Schémas décalés sur grilles non structurées Application aux écoulements multifluides incompressibles

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Joined work with C. Ghosn, T. Goudon and J. Llobell

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Hommage à Jean-Claude Latché,

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Projection method

- Estimate of the splitting error for non conformal finite element
- Algebraic formulation of the pressure elliptic operator
  - Finite volume scheme
  - Relaxation of Dirichlet boundary conditions on the pressure for outflow boundary conditions

[Dardhalon, Latché, Minjeaud, 2013]

Staggered scheme for Euler system

- Discretization of a non-conservative version + corrective terms

[Herbin, Kheriji, Latché, 2013]

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho \mathbf{u}\right) = 0, \\ \partial_t\left(\rho \mathbf{u}\right) + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u}\right) + \nabla p = 0, \\ \partial_t(\rho e) + \operatorname{div}\left(\rho e \mathbf{u}\right) = -p \operatorname{div} \mathbf{u}. \end{cases} \qquad p = (\gamma - 1)\rho e, \quad \gamma > 1$$

- Transfert of discrete fields and operators between the different grids

$\rightsquigarrow$ *ad hoc* averaged quantities and operators

[Goudon, Krell, Llobell, Minjeaud, 2021]

- Incompressible "multifluid" flows (a dispersed phase in a fluid phase)

$$\begin{cases} \partial_t \alpha_p + \mathrm{div}\,(\alpha_p \mathbf{u}_p) = 0, \\ \partial_t(\alpha_p \mathbf{u}_p) + \mathrm{div}\,\big(\alpha_p \mathbf{u}_p \otimes \mathbf{u}_p + \pi(\alpha_p)\big) = D_p \alpha_p (\mathbf{u}_f - \mathbf{u}_p) - \dfrac{\alpha_p}{m_p} \nabla p + \alpha_p \mathbf{g}, \end{cases}$$

$$\begin{cases} \partial_t \alpha_f + \mathrm{div}\,(\alpha_f \mathbf{u}_f) = 0, \\ \partial_t(\alpha_f \mathbf{u}_f) + \mathrm{div}\,(\alpha_f \mathbf{u}_f \otimes \mathbf{u}_f) - \dfrac{1}{m_f} \mathrm{div}\,(\alpha_f \boldsymbol{\tau}_f) = \dfrac{m_p}{m_f} D_p \alpha_p (\mathbf{u}_p - \mathbf{u}_f) - \dfrac{\alpha_f}{m_f} \nabla p + \alpha_f \mathbf{g}, \end{cases}$$

$$\alpha_f + \alpha_p = 1 \qquad \Longleftrightarrow \qquad \boxed{\mathrm{div}\,\big(\alpha_p \mathbf{u}_p + \alpha_f \mathbf{u}_f\big) = 0}\,.$$

[Berthelin, Goudon, Minjeaud, 2016] [Ghosn, Goudon, Minjeaud, soumis]

- close packing term: $\pi(\alpha_p) = c\,\dfrac{\alpha_p^\beta}{\alpha_p^* - \alpha_p}$, $c > 0$, $\beta > 1$.
- Viscous tensor: $\boldsymbol{\tau}_f = \mu_f \Big(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^\intercal - \dfrac{2}{3} (\mathrm{div}\,\mathbf{u}_f) \mathrm{Id} \Big)$.
- Fluid-particle interaction modeled by a drag force $\propto (\mathbf{u}_f - \mathbf{u}_p)$

Assume $(\alpha_p, \alpha_f, \mathbf{u}_p, \mathbf{u}_f, p)$ given (by previous time step) s.t. :

$$\operatorname{div}\left(\alpha_p \mathbf{u}_p\right) + \operatorname{div}\left(\alpha_f \mathbf{u}_f\right) = 0, \qquad \alpha_p + \alpha_f = 1.$$

Explicit computation of volume fractions $\overline{\alpha}_p$ and $\overline{\alpha}_f$

$$\frac{\overline{\alpha}_p - \alpha_p}{\Delta t} + \operatorname{div}\left(\alpha_p \mathbf{u}_p\right) = 0, \qquad \frac{\overline{\alpha}_f - \alpha_f}{\Delta t} + \operatorname{div}\left(\alpha_f \mathbf{u}_f\right) = 0.$$

$$\Longrightarrow \overline{\alpha}_p + \overline{\alpha}_f = 1.$$

Prediction of velocities $\widetilde{\mathbf{u}}_p$ and $\widetilde{\mathbf{u}}_f$

$$\frac{\overline{\alpha}_p \widetilde{\mathbf{u}}_p - \alpha_p \mathbf{u}_p}{\Delta t} + \operatorname{div}\left(\alpha_p \mathbf{u}_p \otimes \mathbf{u}_p + \pi(\alpha_p)\right) = D\overline{\alpha}_p(\widetilde{\mathbf{u}}_f - \widetilde{\mathbf{u}}_p) - \frac{\overline{\alpha}_p}{m_p}\nabla p + \overline{\alpha}_p \mathbf{g},$$

$$\frac{\overline{\alpha}_f \widetilde{\mathbf{u}}_f - \alpha_f \mathbf{u}_f}{\Delta t} + \operatorname{div}\left(\alpha_f \mathbf{u}_f \otimes \mathbf{u}_f\right) - \frac{1}{m_f}\operatorname{div}\left(\overline{\alpha}_f \widetilde{\mathbf{u}}_f\right) = \frac{m_p}{m_f} D\overline{\alpha}_p(\widetilde{\mathbf{u}}_p - \widetilde{\mathbf{u}}_f) - \frac{\overline{\alpha}_f}{m_f}\nabla p + \overline{\alpha}_f \mathbf{g},$$

- No divergence free constraint
- Explicit pressure p

Correction of velocities $\overline{\mathbf{u}}_p$, $\overline{\mathbf{u}}_f$; Pressure computation $\overline{p}$

$$\frac{\overline{\mathbf{u}}_p - \widetilde{\mathbf{u}}_p}{\Delta t} = -\frac{1}{m_p}\nabla(\overline{p} - p), \qquad \frac{\overline{\mathbf{u}}_f - \widetilde{\mathbf{u}}_f}{\Delta t} = -\frac{1}{m_f}\nabla(\overline{p} - p),$$

$$\mathrm{div}\left(\overline{\alpha}_p \overline{\mathbf{u}}_p\right) + \mathrm{div}\left(\overline{\alpha}_f \overline{\mathbf{u}}_f\right) = 0$$

OR EQUIVALENTLY

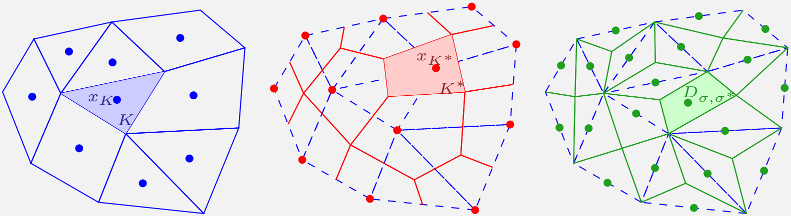
Pressure computation $\overline{p}$

$$\mathrm{div}\left(\overline{\alpha}_p\left(\widetilde{\mathbf{u}}_p - \frac{\Delta t}{m_p}\nabla(\overline{p} - p)\right)\right) + \mathrm{div}\left(\overline{\alpha}_f\left(\widetilde{\mathbf{u}}_f - \frac{\Delta t}{m_f}\nabla(\overline{p} - p)\right)\right) = 0$$

Correction of velocities $\overline{\mathbf{u}}_p$, $\overline{\mathbf{u}}_f$

$$\overline{\mathbf{u}}_p = \widetilde{\mathbf{u}}_p - \frac{\Delta t}{m_p}\nabla(\overline{p} - p), \qquad \overline{\mathbf{u}}_f = \widetilde{\mathbf{u}}_f - \frac{\Delta t}{m_f}\nabla(\overline{p} - p)$$

▶ The discrete unknowns are constant on different meshes

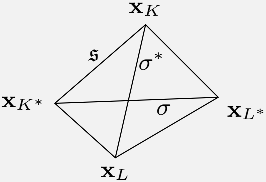


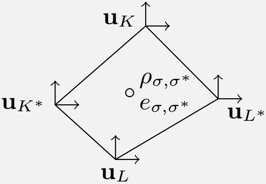
Primal Mesh Dual mesh Diamond mesh.

- Density ρ_{σ,σ^*} and internal energy e_{σ,σ^*} on the diamond cell D_{σ,σ^*}

$$p_{\sigma,\sigma^*} = (\gamma - 1)\rho_{\sigma,\sigma^*} e_{\sigma,\sigma^*}.$$

- Velocity fields $(\mathbf{u}_K, \mathbf{u}_{K^*})$ on the primal cell K and on the dual cell K^*.



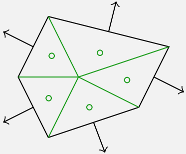


Discrete operators

- discrete pressure gradient (on primal and dual cells)

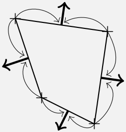
$$(\boldsymbol{\nabla} p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma},$$

$$(\boldsymbol{\nabla} p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| p_{\sigma,\sigma^*} \mathbf{n}_{K^*,\sigma^*},$$



- discrete divergence operator (on diamond cells)

$$(\boldsymbol{\nabla} \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| u_{D_{\sigma,\mathfrak{s}}},$$



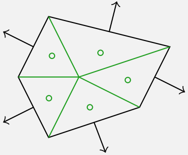
with $u_{D_{\sigma,\mathfrak{s}}} := \frac{\mathbf{u}_K + \mathbf{u}_{K^*}}{2} \cdot \mathbf{n}_{D_{\sigma,\mathfrak{s}}}.$

Discrete operators

- discrete pressure gradient (on primal and dual cells)

$$(\nabla p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma},$$

$$(\nabla p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| p_{\sigma,\sigma^*} \mathbf{n}_{K^*,\sigma^*},$$



- discrete divergence operator (on diamond cells)

$$(\nabla \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{2|D_{\sigma,\sigma^*}|} \Big(|\sigma| \big(\mathbf{u}_L - \mathbf{u}_K\big) \cdot \mathbf{n}_{K,\sigma} + |\sigma^*| \big(\mathbf{u}_{L^*} - \mathbf{u}_{K^*}\big) \cdot \mathbf{n}_{K^*,\sigma^*} \Big),$$

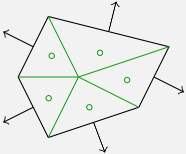
[Domelevo, Omnes 2005]

DISCRETE OPERATORS

- discrete pressure gradient (on primal and dual cells)

$$(\nabla p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma},$$

$$(\nabla p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| p_{\sigma,\sigma^*} \mathbf{n}_{K^*,\sigma^*},$$



- discrete divergence operator (on diamond cells)

$$(\nabla \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{2|D_{\sigma,\sigma^*}|} \Big(|\sigma| \big(\mathbf{u}_L - \mathbf{u}_K\big) \cdot \mathbf{n}_{K,\sigma} + |\sigma^*| \big(\mathbf{u}_{L^*} - \mathbf{u}_{K^*}\big) \cdot \mathbf{n}_{K^*,\sigma^*} \Big),$$

[Domelevo, Omnes 2005]

LOCAL DUALITY RELATIONSHIP

There exists conservative fluxes $q_{D_\sigma,\mathfrak{s}}$ for all D_σ, for all $\mathfrak{s}$, $(\sigma = K|L)$,

$$|K \cap D_\sigma| \mathbf{u}_K \cdot (\nabla p)_K + |L \cap D_\sigma| \mathbf{u}_L \cdot (\nabla p)_L + p_{\sigma,\sigma^*} |\sigma| \big(\mathbf{u}_L - \mathbf{u}_K\big) \cdot \mathbf{n}_{K,\sigma} = \sum_{\mathfrak{s} \in \partial D_\sigma} |\mathfrak{s}| q_{D_\sigma,\mathfrak{s}}$$

[Goudon, Krell, Llobell, Minjeaud, 2020]

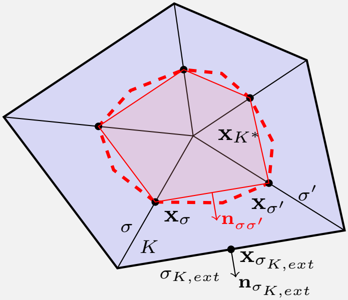
- If $\mathbf{x}_\sigma$ is the **middle point** of σ, for an **affine** function $\bar{p}$,

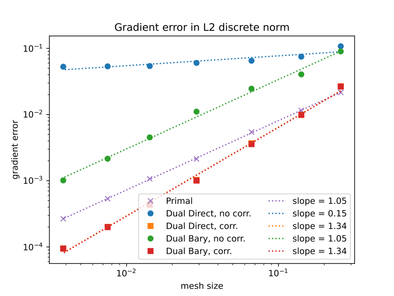
$$\frac{1}{|K|} \sum_{D_\sigma \in \mathfrak{D}_K} |\sigma| \bar{p}(\mathbf{x}_\sigma) \mathbf{n}_{K,\sigma} = \nabla \bar{p}$$

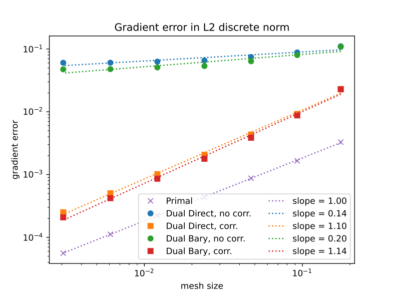
[Eymard, Gallouet, Herbin 2006]

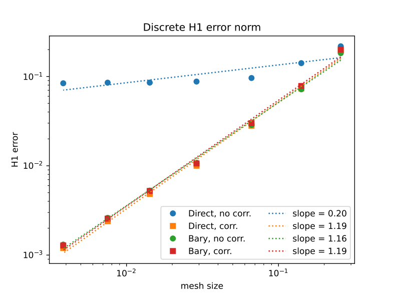
- Weak consistency of $\nabla\left(\mathcal{P}_{\mathrm{X}_{\mathrm{diag}}} p\right)$

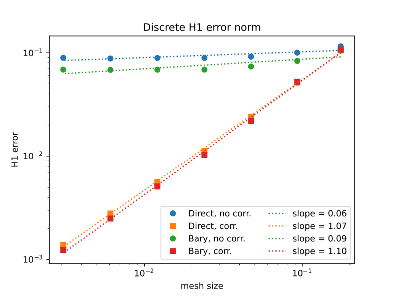
- Strong consistency of $\nabla\left(\mathcal{P}_{\mathrm{X}_\sigma} p\right)$ only
  - on triangle meshes
  - using the barycentric dual mesh
  - if x_K is the center of mass of K




Gradient error in L2 discrete norm
gradient error
mesh size
Primal
Dual Direct, no corr.
Dual Direct, corr.
Dual Bary, no corr.
Dual Bary, corr.
slope = 1.05
slope = 0.15
slope = 1.34
slope = 1.05
slope = 1.34


Gradient error in L2 discrete norm
gradient error
mesh size
Primal
Dual Direct, no corr.
Dual Direct, corr.
Dual Bary, no corr.
Dual Bary, corr.
slope = 1.00
slope = 0.14
slope = 1.10
slope = 0.20
slope = 1.14


Discrete H1 error norm
H1 error
mesh size
Direct, no corr.
Direct, corr.
Bary, no corr.
Bary, corr.
slope = 0.20
slope = 1.19
slope = 1.16
slope = 1.19


Discrete H1 error norm
H1 error
mesh size
Direct, no corr.
Direct, corr.
Bary, no corr.
Bary, corr.
slope = 0.06
slope = 1.07
slope = 0.09
slope = 1.10

▶ The particle volume fractions are updated as follows

$$\frac{\overline{\alpha}_{p,\sigma,\sigma^*} - \alpha_{p,\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{F}^p_{\sigma,\mathfrak{s}} = 0,$$

where the volume fluxes $\mathcal{F}^p_{\sigma,\mathfrak{s}} = \mathcal{F}^{p,+}_{\sigma,\mathfrak{s}} + \mathcal{F}^{p,-}_{\sigma,\mathfrak{s}}$ $(\sim \alpha_p \mathbf{u}_p \cdot \mathbf{n})$ with

$$\mathcal{F}^{p,+}_{\sigma,\mathfrak{s}} = \mathcal{F}^{p,+}\big(\alpha_{p,\sigma,\sigma^*}, u_{p,\sigma,\mathfrak{s}}\big) \quad \text{and} \quad \mathcal{F}^{p,-}_{\sigma,\mathfrak{s}} = \mathcal{F}^{p,-}\big(\alpha_{p,\sigma',\sigma^{*\prime}}, u_{p,\sigma,\mathfrak{s}}\big).$$

▶ Normal velocity: $u_{p,\sigma,\mathfrak{s}} = \dfrac{1}{2}\Big(\mathbf{u}_{p,K} + \mathbf{u}_{p,K^*}\Big) \cdot \mathbf{n}_{\sigma,\mathfrak{s}}$ with $\mathfrak{s} = [\mathbf{x}_K, \mathbf{x}_{K^*}]$.

▶ “Sound” speed $c(\alpha_p) = \sqrt{\pi'(\alpha_p)}$ linked to close packing term π.

▶ Flux splitting functions $\mathcal{F}^{p,+}$ and $\mathcal{F}^{p,-}$ inspired from the kinetic framework

$$\mathcal{F}^{p,+}(\alpha, u) = \frac{\alpha}{2c} \int_{\xi>0} \xi \mathbb{1}_{|\xi-u|\leqslant c}(\xi) d\xi$$

- The particle volume fractions are updated as follows

$$\frac{\overline{\alpha}_{p,\sigma,\sigma^*} - \alpha_{p,\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{F}^{p}_{\sigma,\mathfrak{s}} = 0,$$

where the volume fluxes $\mathcal{F}^{p}_{\sigma,\mathfrak{s}} = \mathcal{F}^{p,+}_{\sigma,\mathfrak{s}} + \mathcal{F}^{p,-}_{\sigma,\mathfrak{s}}$ $(\sim \alpha_p \mathbf{u}_p \cdot \mathbf{n})$ with

$$\mathcal{F}^{p,+}_{\sigma,\mathfrak{s}} = \mathcal{F}^{p,+}\big(\alpha_{p,\sigma,\sigma^*}, u_{p,\sigma,\mathfrak{s}}\big) \quad \text{and} \quad \mathcal{F}^{p,-}_{\sigma,\mathfrak{s}} = \mathcal{F}^{p,-}\big(\alpha_{p,\sigma',\sigma^{*\prime}}, u_{p,\sigma,\mathfrak{s}}\big).$$

- Normal velocity: $u_{p,\sigma,\mathfrak{s}} = \dfrac{1}{2}\Big(\mathbf{u}_{p,K} + \mathbf{u}_{p,K^*}\Big) \cdot \mathbf{n}_{\sigma,\mathfrak{s}}$ with $\mathfrak{s} = [\mathbf{x}_K, \mathbf{x}_{K^*}]$.

- "Sound" speed $c(\alpha_p) = \sqrt{\pi'(\alpha_p)}$ linked to close packing term π.

- Flux splitting functions $\mathcal{F}^{p,+}$ and $\mathcal{F}^{p,-}$ inspired from the kinetic framework

$$\mathcal{F}^{p,+}(\alpha, u) = \begin{cases} 0 & \text{if } u \leqslant -c, \\ \dfrac{\alpha}{4c}\big(u+c\big)^2 & \text{if } |u| \leqslant c, \\ \rho u & \text{if } u \geqslant c, \end{cases} \quad \text{and } \mathcal{F}^{p,-}(\rho, u) = -\mathcal{F}^{p,+}(\rho, -u).$$

- The fluid volume fractions are updated as follows

$$\frac{\overline{\alpha}_{f,\sigma,\sigma^*} - \alpha_{f,\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{F}^f_{\sigma,\mathfrak{s}} = 0,$$

where the volume fluxes $\mathcal{F}^f_{\sigma,\mathfrak{s}} = \mathcal{F}^{f,+}_{\sigma,\mathfrak{s}} + \mathcal{F}^{f,-}_{\sigma,\mathfrak{s}}$ $(\sim \alpha_f \mathbf{u}_f \cdot \mathbf{n})$ with

$$\mathcal{F}^{f,+}_{\sigma,\mathfrak{s}} = \alpha_{f,\sigma,\sigma^*} \left[u_{f,\sigma,\mathfrak{s}}\right]^+ \quad \text{and} \quad \mathcal{F}^{f,-}_{\sigma,\mathfrak{s}} = -\alpha_{f,\sigma',\sigma^{*\prime}} \left[u_{f,\sigma,\mathfrak{s}}\right]^-.$$

- Normal velocity: $u_{f,\sigma,\mathfrak{s}} = \dfrac{1}{2}\left(\mathbf{u}_{f,K} + \mathbf{u}_{f,K^*}\right) \cdot \mathbf{n}_{\sigma,\mathfrak{s}}$ with $\mathfrak{s} = [\mathbf{x}_K, \mathbf{x}_{K^*}]$.

▶ Transfer of volume balance equations on primal and dual cells [Ansanay, Babik, Latché, Vola, '11][Goudon, Krell, Llobell, Minjeaud, '20]

- Average volume fractions on a primal cell K

$$\alpha_K = \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} \frac{|D_{\sigma,\sigma^*} \cap K|}{|K|} \alpha_{\sigma,\sigma^*},$$

- Average volume fluxes $\mathcal{F}_{K,\sigma}$ outgoing from a primal cell K

$$\mathcal{F}_{K,\sigma} = \mathcal{F}^+_{K,\sigma} + \mathcal{F}^-_{K,\sigma},$$

with

$$\mathcal{F}^\pm_{K,\sigma} = \frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^\pm_{D_{\sigma,\mathfrak{s}}} - \frac{|D_{\sigma,\sigma^*} \cap L|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^\mp_{D_{\sigma,\mathfrak{s}}}.$$

[Goudon, Krell, 2014]

- The fluxes $\mathcal{F}_{K,\sigma}$ are conservative.
- The average densites α_K satisfy the following conservative equations

$$\frac{\overline{\alpha}_K - \alpha_K}{\delta t} + \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| \mathcal{F}_{K,\sigma} = 0.$$

▶ The velocities are updated as follows

$$\frac{\overline{\alpha}_{p,K}\widetilde{\mathbf{u}}_{p,K} - \alpha_{p,K}\mathbf{u}_{p,K}}{\delta t} + \frac{1}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K}|\sigma|\mathcal{G}^p_{K,\sigma}$$
$$+ D_p\left(\overline{\alpha}_{p,K}\widetilde{\mathbf{u}}_{p,K} - \overline{\alpha}_{f,K}\widetilde{\mathbf{u}}_{f,K}\right) = -\frac{\alpha_{p,K}}{m_p}(\nabla\Phi)_K + \overline{\alpha}_{p,K}\mathbf{g}.$$

$$\frac{\overline{\alpha}_{p,K^*}\widetilde{\mathbf{u}}_{p,K^*} - \alpha_{p,K^*}\mathbf{u}_{p,K^*}}{\delta t} + \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma^*|\mathcal{G}^p_{K^*,\sigma^*}$$
$$+ D_p(\overline{\alpha}_{p,K^*}\widetilde{\mathbf{u}}_{p,K} - \overline{\alpha}_{f,K^*}\widetilde{\mathbf{u}}_{f,K^*}) = -\frac{\alpha_{p,K^*}}{m_p}(\nabla\Phi)_{K^*} + \overline{\alpha}_{p,K^*}\mathbf{g}$$

where the momentum fluxes $\mathcal{G}^p_{K,\sigma}$ and $\mathcal{G}^p_{K^*,\sigma^*}$ ($\sim \alpha_p(\mathbf{u}_p\cdot\mathbf{n})\mathbf{u}_p$) are defined by

$$\mathcal{G}^p_{K,\sigma} = \mathcal{F}^{p,+}_{K,\sigma}\mathbf{u}_{p,K} + \mathcal{F}^{p,-}_{K,\sigma}\mathbf{u}_{p,L} \quad \text{and} \quad \mathcal{G}^p_{K^*,\sigma^*} = \mathcal{F}^{p,+}_{K^*,\sigma^*}\mathbf{u}_{p,K^*} + \mathcal{F}^{p,-}_{K^*,\sigma^*}\mathbf{u}_{p,L^*}.$$

- The velocities are updated as follows

$$\frac{\overline{\alpha}_{f,K}\widetilde{\mathbf{u}}_{f,K}-\alpha_{f,K}\mathbf{u}_{f,K}}{\delta t}+\frac{1}{|K|}\sum_{D_\sigma\in\mathfrak{D}_K}|\sigma|\mathcal{G}^f_{K,\sigma}-\frac{\overline{\alpha}_{f,K}}{m_f}\frac{1}{|K|}\sum_{D_\sigma\in\mathfrak{D}_K}|\sigma|(\widetilde{\boldsymbol{\tau}}_f)_{\sigma,\sigma^*}\mathbf{n}_{K,\sigma}$$
$$+\frac{m_p}{m_f}D_p\Big(\overline{\alpha}_{f,K}\widetilde{\mathbf{u}}_{f,K}-\overline{\alpha}_{p,K}\widetilde{\mathbf{u}}_{p,K}\Big)=-\frac{\alpha_{f,K}}{m_f}(\nabla\Phi)_K+\overline{\alpha}_{f,K}\mathbf{g}.$$
$$\frac{\overline{\alpha}_{f,K^*}\widetilde{\mathbf{u}}_{f,K^*}-\alpha_{f,K^*}\mathbf{u}_{f,K^*}}{\delta t}+\frac{1}{|K^*|}\sum_{D_{\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma^*|\mathcal{G}^f_{K^*,\sigma^*}-\frac{\overline{\alpha}_{f,K^*}}{m_f}\frac{1}{|K^*|}\sum_{D_{\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma|(\widetilde{\boldsymbol{\tau}}_f)_{\sigma,\sigma^*}\mathbf{n}_{K^*,\sigma^*}$$
$$+\frac{m_p}{m_f}D_p(\overline{\alpha}_{f,K^*}\widetilde{\mathbf{u}}_{f,K}-\overline{\alpha}_{p,K^*}\widetilde{\mathbf{u}}_{p,K^*})=-\frac{\alpha_{f,K^*}}{m_f}(\nabla\Phi)_{K^*}+\overline{\alpha}_{f,K^*}\mathbf{g}.$$

- The fluxes $\mathcal{G}^f_{K,\sigma}$ and $\mathcal{G}^f_{K^*,\sigma^*}$ $(\sim\alpha_f(\mathbf{u}_f\cdot\mathbf{n})\mathbf{u}_f)$ are defined by

$$\mathcal{G}^f_{K,\sigma}=\mathcal{F}^{f,+}_{K,\sigma}\mathbf{u}_{f,K}+\mathcal{F}^{f,-}_{K,\sigma}\mathbf{u}_{f,L}\quad\text{and}\quad\mathcal{G}^f_{K^*,\sigma^*}=\mathcal{F}^{f,+}_{K^*,\sigma^*}\mathbf{u}_{f,K^*}+\mathcal{F}^{f,-}_{K^*,\sigma^*}\mathbf{u}_{f,L^*}.$$

- The viscous tensor is defined by

$$(\boldsymbol{\tau}_f)_{\sigma,\sigma^*}=\mu_f\Big((\nabla\mathbf{u}_f)_{\sigma,\sigma^*}+(\nabla\mathbf{u}_f)^{\mathsf{T}}_{\sigma,\sigma^*}-\frac{2}{3}\mathrm{Tr}(\nabla\mathbf{u}_f)_{\sigma,\sigma^*}\mathrm{Id}\Big),$$

thanks to velocity gradient

$$(\nabla\mathbf{u}_f)_{\sigma,\sigma^*}=\frac{1}{2|D_{\sigma,\sigma^*}|}\Big(|\sigma|\big(\mathbf{u}_{f,L}-\mathbf{u}_{f,K}\big)\otimes\mathbf{n}_{K,\sigma}+|\sigma^*|\big(\mathbf{u}_{f,L^*}-\mathbf{u}_{f,K^*}\big)\otimes\mathbf{n}_{K^*,\sigma^*}\Big).$$

- The system for the velocities couples

$$\mathrm{U}_f = \left(\widetilde{\mathbf{u}}^{(1)}_{f,K},\, \widetilde{\mathbf{u}}^{(2)}_{f,K},\, \widetilde{\mathbf{u}}^{(1)}_{f,K^*},\, \widetilde{\mathbf{u}}^{(2)}_{f,K^*}\right) \text{ and } \mathrm{U}_p = \left(\widetilde{\mathbf{u}}^{(1)}_{p,K},\, \widetilde{\mathbf{u}}^{(2)}_{p,K},\, \widetilde{\mathbf{u}}^{(1)}_{p,K^*},\, \widetilde{\mathbf{u}}^{(2)}_{p,K^*}\right).$$

via drag forces and viscous term.

- Drag forces are diagonal !

$$\begin{pmatrix} \mathcal{M}_{f,f} & \mathcal{M}_{f,p} \\ \mathcal{M}_{p,f} & \mathcal{M}_{p,p} \end{pmatrix} \begin{pmatrix} \mathrm{U}_f \\ \mathrm{U}_p \end{pmatrix} = \begin{pmatrix} \mathrm{S}_f \\ \mathrm{S}_p \end{pmatrix}$$

with $\mathcal{M}_{f,p},\ \mathcal{M}_{p,f},\ \mathcal{M}_{p,p}$ diagonal matrices.

- We solve for the fluid velocity

$$\mathrm{U}_f = \left[\mathcal{M}_{f,f} - \mathcal{M}_{f,p}\mathcal{M}^{-1}_{p,p}\mathcal{M}_{p,f}\right]^{-1}\left(\mathrm{S}_f - \mathcal{M}_{f,p}\mathcal{M}^{-1}_{p,p}\mathrm{S}_p\right).$$

- And the particle velocity

$$\mathrm{U}_p = \mathcal{M}^{-1}_{p,p}\left(\mathrm{S}_p - \mathcal{M}_{p,f}\mathrm{U}_f\right).$$

Pressure increment $\Phi_{\sigma,\sigma^*} = p_{\sigma,\sigma^*} - p^*_{\sigma,\sigma^*}$

$$\mathrm{div}\left(\overline{\alpha}_p\left(\widetilde{\mathbf{u}}_p - \frac{\Delta t}{m_p}\nabla(\overline{p} - p)\right)\right) + \mathrm{div}\left(\overline{\alpha}_f\left(\widetilde{\mathbf{u}}_f - \frac{\Delta t}{m_f}\nabla(\overline{p} - p)\right)\right) = 0$$

- We denote

$$\left[\mathbb{D}^{(p)}(\alpha_p^k, \mathbf{u}_p^k)\right]_{\sigma,\sigma^*} = \frac{1}{|D_\sigma|}\sum_{\mathfrak{s}\in\partial D_\sigma}|\mathfrak{s}|\mathcal{F}^p_{\sigma,\mathfrak{s}}, \qquad \left[\mathbb{D}^{(f)}(\alpha_f^k, \mathbf{u}_f^k)\right]_{\sigma,\sigma^*} = \frac{1}{|D_\sigma|}\sum_{\mathfrak{s}\in\partial D_\sigma}|\mathfrak{s}|\mathcal{F}^f_{\sigma,\mathfrak{s}},$$

- Pressure increment computation

$$\mathbb{D}^{(p)}\left(\overline{\alpha}_p, \widetilde{\mathbf{u}}_p - \frac{\delta t}{m_p}\nabla\Phi\right) + \mathbb{D}^{(f)}\left(\overline{\alpha}_f, \widetilde{\mathbf{u}}_f - \frac{\delta t}{m_f}\nabla\Phi\right) = 0.$$

- Nonlinear problem, very few iterations of the nonlinear solver (2 or 3)

Pressure and velocity correction

$$\overline{p} = p + \Phi, \qquad \overline{\mathbf{u}}_p = \widetilde{\mathbf{u}}_p - \frac{\Delta t}{m_p}\nabla\Phi, \qquad \overline{\mathbf{u}}_f = \widetilde{\mathbf{u}}_f - \frac{\Delta t}{m_f}\nabla\Phi$$

- Conservation of positivity of volume fraction under CFL: conditions

$$\frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \left[\lambda_+(\alpha_{p,\sigma,\sigma^*}, u_{p,\sigma,\mathfrak{s}})\right]^+ \leqslant 1,$$

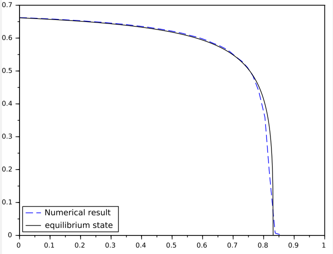
$$\frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \left[u_{f,\sigma,\mathfrak{s}}\right]^+ \leqslant 1.$$

- The close packing constraint, $\alpha_{p,\sigma,\sigma^*} < \alpha^*$, is preserved under CFL conditions

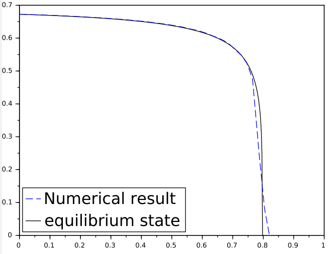
$$\frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \left[\lambda_-(\alpha_{p,\sigma,\sigma^*}, u_{p,\sigma,\mathfrak{s}})\right]^- \leqslant 1 - \frac{\alpha_{p,\sigma,\sigma^*}}{\alpha^*},$$

- At least one solution exists for the nonlinear pressure increment problem.

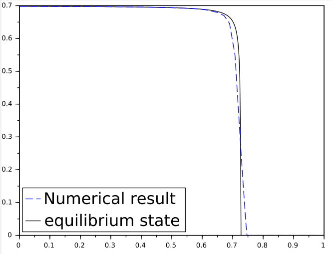
- Discrete energy estimates remain open due to numerical diffusion
(in the divergence-free constraint operator).



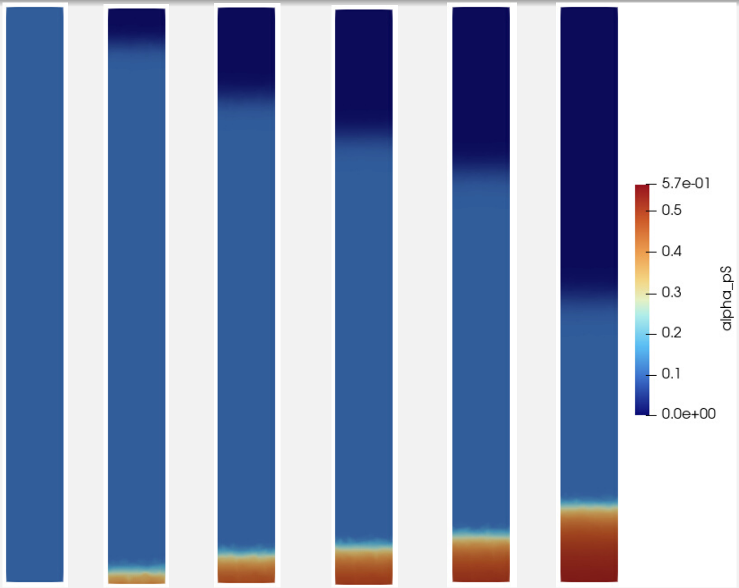
$c = 1$ and $\beta = 4$



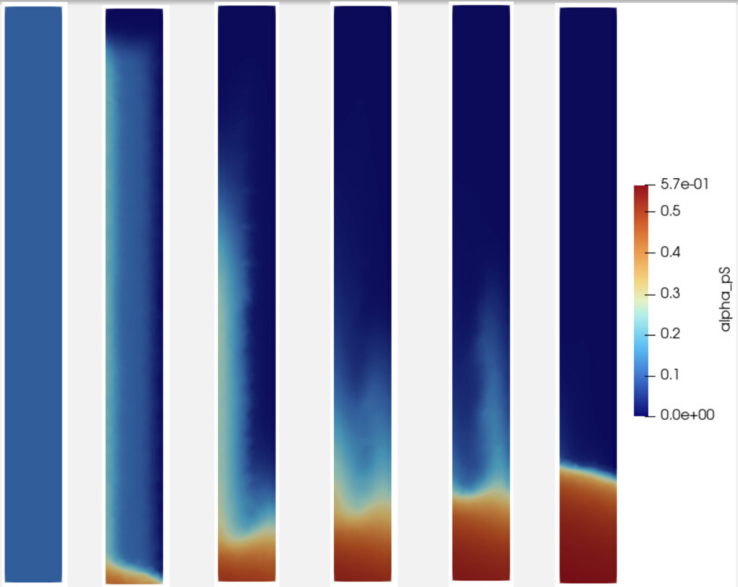
$c = 1$ and $\beta = 5$



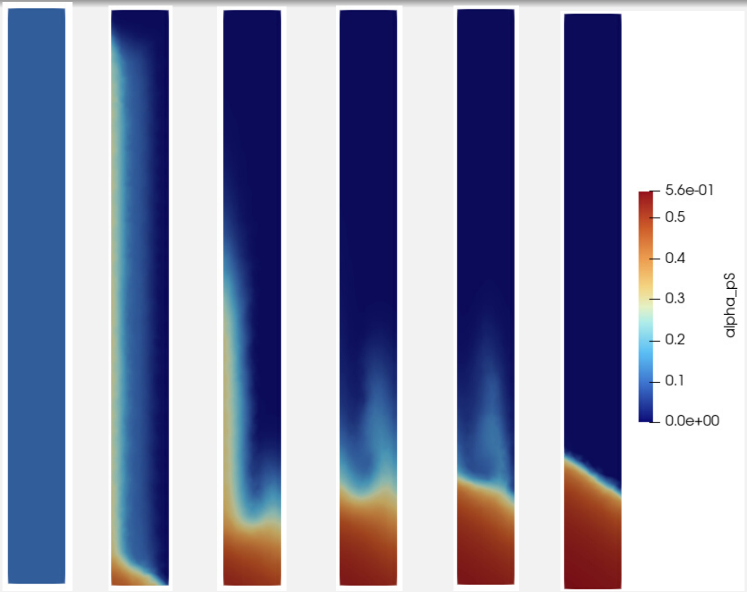
$c = 0.2$ and $\beta = 4$



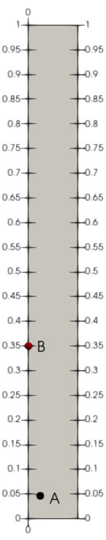
$t = 0$ $t = 0.5$ $t = 1.25$ $t = 1.75$ $t = 2.25$ $t = 4$



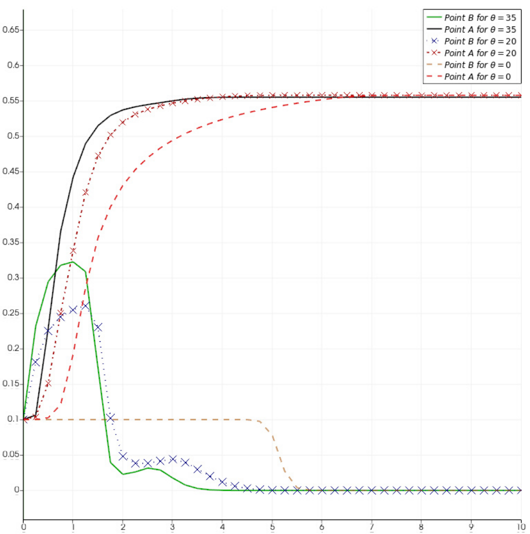
$t = 0$ $t = 0.5$ $t = 1.25$ $t = 1.75$ $t = 2.25$ $t = 4$



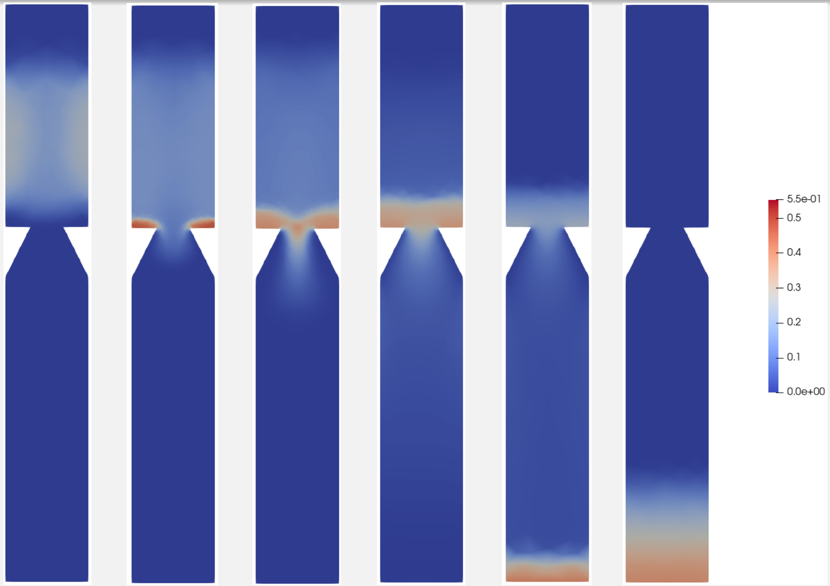
$t = 0$ $t = 0.5$ $t = 1.25$ $t = 1.75$ $t = 2.25$ $t = 4$



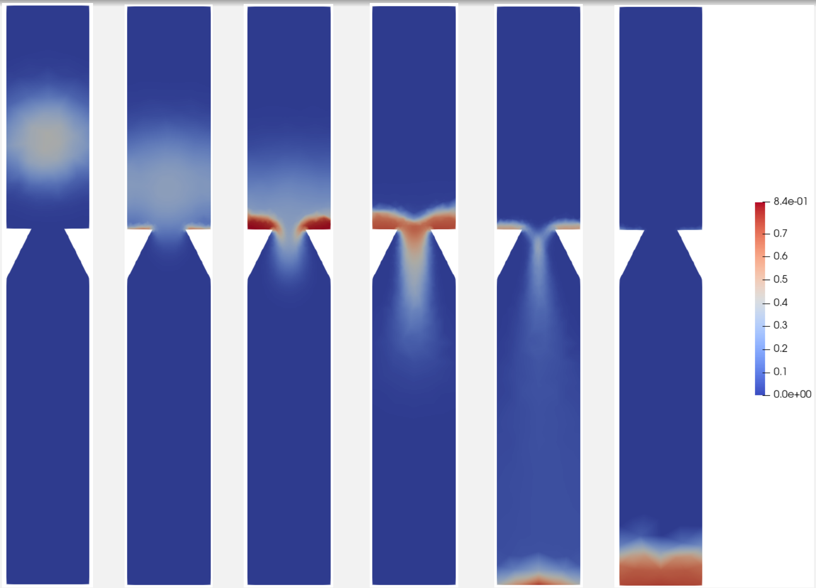
Selected points



Plot over time

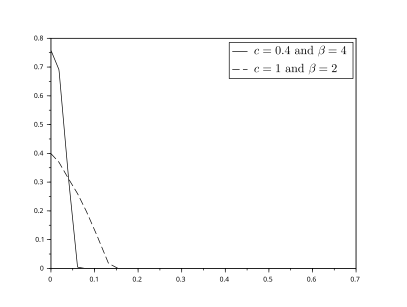


$t = 0.05$ $t = 0.083$ $t = 0.12$ $t = 0.215$ $t = 0.38$ $t = 1.6$

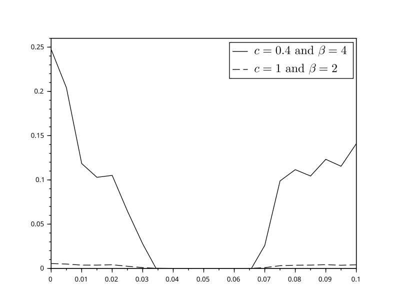


$t = 0.05$ $t = 0.12$ $t = 0.15$ $t = 0.215$ $t = 0.38$ $t = 1.6$

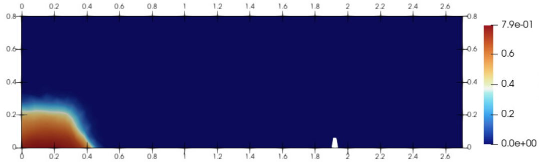
Cutlines at time $t = 0.6$



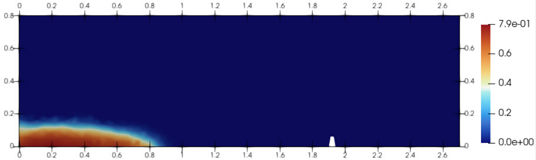
Vertical cutline at $x = 0.05$



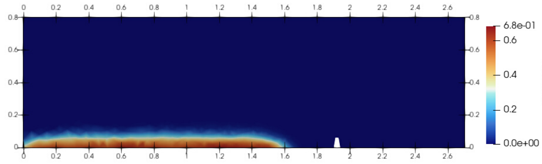
Horizontal cutline at $y = 0.43$



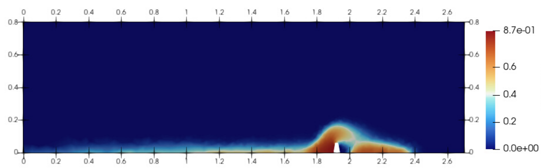
$t = 0.21$



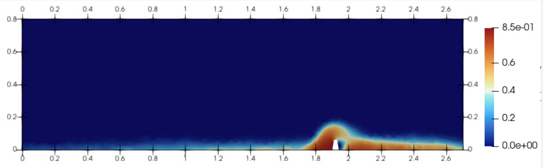
$t = 0.55$


0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6
0.8
0.6
0.4
0.2
0
6.8e-01
0.6
0.4
0.2
0.0e+00

$t = 1.08$



$t = 2$



$t = 2.3$

Conclusion

- A numerical scheme for a multifluid system
(including drag forces and close packing term)

- Preserve the positivity of α and the close packing constraint
(under CFL conditions)

- A global strategy inspired by the work of Jean-Claude and Raphaèle.

Perspective

- Second order extension
(MUSCL techniques already implemented for the Euler system)

- Modification of the scheme to ensure compatibility with energy estimates