Asymptotic preserving schemes

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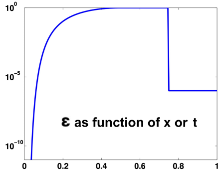
Model : S_ε depends on a parameter ε

- ε takes different orders of magnitude in the space/time domain
- Regular or rough transitions
- Limit model $S_0 = \lim_{\varepsilon \to 0} S_\varepsilon$



Multiscale problem : microscopic $\rightarrow$ macroscopic

Difficulties :

- Classical explicit schemes for S_ε : stable, consistent and accurate if the mesh resolves all the scales $\Rightarrow$ **huge cost**
- Schemes for S_0 with meshes independent of ε **but** S_0 is not valid everywhere and difficult to locate the interface

A solution :

Develop schemes for S_ε which preserve the limit $\varepsilon \to 0$

Asymptotic Preserving schemes (AP)

Objectives of Asymptotic Preserving schemes

✓ **Use only one model** in the space/time domain : S_ε

✓ **Preserve the limit** $\varepsilon \to 0$:

- We can use meshes independent of ε

Asymptotic stability

- Recover an approximate solution of the limit model S_0 when $\varepsilon \to 0$

Asymptotic consistency

Asympt. stable and consistent = Asymptotic Preserving

[S. Jin, 1999], kinetic $\to$ fluid

✓ **Preserve some properties of classic explicit schemes :**

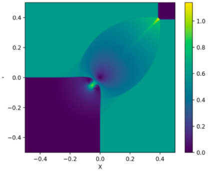
- Cost of one iteration in time

- Preserve stationary solutions...

➠ Necessary to well understand the asymptotic limit $(S_\varepsilon) \to (S_0)$

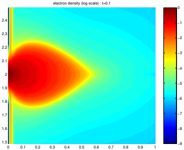
Low-Mach number limit

- Fluid mechanics : Euler, Navier-Stokes.



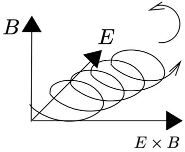
Quasi-neutral limit

- Plasmas : Euler-Poisson, Vlasov-Poisson.
- Semiconductors : Drift diffusion-Poisson.



Gyromagnetic limit

- Magnetic confinement : Euler-Lorentz.



In this talk

- Low Mach number limit for the Euler eqs,

P. Allegrini G. Dimarco, R. Loubère, V. Michel-Dansac.

- Quasi-neutral limit for the Vlasov-Poisson system,

A. Blaustein, G. Dimarco, F. Filbet.

Rescaled Euler system :

$$(S_\varepsilon)\begin{cases}\partial_t\rho^\varepsilon+\nabla\cdot(\rho^\varepsilon u^\varepsilon)=0,\\ \partial_t(\rho^\varepsilon u^\varepsilon)+\nabla\cdot(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon)+\dfrac{1}{\varepsilon}\nabla p^\varepsilon=0,\\ \partial_t E^\varepsilon+\nabla\cdot((E^\varepsilon+p^\varepsilon)u^\varepsilon)=0,\\ E^\varepsilon=\dfrac{p^\varepsilon}{\gamma-1}+\varepsilon\,\dfrac{\rho^\varepsilon|u^\varepsilon|^2}{2},\end{cases}\qquad x\in\Omega\subset\mathbb{R}^d,\ t>0.$$

Parameter : $\varepsilon=\gamma M_0^2>0$ with $M_0=\dfrac{|u_0|}{c_0}=\dfrac{\text{Fluid velocity}}{\text{Pressure wave velocity}}$

Initial and boundary conditions

$u^\varepsilon\cdot\nu=0,$ on $\partial\Omega$, et $\rho(0,x),u(0,x),p(0,x)$ given and no condition

Vector form $\quad \partial_t W^\varepsilon+\nabla\cdot F^\varepsilon(W^\varepsilon)=0,$

with $W=(\rho,q,E)$, $q=\rho u$ and $F^\varepsilon(W)=\left(q,\,\rho u\otimes u+\frac{1}{\varepsilon}p\,\mathrm{Id}_{\mathbb{R}^d},\,(E+p)u\right)$.

Eigen values of $DF^\varepsilon(W^\varepsilon)$: ➠ u^ε : fluid velocity

➠ $u^\varepsilon-\sqrt{\dfrac{\gamma p^\varepsilon}{\varepsilon\rho^\varepsilon}}$ and $u^\varepsilon+\sqrt{\dfrac{\gamma p^\varepsilon}{\varepsilon\rho^\varepsilon}}\to$ tend to $\pm\infty$ when ε tends to 0

Rigorous limit

- Isentropic case $p = p(\rho) = \rho^\gamma$
  [Klainerman, Majda, 81, 82], [Schochet, 86], [Lions, Masmoudi, 98]
- Full Euler case
  [Métivier,Schochet, 01, 03], [Alazard, 05]

Formal limit *with ill-prepared initial conditions*

$$\partial_t \rho^\varepsilon + \nabla\cdot(\rho^\varepsilon u^\varepsilon) = 0, \quad (1_\varepsilon) \qquad \partial_t(\rho^\varepsilon u^\varepsilon) + \nabla\cdot(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{1}{\varepsilon}\nabla p^\varepsilon = 0, \quad (2_\varepsilon)$$

$$\partial_t E^\varepsilon + \nabla\cdot((E^\varepsilon + p^\varepsilon)u^\varepsilon) = 0, \quad (3_\varepsilon) \qquad E^\varepsilon = \frac{p^\varepsilon}{\gamma-1} + \varepsilon\frac{\rho^\varepsilon|u^\varepsilon|^2}{2}. \quad (4_\varepsilon)$$

If $\lim_{\varepsilon\to 0}(\rho^\varepsilon, u^\varepsilon, p^\varepsilon, E^\varepsilon) = (\rho, u, p, E)$, then

$(2_\varepsilon), (4_\varepsilon) \Rightarrow \nabla p = 0$ and $E = \frac{p}{\gamma-1} \Rightarrow p(x,t) = p(t)$ and $E(x,t) = E(t)$,

$$\int_\Omega (3_\varepsilon) \;\Rightarrow\; |\Omega|E(t) = \int_\Omega E(x,0) - \int_0^t (E(t)+p(t))\int_{\partial\Omega} u.\nu = |\Omega| < E(\cdot,0) >,$$

$\Rightarrow$ E and p constant for all $t > 0$.

$(3_\varepsilon) \quad \Rightarrow \nabla\cdot u(\cdot,t) = 0,$ incompressible regime.

$$(S_0)\begin{cases}\partial_t\rho+\nabla\cdot(\rho u)=0, & (1_0)\\ \partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla p_1=0, & (2_0)\\ \nabla\cdot u=0, & (3_0)\\ E=\dfrac{p}{\gamma-1}=\dfrac{1}{|\Omega|}\displaystyle\int_\Omega E(x,0)\,dx, & (4_0)\end{cases}$$

$p^\varepsilon(x,t)=p+\varepsilon p_1(x,t)+o(\varepsilon)$ and p_1 is impl. given by the constraint $\nabla\cdot u=0$

Elliptic eq. for p_1 :

$$(1_0) \text{ in } (2_0) \quad \Rightarrow \rho\partial_t u+(1_0)u+\rho(u.\nabla)u+\nabla p_1=0 \quad (*)$$

$$\nabla\cdot((*)/\rho) \quad \Rightarrow \partial_t(3_0)+\nabla\cdot((u.\nabla)u)+\nabla\cdot\left(\tfrac{1}{\rho}\nabla p_1\right)=0$$

$$\Rightarrow -\nabla\cdot\left(\frac{1}{\rho}\nabla p_1\right)=\nabla\cdot((u\cdot\nabla)u)\,. \quad (R3_0)$$

Reformulated limit model

$$(S_0) \quad \Rightarrow \quad (RS_0) \quad = \quad (1_0),(2_0),(R3_0),(4_0)$$

Understand the singularity of the limit : reformulate S_ε

$$\begin{tikzcd} S_\varepsilon \arrow[r, "\varepsilon \longrightarrow 0"] \arrow[d] & S_0 \arrow[d] \\ RS_\varepsilon \arrow[r, "\varepsilon \longrightarrow 0"'] & RS_0 \end{tikzcd}$$

We complete the diagram

Reformulation of S_ε like for S_0

$$(S_\varepsilon) \quad \Rightarrow \quad (RS_\varepsilon) \begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, & (1_\varepsilon) \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \dfrac{1}{\varepsilon} \nabla p = 0, & (2_\varepsilon) \\ \partial_t \left(\dfrac{\partial_t p}{\gamma p} + \dfrac{u \cdot \nabla p}{\gamma p} \right) - \dfrac{1}{\varepsilon} \nabla \cdot \left(\dfrac{1}{\rho} \nabla p \right) = \nabla \cdot ((u.\nabla) u), & (R3_\varepsilon) \\ E = \dfrac{p}{\gamma - 1} + \varepsilon \dfrac{\rho |u|^2}{2} & (4_\varepsilon) \end{cases}$$

Eq. $(R3_\varepsilon)$ is a wave equation with a velocity $O(1/\sqrt{\varepsilon})$

✓ An **implicit** discretization is necessary otherwise $\Delta t = O(\sqrt{\varepsilon})$

✓ $(RS_\varepsilon) \Rightarrow \quad (S_\varepsilon)$ iff the init. cond. are well-prepared

Conclusion : - We discretize (S_ε) in time and **after** we reformulate

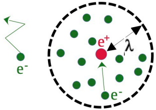
- We must **well choose** the implicit terms

Vlasov-Poisson system in the quasi-neutral limit

Rescaled Vlasov-Poisson system : (One species model for clarity)

$$(S_\lambda) \begin{cases} \partial_t f^\lambda + v \cdot \partial_x f^\lambda - \partial_x \varphi^\lambda \partial_v f^\lambda = 0, \\ -\lambda^2 \partial_{xx}^2 \varphi^\lambda = \rho^\lambda - 1 = \displaystyle\int_{\mathbb{R}} f^\lambda \, dv - 1. \end{cases} \qquad x \in \mathbb{T},\ v \in \mathbb{R},\ t \geq 0.$$

Unknowns : ✓ $f^\lambda(t,x,v)$: electron distribution function,
✓ $-\varphi^\lambda$: electric potential.



Parameter :

λ = rescaled Debye length = $\dfrac{\text{characteristic scale of electric interactions}}{\text{Size of the domain}}$

Initial conditions

$$\int_{\mathbb{T}} \rho^\lambda(0,x)\, dx = \int_{\mathbb{T}} 1 \, dx = |\mathbb{T}|, \qquad \int_{\mathbb{T}} \overbrace{\int_{\mathbb{R}} f^\lambda(0,x,v)\, dv}^{=j^\lambda(0,x)} \, dx = 0,$$

$$\int_{\mathbb{R}} \Bigg(\underbrace{\int_{\mathbb{R}} f^\lambda(0,x,v)\, v^2 \, dv}_{=K^\lambda(0,x)} + \lambda^2 \big| \underbrace{-\partial_x \varphi^\lambda(0,x)}_{=E^\lambda(0,x)} \big|^2 \Bigg) dx < +\infty. \ \Rightarrow\ |\partial_x \varphi^\lambda(0,x)| \lesssim \frac{1}{\lambda}$$

Formal limit :

$$(S_0) \begin{cases} \partial_t f + v\partial_x f - \partial_x \varphi \partial_v f = 0, \\ \rho - 1 = \int_{\mathbb{R}} f\,dv - 1 = 0. \end{cases}$$

Now, φ is implicitly given by the quasi-neutrality constraint $\rho = 1$

Reformulation : Moment eqs of Vlasov with

$$j = \int f\,v\,dv, \quad K = \int f\,v^2\,dv.$$

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \partial_t j + \partial_x K = -\rho \partial_x \varphi. \end{cases} \Leftrightarrow \begin{cases} j = 0, \\ E = -\partial_x \varphi = \partial_x K. \end{cases}$$

Reformulated limit model :

$$(RS_0) \begin{cases} \partial_t f + v\partial_x f + E\,\partial_v f = 0, \\ E = \partial_x K. \end{cases}$$

Understand the singularity of the limit : reformulate S_λ

$$\begin{tikzcd} S_\lambda \arrow[r, "\lambda \longrightarrow 0"] \arrow[d] & S_0 \arrow[d] \\ RS_\lambda \arrow[r, "\lambda \longrightarrow 0"'] & RS_0 \end{tikzcd}$$

We complete the diagram λ

Reformulation of S_λ like for S_0

From the Vlasov-Poisson eqs.

$$\partial_t f^\lambda + v \cdot \partial_x f^\lambda - \partial_x \varphi^\lambda \partial_v f^\lambda = 0, \quad -\lambda^2 \partial^2_{xx} \varphi^\lambda = \rho^\lambda - 1.$$

Use the moment eqs. of the Vlasov eq.

$$\begin{cases} \partial_t \rho^\lambda + \partial_x j^\lambda = 0, \\ \partial_t j^\lambda + \partial_x K^\lambda = -\rho^\lambda \partial_x \varphi^\lambda. \end{cases} \Rightarrow \partial^2_{tt} \rho^\lambda - \partial^2_{xx} K^\lambda = \partial_x \left(\rho^\lambda \partial_x \varphi^\lambda \right)$$

Reformulated Poisson eq. : dispersive eq.

$$\lambda^2 \partial^2_{tt} (\partial^2_{xx} \varphi^\lambda) + \partial_x \left(\rho^\lambda \partial_x \varphi^\lambda \right) = -\partial^2_{xx} K^\lambda$$

Time-oscillatory sol. at frequency of order $1/\lambda \Rightarrow$ Implicit discretization

Strongly time-oscillatory solution :

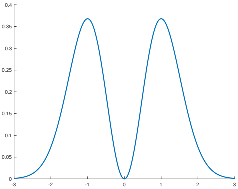
Frequency and amplitude of order $1/\lambda$

Rigorous quasi-neutral limit : Not always valid

[Grenier 96], [Brenier 00], [Han-Kwan,Hauray 15], [Han-Kwan,Hauray 16], [Bobylev,Potatenko 19,23]...

- (S_0) can be ill-posed for some initial data

- Existence of unstable cases (Penrose) : limit not valid



Toy equation

$$(S^\varepsilon)\begin{cases}\partial_t w^\varepsilon(t,x)+\frac{1}{\varepsilon}\partial_x w^\varepsilon(t,x)=0, \quad x\in]0,1[,\ t\in]0,T[,\\ w^\varepsilon(\cdot,0)=w^\varepsilon(\cdot,1),\\ w^\varepsilon(0,\cdot)=w_0^\varepsilon(\cdot)=\cos(2\pi\cdot).\end{cases}$$

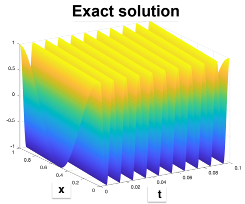
Formal limit $\varepsilon\to 0$ **:** asymptotic expansion : $w^\varepsilon=w^0+\varepsilon w^1+o(\varepsilon^2)$

$$\begin{cases}\partial_x w^0(t,x)=0,\\ \partial_t w^0(t,x)+\partial_x w^1(t,x)=0,\\ w^0(\cdot,0)=w^0(\cdot,1),\\ w^1(\cdot,0)=w^1(\cdot,1),\\ w^0(0,\cdot)=\cos(2\pi\cdot).\end{cases} \Rightarrow (S^0)\begin{cases}\partial_x w^0(t,x)=0,\\ \int_0^1 \partial_t w^0(t,x)\,dx=0,\\ w^0(\cdot,0)=w^0(\cdot,1),\\ w^0(0,\cdot)=\cos(2\pi\cdot).\end{cases}$$

ill-prepared initial cond. : $\partial_x(\lim_{\varepsilon\to 0} w_0^\varepsilon)=\partial_x w_0^0=\partial_x\cos(2\pi\cdot)\neq 0$

$$w^0(t,x)=w^0(t),\ \forall t>0\Rightarrow w^0(t)=\int_0^1 w_0^0(0,x)=0,\ \forall t>0$$

Exact solution of (S^ε) **:** $w^\varepsilon(t,x) = w_0^\varepsilon\left(x - \frac{t}{\varepsilon}\right)$



ill-prepared initial condition : $\partial_x(\lim_{\varepsilon\to 0} w_0^\varepsilon) = \partial_x w_0^0 \neq 0$

$$w^\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} \int_0^1 w_0^0(x)\,dx = 0, \quad \text{in } L^p(]0,1[\times]0,T[) \text{ for all } p \geq 2$$

Asymptotic consistency :

The scheme must capture this weak limit when $\varepsilon \to 0$

Explicit **Implicit**

$$\frac{w_j^{n+1}-w_j^n}{\Delta t}+\frac{1}{\varepsilon}\frac{w_j^n-w_{j-1}^n}{\Delta x}=0 \quad \text{vs} \quad \frac{w_j^{n+1}-w_j^n}{\Delta t}+\frac{1}{\varepsilon}\frac{w_j^{n+1}-w_{j-1}^{n+1}}{\Delta x}=0$$

L^∞ **stability of the explicit scheme :** If $\Delta t \leq \varepsilon \Delta x$

$$|w_j^{n+1}| \leq \left(1-\frac{\Delta t}{\varepsilon\Delta x}\right)|w_j^n|+\frac{\Delta t}{\varepsilon\Delta x}|w_{j-1}^n| \leq \max|w_j^n| \leq \|w_0\|_{L^\infty}$$

$\Delta t \overset{\varepsilon\to 0}{\to} 0 \quad \Rightarrow \quad$ the scheme is not asymptotically stable

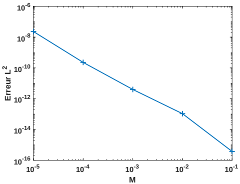
L^∞ **stability of the implicit scheme :**

$$w_{jmax}^{n+1}=\max_j w_j^{n+1} \leq w_{jmax}^{n+1}+\frac{\Delta t}{\varepsilon\Delta x}\overbrace{w_{jmax}^{n+1}-w_{jmax-1}^{n+1}}^{\geq 0}=w_{jmax}^n \leq \|w_0\|_{L^\infty}$$

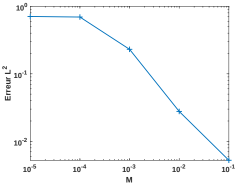
No condition on $\Delta t \Rightarrow$ asymptotically stable scheme

Asymptotic consistency :

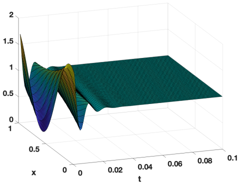
For both schemes : consistency error $= O\left(\frac{\Delta t}{\varepsilon^2} + \frac{\Delta x}{\varepsilon}\right) \left(= O\left(\frac{\Delta x}{\varepsilon}\right)\right)$



Explicit



Implicit $\Delta t = \varepsilon \Delta x$



Implicit $\Delta t = 10\,\varepsilon \Delta x$

➟ Not the key point for the asymptotic consistency

We want to recover $\partial_x w^{n+1} = 0$ for all $n \geq 0$

Explicit : $\dfrac{w_j^{n+1} - w_j^n}{\Delta t} + \dfrac{1}{\varepsilon}\dfrac{w_j^n - w_{j-1}^n}{\Delta x} = 0 \not\Rightarrow w_j^n = w_{j-1}^n \rightarrow$ not AP

Implicit : $\dfrac{w_j^{n+1} - w_j^n}{\Delta t} + \dfrac{1}{\varepsilon}\dfrac{w_j^{n+1} - w_{j-1}^{n+1}}{\Delta x} = 0 \Rightarrow w_j^{n+1} = w_{j-1}^{n+1} \rightarrow$ AP

IMEX (Implicit-Explicit) schemes

$$\frac{W^{n+1}-W^n}{\Delta t}+\nabla\cdot F_e(W^n)+\nabla\cdot F_i^{\varepsilon}(W^{n+1})=0$$

Question : Which terms in F_e and F_i^{ε} ?

We need :

- Jacobian matrices DF_e and DF_i^{ε} : real eigen values
- Eigen values of DF_e uniformly bounded below when $\varepsilon \to 0$.
- Asymptotic consistency :

$$\varepsilon \to 0 \quad \Rightarrow \quad \nabla_x p^{n+1}=\nabla_x E^{n+1}=0, \quad \nabla_x \cdot u^{n+1}=0.$$

- Preservation of contact discontinuities :

$$\left\{\begin{array}{l} Du^n(x)=0, \\ \nabla p^n(x)=0, \end{array}\right. \Rightarrow \exists W^{n+1} \text{ such that } \left\{\begin{array}{l} Du^{n+1}(x)=0, \\ \nabla p^{n+1}(x)=0, \end{array}\right.$$

$Du(x)=(\partial_{x_j} u_i(x))_{1\le i,j\le d}$.

- When $\varepsilon=O(1)$, cost similar to the explicit scheme

- Classical explicit semi-discretization in time

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^n = 0, \qquad (1_n)$$

$$\frac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u\otimes u)^n + \frac{1}{\varepsilon}\nabla p^n = 0, \qquad (2_n)$$

$$\frac{E^{n+1}-E^n}{\Delta t} + \nabla\cdot((E+p)\, u)^n = 0, \qquad (3_n)$$

$$E^n = \frac{p^n}{\gamma-1} + \varepsilon\, \frac{\rho^n\,|u^n|^2}{2}. \qquad (4_n)$$

- $\varepsilon \to 0$ (formally) gives $\nabla p^n = 0$ when W^{n+1} is updated

Nothing on $\nabla p^{n+1} \Rightarrow$ only a constraint on the initial conditions,
not asymptotically consistent

- To recover $\nabla\cdot u^{n+1} = 0$,

an Implicit treatment of u in the energy flux is necessary

State of the art for the full Euler eqs.

[Cordier, Degond,Kumbaro,12], [Chalons,Girardin,Kokh, 13, 16], [Tavelli,Dumbser,17], [Dimarco,Loubère,MHV, 17], [Boscarino, Russo, Scandurra, 17], [Thomann, Zenk, Puppo, Klingenberg, 19]...

➠ **Several choices of flux splitting** $F^\varepsilon(W) = F_e(W) + F_i^\varepsilon(W)$

Choosen splitting : [Toro,Vazquez,12]

$$\frac{E^{n+1} - E^n}{\Delta t} + \nabla \cdot (k_\varepsilon^n \, u^n) + \nabla \cdot \left(h^{n+1} \, u^{n+1}\right) = 0,$$

with

$$k_\varepsilon = \varepsilon \frac{\rho \, |u|^2}{2} = \text{kinetic energy} \qquad h = \frac{\gamma p}{\gamma - 1} = \text{specific enthalpy}$$

Eigen values : $\lambda_e = u, u, u, \quad \lambda_i = 0, u/2 \pm \sqrt{u^2/4 + \frac{\gamma p}{\varepsilon \rho}}$

Non linear AP scheme **[Boscheri,Dimarco,Loubère,Tavelli,MHV,20]**

Reformulated pressure eq., prepared to the low-Mach number regime

$$
\begin{aligned}
E^{n+1} = \frac{p^{n+1}}{\gamma-1} + k_\varepsilon^{n+1} &= E^n - \Delta t\, \nabla\cdot\left(k_\varepsilon^n\, u^n\right) \\
&\quad - \Delta t\, \nabla\cdot\left(\frac{h^{n+1}}{\rho^{n+1}}\left(q^n - \Delta t \nabla\cdot\left(\rho^n\, u^n \otimes u^n\right) - \frac{\Delta t}{\varepsilon}\nabla p^{n+1}\right)\right)
\end{aligned}
\qquad (1)
$$

- Invertible matrix $\Rightarrow \exists\,!$ of p^{n+1}

Algorithm

➠ Calculation of ρ^{n+1} using the explicit mass equation,

➠ Calcul. of p^{n+1} and $q^{n+1} = (\rho\, u)^{n+1}$, with (1) and the moment. eq.

$$\text{Picard algorithm} \Rightarrow \begin{cases} \text{Important cost} \\ \text{Non symmetric matrix, convergence pb.} \end{cases}$$

➠ Calcul. of E^{n+1} with the energy eq. (otherwise non entropic).

Algorithme [Allegrini, MHV, SISC 2024]

- Calculation of ρ^{n+1} using the explicit mass eq.,
- Calculation of p^{n+1} with

$$\frac{p^{n+1}}{\gamma-1} - \Delta t^2 \nabla \cdot \left(\frac{1}{\varepsilon} \nabla p^{n+1} \right) = E^n - \Delta t \nabla \cdot (k_\varepsilon^n u^n) - k_\varepsilon^{n+1,exp}$$
$$-\Delta t \nabla \cdot \left(\frac{h^{n+1,exp}}{\rho^{n+1}} (q^n - \Delta t \nabla \cdot (\rho^n u^n \otimes u^n)) \right),$$

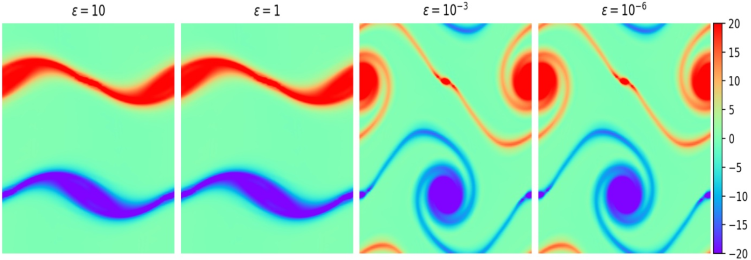
- Calculation of $q^{n+1} = (\rho u)^{n+1}$ using the momentum eq.
- Calculation of E^{n+1} with the energy eq. (otherwise non entropic).

Lemma : [Allegrini, MHV, 2024]

Asymptotically consistent semi-discretization

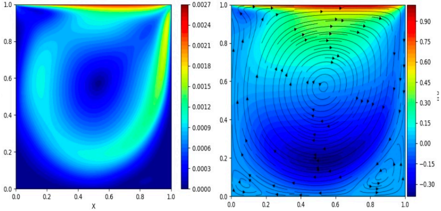
Linearly stable scheme under a uniform CFL : $|u| \Delta t \leq \Delta x$

- **Double shear layer test case**



Convergence towards the incompressible solution

- **Extension to Navier-Stokes eqs.** [Allegrini,MHV, submitted]



Lid-driven cavity incompressible flow

[A. Blaustein, G. Dimarco, F. Filbet,MHV, submitted]

Objectives : To design and analyze a scheme that :

✓ preserves the limit $\lambda \to 0$, i.e. is asymptotic-preserving

✓ does not project the solution onto the quasi-neutral limit

✓ provides a better understanding of unstable cases

Difficulties :

- **Nonlinear problem** due to the coupling in the Vlasov–Poisson system
- Need for sufficient accuracy to capture the **oscillations of period λ**

IMEX (Implicit-Explicit) schemes

$$\frac{f^{\lambda,n+1} - f^{\lambda,n}}{\Delta t} + v \cdot \partial_x f^{\lambda,n+1} - \partial_x \varphi^{\lambda,n+1} \partial_v f^{\lambda,n} = 0,$$

$$-\lambda^2 \partial^2_{xx} \varphi^{\lambda,n+1} = \rho^{\lambda,n+1} - 1.$$

Context : Spectral methods (Hermite basis Ψ_k)

$$f^\lambda(t,x,v) = \sum_{k\in\mathbb{N}} C_k^\lambda(t,x)\,\Psi_k(v) \quad \Rightarrow \quad (C_k^\lambda)_{k\in\mathbb{N}}\text{linear hyperbolic system}$$

$$\rho^\lambda = C_0^\lambda, \quad j^\lambda = C_1^\lambda, \quad K^\lambda = \sqrt{2}\,C_2^\lambda + C_0^\lambda, \ldots$$

Continuous system : linear hyperbolic syst. coupled with Poisson eq.

$$\begin{cases} \partial_t C_k^\lambda + \sqrt{k}\,\partial_x C_{k-1}^\lambda + \sqrt{k+1}\,\partial_x C_{k+1}^\lambda - \sqrt{k}\,E^\lambda C_{k-1}^\lambda = 0, \quad \forall k \in \mathbb{N}, \\ E^\lambda = -\partial_x \phi^\lambda, \qquad -\lambda^2 \partial_{xx}\phi^\lambda = C_0^\lambda - 1. \end{cases}$$

- Explicit determination of the oscillatory parts of $C_1^\lambda = j^\lambda$ and E^λ.

$$\begin{cases} E_{\rm osc}^\lambda(t,x) & := \cos\left(\frac{t}{\lambda}\right)\left(E^\lambda - \partial_x K^\lambda(t)\right)(0,x) \,-\, \frac{1}{\lambda}\sin\left(\frac{t}{\lambda}\right) C_1^\lambda(0,x)\,, \\ C_{1,\rm osc}^\lambda(t,x) & := \cos\left(\frac{t}{\lambda}\right) C_1^\lambda(0,x) \,+\, \lambda \sin\left(\frac{t}{\lambda}\right)\left(E^\lambda - \partial_x K^\lambda(t)\right)(0,x)\,. \end{cases}$$

- Continuous and discrete "error estimates"

Proposition (continuous)

If $\int (C_0^\lambda(0,x) - 1)\,dx = 0$ and $\int C_1^\lambda(0,x)\,dx = 0$ and if $\exists\, \alpha \in [0,1[$ s.t.

$$|C_1^\lambda(t=0)| \lesssim \lambda^{1-\alpha} \quad \text{and} \quad |E^\lambda(t)| \lesssim \frac{1}{\lambda^\alpha}$$

then

$$\mathscr{E}_0^\lambda = \|E^\lambda(t) - E_{osc}^\lambda(t) - \partial_x K^\lambda(t)\|_{L^2} \leq D\,\lambda^{1-\alpha},$$

$$\text{and} \qquad \mathscr{E}_1^\lambda = \|C_1^\lambda - C_{1,osc}^\lambda - 0\|_{L^2} \leq D\,\lambda^{2-\lambda}.$$

Continuous system : linear hyperbolic syst. coupled with Poisson eq.

$$\begin{cases} \partial_t C_k^\lambda + \sqrt{k}\,\partial_x C_{k-1}^\lambda + \sqrt{k+1}\,\partial_x C_{k+1}^\lambda - \sqrt{k}\,E^\lambda C_{k-1}^\lambda = 0, \quad \forall k \in \mathbb{N}, \\ E^\lambda = -\partial_x \phi^\lambda, \qquad -\lambda^2 \partial_{xx}\phi^\lambda = C_0^\lambda - 1. \end{cases}$$

Discretization :

- In velocity : fix a number of Hermite modes $N_H > 0$,

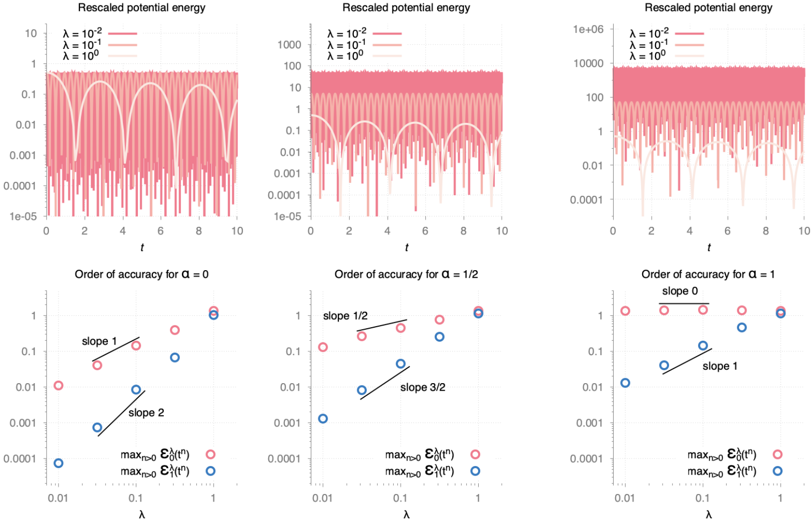
then $k = 0, \cdots, N_H$ with $C_{N_H+1}^\lambda = 0$

- In time : fully implicit order 2 scheme & splitting

$$\begin{cases} \partial_t C_0^\lambda + \partial_x C_1^\lambda = 0, \\ \partial_t C_1^\lambda + \partial_x C_0^\lambda + \sqrt{2}\,\partial_x C_2^\lambda - 1\,C_0^\lambda = 0, \\ \partial_t C_k^\lambda + \sqrt{k}\,\partial_x C_{k-1}^\lambda + \sqrt{k+1}\,\partial_x C_{k+1}^\lambda = 0, \\ E^\lambda = -\partial_x \phi^\lambda, \qquad -\lambda^2 \partial_{xx}\phi^\lambda = C_0^\lambda - 1. \end{cases} \qquad \begin{cases} \partial_t C_1^\lambda - (E^\lambda - 1)\,C_0^\lambda = 0, \\ \partial_t C_k^\lambda - \sqrt{k}\,E^\lambda C_{k-1}^\lambda = 0, \end{cases}$$

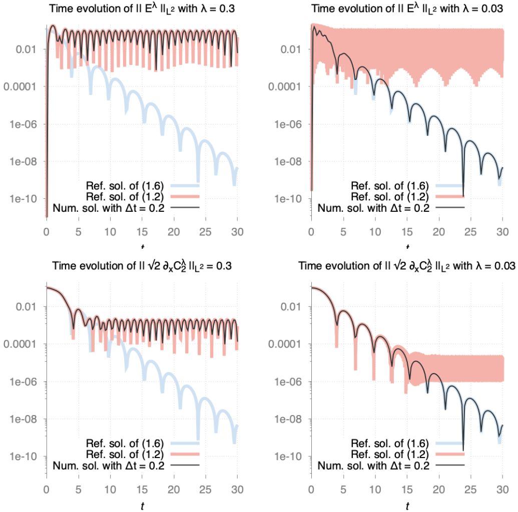
- In space : centered approximation (order 2) on a uniform grid.

Stable case : small $(0.1\,\lambda^{2-\alpha})$ perturbation on ρ of a Maxwellian distribution



$$\mathscr{E}_0^\lambda \lesssim \lambda^{1-\alpha} \quad \text{and} \quad \mathscr{E}_1^\lambda \lesssim \lambda^{2-\lambda}.$$

Intermediate case : perturbation on T of a Maxwellian distribution



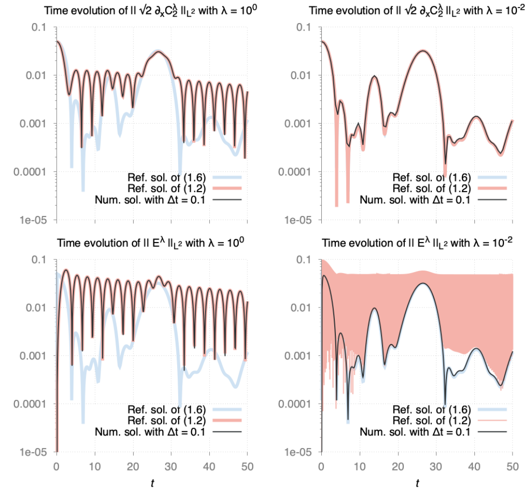
$$f_{\mathrm{in}}(x,v) = \frac{1}{\sqrt{2\pi T_{\mathrm{in}}}} \exp\left(-\frac{|v|^2}{2T_{\mathrm{in}}}\right)$$

$$T_{\mathrm{in}}(x) = 1 + 0.1\cos\left(\frac{\pi}{10}x\right)$$

(1.2)=Vlasov-Poisson

(1.6)=QN model

Intermediate case : perturbation on ρ and T of a Maxwellian distribution



$$f_{\text{in}}(x,v) = \frac{1 + 0.1\cos\left(\frac{\pi}{10}x\right)\sin(3\pi v)}{\sqrt{2\pi T_{\text{in}}}} \exp\left(-\frac{|v|^2}{2T_{\text{in}}}\right)$$

$$T_{\text{in}}(x) = 1 + 0.1\,\cos\left(\frac{\pi}{10}x\right)$$

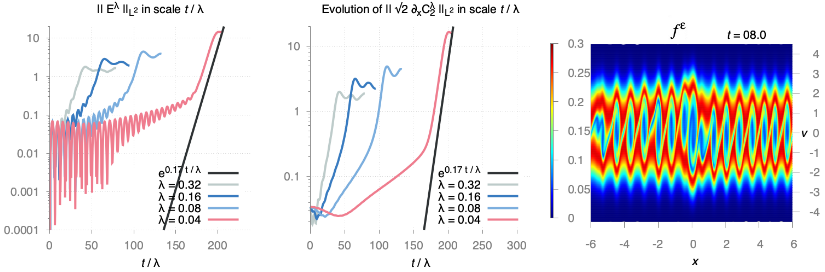
(1.2)=Vlasov-Poisson

(1.6)=QN model

Unstable case, two stream instability

$$f_{\text{in}}(x,v) = \frac{1}{6\sqrt{2\pi T_{\text{in}}}}\left(1+\frac{5\,|v|^2}{T_{\text{in}}}\right)\exp\left(-\frac{|v|^2}{2T_{\text{in}}}\right),$$

$$T_{\text{in}}(x) = 1+0.01\cos\left(k_x x\right).$$



Works in progress and perspectives

- **Low Mach number limit**
  - ✓ Domain decomposition : Use the AP scheme only in intermediate zones
  - ✓ Other models : Tabulated pressure, Baer-Nunziato, MHD
  - ✓ Numerical analysis :
    - Asymptotic properties, ex. :J.C. Latché, R. Herbin, K. Saleh (Marseille)
    - Asymptotically accurate schemes, ex. : J. Jung, V. Perrier (Pau)
    - Entropic stability, ex. : C. Berthon (Nantes), L. Martaud (Rennes)
    - Discrete spectral analysis : T. Crin-Barat (Toulouse)
- **Quasi-neutral (QN) limit**
  - ✓ More test cases
  - ✓ Multi-D, other schemes
- **Boundary and initial cond. :** Low Mach number and QN limits
- **Preservation of several asymptotic regimes :**

  Low Mach number & QN, QN & gyro...