/2} u \right\|_{L^2}^2 = \sum_{k\neq 0} |k||\hat{u}_k|^2 \geq 0.$$

Bonus bound & failed compactness. . .

- $\mathcal{L}_{1/2} = L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^{1/2}_x)$ a-priori bound for "Euler-style" (NB) !

$$\int_0^T \|u\|^2_{\dot{H}^{1/2}} = \int_{\mathbb{T}^d} u(T,\cdot) - \int_{\mathbb{T}^d} u_0 \leq 2|\mathbb{T}^d|^{1/2}\|u_0\|_{L^2}$$

  $+$ usual $\mathcal{L}_1 = L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^1_x)$ for $(\mathrm{NB})_\nu$.

- Galerkine scheme : u_n approximate solutions, unif. bound in $\mathcal{L}_{1/2}$, compactness in weak-$*$ topology. *If $u_n|D|u_n \rightharpoonup u|D|u$, then testing against* $\mathbf{1}$ *we would get*

$$u_n \to u \qquad \dot{H}^{1/2} \quad \textit{strong} \qquad (\textit{unrealistic}).$$

  Leray's method for (NB) is just out of reach!

Momentum + fluctuations decomposition

- $[v+\alpha, |D|](v+\alpha) = [v, |D|]v - \alpha|D|v \qquad \alpha \in \mathbb{R}$
- Average zero projector: $v = \mathbb{P}_0(u) := u - \fint_{\mathbb{T}^d} u \quad$ (fluctuations)

$$\partial_t v + p(t)|D|v = \mathbb{P}_0[v, |D|]v \tag{NB}$$

$$p(t) = p_0 + \frac{1}{|\mathbb{T}^d|}\int_0^t \|v\|^2_{\dot{H}^{1/2}} \in p_0 + \left[0, \frac{2\,\|u_0\|_{L^2}}{|\mathbb{T}^d|^{1/2}}\right]$$

- Frozen version:

$$\partial_t v - \nu\Delta v + p|D|v = \mathbb{P}_0[v, |D|]v \qquad p > 0 \tag{FNB$_{p,\nu}$}$$

with constraint $\fint_{\mathbb{T}^d} v = 0$. A-priori bounds in $p\mathcal{L}_{1/2} \cap \nu\mathcal{L}_1$.

$\rightsquigarrow$ *external force, constant in x, that keeps the momentum constant.*

Scaling invariance

- The sign controls the flow of time: $u_*(t,x) = -u(-t,x)$.
- Scaling laws for (NB) on rescaled domains:

$$u_\lambda(t,x) = \lambda^\alpha u(\lambda^{\alpha+\beta} t, \lambda^\beta x)$$

- $u_\lambda(t,x) = u(\lambda t, \lambda x)$ is the only scaling law that conserves the energy density and the momentum:

$$\frac{d}{d\lambda}\left(\frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} u_\lambda(t,x)^2 dx\right) = 0 = \frac{d}{d\lambda}\left(\frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} u_\lambda(t,x) dx\right).$$

  Compatible with additional (frozen) term $p|D|u$.
- $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ is the only one compatible with $-\nu\Delta u$

$\rightsquigarrow$ critical spaces for u :

(NB)	:	L^∞, $\dot{H}^{d/2}$, ...
$(\mathrm{NB})_\nu$	:	L^d, $\dot{H}^{d/2-1}$, ...
$(\mathrm{FNB})_{p,\nu}$	:	hybrid space (cut-off at $k_0 \sim p/\nu$)

Part III

Positive solutions

Theorem (C. Imbert, R. Shvydkoy, F. Vigneron – 2016)

*The Cauchy problem for (NB) for a **positive** data admits at least one solution $u \in C^\infty((0,\infty) \times \mathbb{T}^d)$ such that*

1. $u \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \cap L^2(\mathbb{R}_+; \dot{H}^{1/2}(\mathbb{T}^d))$ *and* $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$,
2. *the energy density is continuous:* $u(t) \underset{t\to 0^+}{\longrightarrow} u_0$ *in* L^2,
3. *the momentum is continuous:* $u(t) \underset{t\to 0^+}{\longrightarrow} u_0$ *weak-$*$ in* L^∞.

It converges towards the constant state

$$u(t) \underset{t\to+\infty}{\longrightarrow} \frac{\|u_0\|_{L^2}}{\sqrt{|\mathbb{T}^d|}}$$

both at large and small scales (decay of oscillation and in Lipschitz norm).

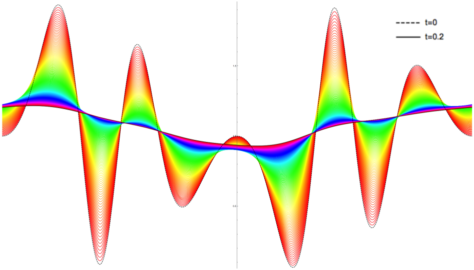
Theorem

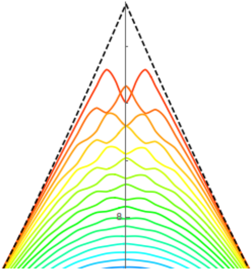
The previous solution is unique as soon as $u_0 \in H^m(\mathbb{T}^d)$ *with* $m > \frac{d}{2} + 1$.

$\rightsquigarrow$ A bifurcation at $t = 0$ for a non-smooth u_0 is still possible.

Sketch of proof

- Local theory ▷ Beale-Kato-Majda survival criterion.





- "Good" equation for **energy density** $w = u^2$:

$$\partial_t w + J_\mu(w) = 0 \qquad \mu = \frac{2u(x)u(y)}{u(x)+u(y)} \qquad x_0\text{-blow-up friendly}$$

▷ regularity (parabolic DeGiorgi + Schauder bootstrap).

- Topology $u^2 \rightsquigarrow u$: exclude the concentration of $\dot{H}^{1/2}$ norm near $t = 0$ (i.e. continuity of the **momentum**).

Sketch of proof

- Local theory ▷ Beale-Kato-Majda survival criterion.

Ricatti estimate for $H^2(\mathbb{T})$, similar to 3D Euler in H^3:

$$\frac{d}{dt}\left\|u''\right\|_{L^2}^2 \leq C\left\|u''\right\|_{L^2}^2 \left\|u', |D|u\right\|_{L^\infty} - \underbrace{\langle uu'' | |D|u''\rangle}_{\text{structure of (NB)}} \leq C\left\|u''\right\|_{L^2}^3 - \frac{1}{2}\underbrace{\langle (u'')^2 | |D|u\rangle}_{L^2\times L^2\times L^\infty}.$$

Log-sobolev inequalities:

$$\int_0^T \|\nabla u(\tau)\|_{L^\infty} \, d\tau < \infty \quad \Longrightarrow \quad T < T^*.$$

- "Good" equation for **energy density** $w = u^2$:

$$\partial_t w + J_\mu(w) = 0 \qquad \mu = \frac{2u(x)u(y)}{u(x)+u(y)} \qquad x_0\text{-blow-up friendly}$$

▷ regularity (parabolic DeGiorgi + Schauder bootstrap).

- Topology $u^2 \rightsquigarrow u$: exclude the concentration of $\dot{H}^{1/2}$ norm near $t = 0$ (i.e. continuity of the **momentum**).

Corollary (Fun observation on momentum law)

The future dissipation (of fluctuations) is the defect in Cauchy-Schwartz:

$$\int_t^\infty \|u(\tau)\|_{\dot{H}^{1/2}}^2 \, d\tau = \sqrt{|\mathbb{T}^d|}\, \|u(t)\|_{L^2} - \int_{\mathbb{T}^d} u(t,x) dx$$

which is bounded by $|\mathbb{T}^d| \operatorname{Osc} u(t)$ *and exponentially decaying.*

Corollary (Dual blow-up result for $u_0 < 0$)

For any $T > 0$*, one can chose* $u_0 \in C^\infty(\mathbb{T}^d)$ *with* $u_0 < 0$ *associated to a classical solution* $u \in L^\infty([0,T) \times \mathbb{T}^d)$ *with*

$$\lim_{t \to T} u(t) \Big|_{L^2 \cap L^\infty_{w*}} \notin C^0(\mathbb{T}^d).$$

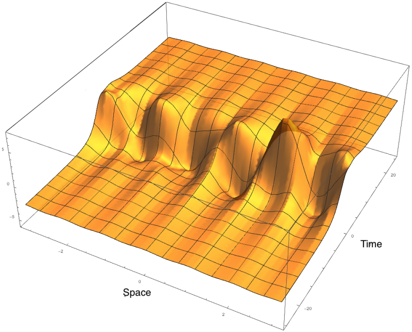
$\rightsquigarrow$ *smooth negative solutions of (NB) may lose most of their regularity.*

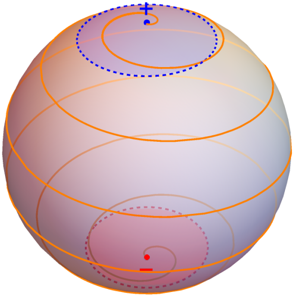
Part IV

Unsigned solutions

Global picture by finite element approximation of (NB)

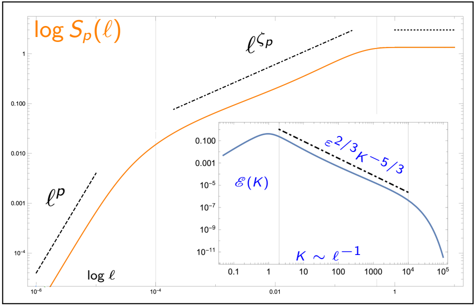
Global trajectory on an energy sphere of $\mathbb{R}^N$ for ODE associated to finite element approximation of (NB): N: degrees of freedom

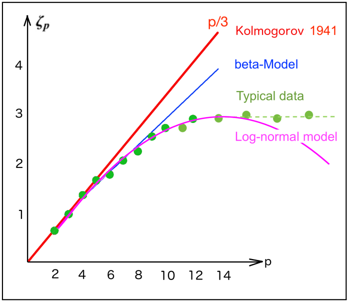




It displays an instability that connects two constant states of opposite signs through a transient unsigned regime (*turbulent?*). One observes an exponential amplification (before) / decay (after) of the solution, that peaks as the momentum vanishes (*intermittency?*).

Turbulence & intermittency in a nutshell





$$S_p(\ell) = \fint_{T_0}^{T_1} \iint_{\mathbb{R}^3\times\mathbb{S}^2} |u(t,x+\ell\theta) - u(t,x)|^p \, dx d\theta dt$$

$$\mathscr{E}(K) = \fint_{T_0}^{T_1} \int_{\mathbb{S}^2} |\hat{u}(t,K\theta)|^2 K^2 d\theta dt \qquad \varepsilon = \nu \int_0^\infty K^2 \mathscr{E}(K) dK$$

Well-posedness (hydrodynamic case: $\nu > 0$)

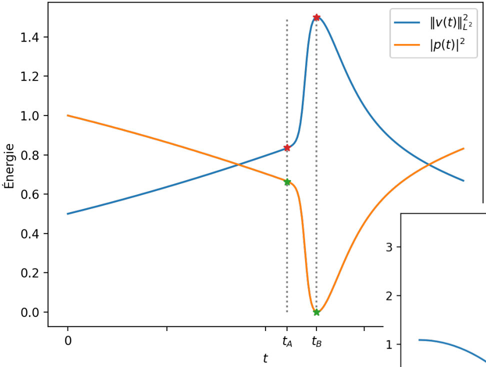
Theorem (F. Lelièvre, P.G. Lemarié – 2010)

The Cauchy problem for $(NB)_\nu$ in $L^3(\mathbb{R}^3)$ admits a strong local solution, which is global if the norm of the initial data is small enough.

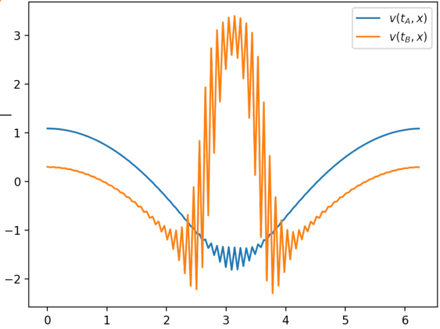
Theorem (K. Verdure, R. Anton, F.V. – 2025)

- *$(FNB)_{p,\nu}$ has global solutions in $L^2(\mathbb{T}^d)$ for $d = 1, 2, 3$ and*
$$\|v_1 - v_2\|_{L^2}^2 + 2p\int_0^t \|v_1 - v_2\|_{\dot{H}^{1/2}}^2 + \nu \int_0^t \|v_1 - v_2\|_{\dot{H}^1}^2 \leq M \left\|v_1^0 - v_2^0\right\|_{L^2}^2$$
with $M = M_{2D}(\nu, \left\|v_j^0\right\|_{L^2})$; stability in 3D requires $L_t^5(\dot{H}^1)$ control.
- *Without restriction in 2D, or as long as v exists in 3D:*
$$\left\|e^{\left(\frac{1}{2}\sqrt{\nu t}+pt\right)|D|} v(t)\right\|_{L^2} \leq C \|v_0\|_{L^2}.$$
so analyticity radius $\geq \frac{1}{2}\sqrt{\nu t/d} + pt$.

Numerical observations for (NB)

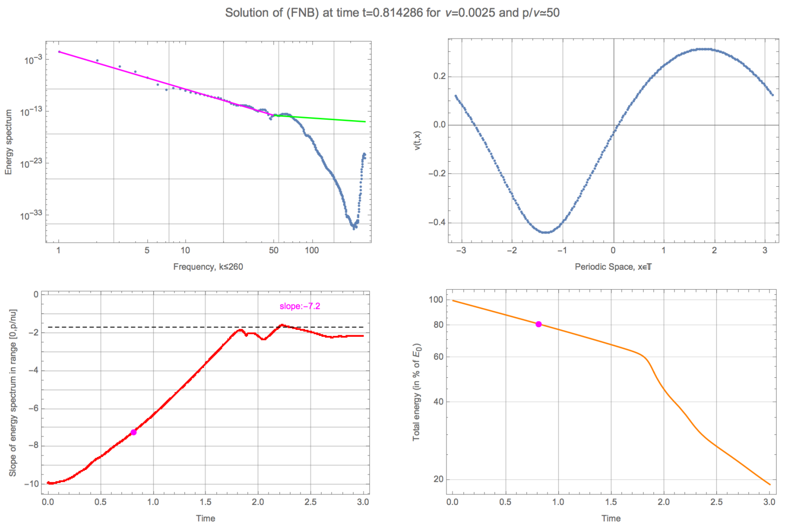


K. Verdure (2025)



Numerical observations for $(\text{FNB})_{p,\nu}$

The rate of decay of the energy (right) is directly correlated to the spectral profile; it peaks when $|\hat{u}_k|^2 \propto |k|^{-2}$ in the range $k \leq p/\nu$.



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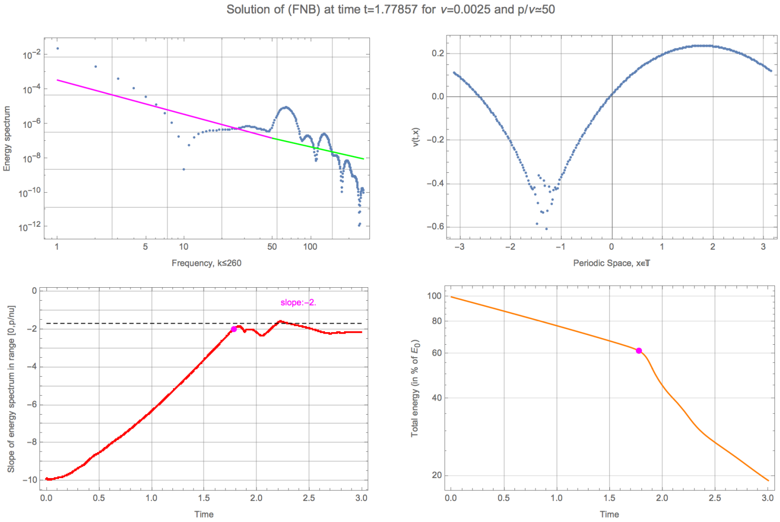

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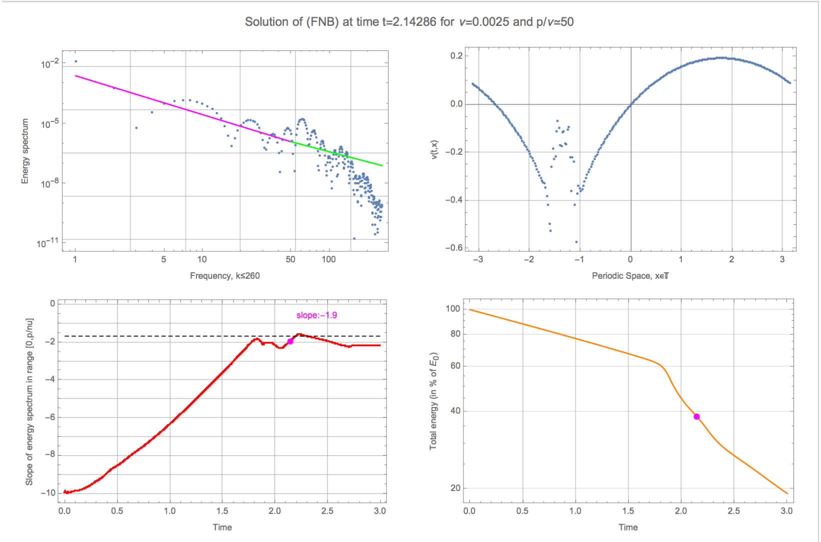
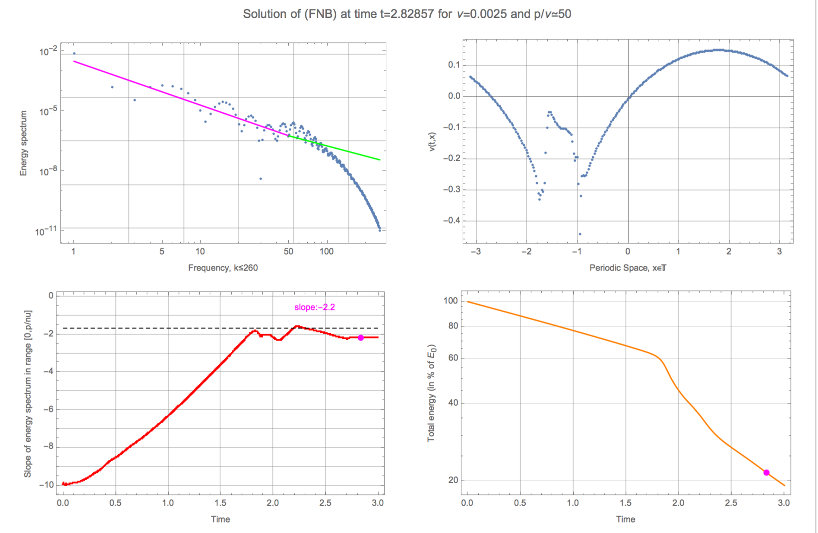


Numerical observations for $(\text{FNB})_{p,\nu}$

The rate of decay of the energy (right) is directly correlated to the spectral profile; it peaks when $|\hat{u}_k|^2 \propto |k|^{-2}$ in the range $k \leq p/\nu$.



Conclusion & perspectives

Non-local models with hydrodynamics flavor

$$\partial_t u - \nu \Delta u = [u, |D|]u \qquad (\mathrm{NB}_\nu)$$

$$\begin{cases} \partial_t v - \nu \Delta v + p|D|v = \mathbb{P}_0[v, |D|]v & p > 0 \\ \fint_{\mathbb{T}^d} v = 0 \end{cases} \qquad (\mathrm{FNB}_{p,\nu})$$

Well-posedness theory (positive case $\nu = 0$ or unsigned case $\nu > 0$)
+ encouraging numerical experiments.

Questions:

- Unsigned theory in the turbulent limit $p, \nu \to 0$?
- Precise large-scale numerical experiments.

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Thank you !