convergence of the incremental projection method using (non)conforming approximations

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the incompressible Navier–Stokes equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \quad \text{in } (0, T) \times \Omega$$

$$\mathrm{div} \boldsymbol{u} = 0 \quad \text{in } (0, T) \times \Omega$$

- $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, Lipschitz boundary
- $\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0$
- $\boldsymbol{f} \in L^2(0, T; H^{-1}(\Omega)^d), \quad \boldsymbol{u}_0 \in L^2(\Omega)^d$

objective: convergence of an approximate solution to a weak solution

divergence-free space

$$\boldsymbol{V}(\Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega)^d : \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla}\xi \mathrm{d}x = 0, \ \forall \xi \in H^1(\Omega) \right\}$$
$$= \left\{ \boldsymbol{v} \in L^2(\Omega)^d : \mathrm{div}\boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \mathrm{n}|_{\partial\Omega} = 0 \right\}$$

weak solution

$\boldsymbol{u} \in L^2\big(0, T; H_0^1(\Omega)^d \cap \boldsymbol{V}(\Omega)\big) \cap L^\infty\big(0, T; L^2(\Omega)^d\big)$ such that

for all $\boldsymbol{v} \in C_c^\infty(\Omega \times [0, T))^d$ such that $\mathrm{div}\boldsymbol{v} = 0$ a.e. in $\Omega \times (0, T)$

$$-\int_0^T (\boldsymbol{u}, \partial_t \boldsymbol{v})\mathrm{d}t + \int_0^T ((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}, \boldsymbol{v})\mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v} \mathrm{d}x\mathrm{d}t$$
$$= (\boldsymbol{u}_0, \boldsymbol{v}(0, \cdot)) + \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} \rangle \mathrm{d}t$$

use of projection scheme

- coupled velocity–pressure systems: **expensive**, ill-conditioned
- projection schemes (Chorin 1969, Temam 1969):
  decoupled elliptic solves
- incremental version (Goda 1979): better pressure accuracy
- many results of convergence under additional regularity hypotheses
- G.H.L.M.: convergence of the fully discrete incremental projection scheme for incompressible flows.
  J. Math. Fluid Mech. **25** (2023)
- E.M.: convergence of the incremental projection method using conforming approximations.
  Comput. Methods Appl. Math. **24** (2024)

remaining of this presentation

- study of discrete-in-time incremental projection scheme (G.H.L.M.)
  convergence owing to Gallouët's trick on time translate
- extension to MAC scheme (G.H.L.M.)
  use of discrete functional analysis tools
- extension to conforming scheme (E.M.)
  inf-sup condition replaced by Fortin-type operator on regular functions

the discrete-in-time incremental projection scheme

time partition: $\delta t_N = T/N, \quad t_N^n = n\,\delta t_N$

prediction — find $\tilde{\boldsymbol{u}}_N^{n+1} \in H_0^1(\Omega)^d$:

$$\frac{\tilde{\boldsymbol{u}}_N^{n+1} - \boldsymbol{u}_N^n}{\delta t_N} + \operatorname{div}(\tilde{\boldsymbol{u}}_N^{n+1} \otimes \boldsymbol{u}_N^n) + \nabla p_N^n - \Delta \tilde{\boldsymbol{u}}_N^{n+1} = \boldsymbol{f}_N^{n+1}$$

correction — find $p_N^{n+1} \in H^1(\Omega)$, then $\boldsymbol{u}_N^{n+1}$:

$$\int_\Omega \nabla(p_N^{n+1} - p_N^n) \cdot \nabla q \, dx = \frac{1}{\delta t_N} \int_\Omega \tilde{\boldsymbol{u}}_N^{n+1} \cdot \nabla q \, dx \quad \forall q \in H^1(\Omega)$$

$$\boldsymbol{u}_N^{n+1} = \tilde{\boldsymbol{u}}_N^{n+1} - \delta t_N \nabla(p_N^{n+1} - p_N^n)$$

- $\boldsymbol{u}_N^{n+1}$ is L^2-divergence free with no boundary flux: $\boldsymbol{u}_N^{n+1} \in \boldsymbol{V}(\Omega)$
- Dirichlet condition on $\tilde{\boldsymbol{u}}_N^{n+1} \in H_0^1(\Omega)^d$, not on $\boldsymbol{u}_N^{n+1}$

goal: prove that both converge to the **same** weak solution without more regularity assumption on this weak solution

a priori estimates

key identity (test prediction with $\tilde{\boldsymbol{u}}_N^{n+1}$, add correction):

$$\frac{\|\boldsymbol{u}_N^{n+1}\|^2 - \|\boldsymbol{u}_N^n\|^2}{2\delta t_N} + \frac{\delta t_N}{2}\left(\|\nabla p_N^{n+1}\|^2 - \|\nabla p_N^n\|^2\right) + \|\nabla \tilde{\boldsymbol{u}}_N^{n+1}\|^2 = \langle \boldsymbol{f}_N^{n+1}, \tilde{\boldsymbol{u}}_N^{n+1}\rangle$$

Cauchy–Schwarz + Poincaré, summing over n:

$$\|\tilde{\boldsymbol{u}}_N\|_{L^2(0,T;\,H_0^1)} \leq C, \qquad \|\boldsymbol{u}_N\|_{L^\infty(0,T;\,L^2)} \leq C$$

$$\|\boldsymbol{u}_N - \tilde{\boldsymbol{u}}_N\|_{L^2(0,T;\,L^2)} \leq C\,\delta t_N^{1/2}$$

$\Rightarrow$ **existence** of the discrete solution at each step

$\Rightarrow$ **compactness** properties of approximate solution

weak convergence

from the a priori estimates (up to a subsequence) and $\|\boldsymbol{u}_N - \tilde{\boldsymbol{u}}_N\|_{L^2} \to 0$

$$\tilde{\boldsymbol{u}}_N \rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } L^2(0,T;H^1_0(\Omega)^d)$$

$$\boldsymbol{u}_N \overset{\star}{\rightharpoonup} \bar{\boldsymbol{u}} \quad \text{in } L^\infty(0,T;L^2(\Omega)^d)$$

limit $\bar{\boldsymbol{u}}$ is divergence free since

$$\int_\Omega \boldsymbol{u}_N^{n+1} \cdot \nabla q \,\mathrm{d}x = 0 \implies \int_\Omega \bar{\boldsymbol{u}} \cdot \nabla q \,\mathrm{d}x = 0 \quad \forall q \in H^1(\Omega)$$

limit $\bar{\boldsymbol{u}}$ satisfies B.C. since $\bar{\boldsymbol{u}} \in L^2(0,T;H^1_0(\Omega)^d)$

weak convergence alone is not sufficient to pass to the limit in the nonlinear term $\mathrm{div}(\tilde{\boldsymbol{u}}_N \otimes \boldsymbol{u}_N)$

Gallouët's trick: time translate estimates

discrete dual norm:

$$\|\boldsymbol{v}\|_{\star,1} = \sup\big\{\langle \boldsymbol{v}, \varphi\rangle_{L^2(\Omega)} : \varphi \in W_0^{1,3}(\Omega) \cap \boldsymbol{V}(\Omega),\ \|\varphi\|_{W^{1,3}} = 1\big\}$$

estimate (using Sobolev inequalities):

$$\int_0^{T-\tau} \|\tilde{\boldsymbol{u}}_N(t+\tau) - \tilde{\boldsymbol{u}}_N(t)\|_{\star,1}^2 \, dt \ \leq\ C\,\tau\big(\tau + \delta t_N\big)$$

(pressure term vanishes for $\boldsymbol{v} \in H_0^1(\Omega)^d \cap \boldsymbol{V}(\Omega)$ in the prediction step)

apply Lions-type lemma: $\|\mathcal{P}_{\boldsymbol{V}(\Omega)}\boldsymbol{v}\|_{L^2} \leq \varepsilon\|\boldsymbol{v}\|_{H^1} + C_\varepsilon\|\boldsymbol{v}\|_{\star,1}$
and use $\boldsymbol{u}_N \in \boldsymbol{V}(\Omega)$ and $\|\boldsymbol{u}_N - \tilde{\boldsymbol{u}}_N\|_{L^2(0,T;\,L^2)} \leq C\,\delta t_N^{1/2}$

then apply Kolmogorov theorem

bounded sequence in $L^2(H_0^1)$ + time-translate estimate in L^2
$\Rightarrow$ relatively compact in $L^2(0,T;L^2(\Omega)^d)$

strong compactness and passage to the Limit

From weak and strong convergence properties, **each term in the scheme converges:**

- **time derivative:** $\|\boldsymbol{u}_N - \tilde{\boldsymbol{u}}_N\|_{L^2} \to 0$ + weak conv.
- **pressure:** vanishes against div-free test functions
- **viscous term:** weak convergence in $L^2(H_0^1)$ suffices
- **convection:** weak conv. grad. and strong conv. gives

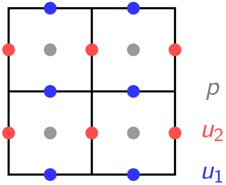
$$\mathrm{div}(\tilde{\boldsymbol{u}}_N \otimes \boldsymbol{u}_N) \rightharpoonup \mathrm{div}(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}})$$

$\Rightarrow$ $\bar{\boldsymbol{u}}$ satisfies the Navier–Stokes weak formulation

spatial discretization: MAC Scheme

staggered finite volumes
on non-uniform rectangular meshes

- u_i located at face centers $\perp$ e_i
- p at cell centers
- no inf-sup condition required
- discrete div, ∇, Δ defined by duality
- non-uniform mesh allowed



fully discrete scheme: same structure, all operators replaced by MAC counterparts

main convergence theorem (G.H.L.M.)

Theorem

let $\Omega \subset \mathbb{R}^d$ *be bounded Lipschitz,* $\boldsymbol{f} \in L^2((0,T)\times\Omega)^d$*,*
$\boldsymbol{u}_0 \in H_0^1(\Omega)^d \cap \boldsymbol{V}(\Omega)$*.*
as $\delta t_N \to 0$ *(semi-discrete) or* $(\delta t_N, h_N) \to 0$ *(fully discrete) with* no restriction on their ratio, *there exists a subsequence and*

$$\bar{\boldsymbol{u}} \in L^2(0,T; H_0^1(\Omega)^d \cap \boldsymbol{V}(\Omega)) \cap L^\infty(0,T; L^2(\Omega)^d)$$

such that $\tilde{\boldsymbol{u}}_N$, $\boldsymbol{u}_N \to \bar{\boldsymbol{u}}$ *strongly in* $L^2(0,T;L^2(\Omega)^d)$*, and* $\bar{\boldsymbol{u}}$ *is a* weak solution*.*

comparison with literature:

- no regularity assumption on $\bar{\boldsymbol{u}}$
- no CFL condition
- **first** convergence proof for the fully discrete incremental projection scheme
- covers non-uniform MAC meshes in 3D

conforming approximations

- $\boldsymbol{X}_N \subset H_0^1(\Omega)^d$ velocity (closed subspace)
- $M_N \subset H^1(\Omega) \cap L_0^2(\Omega)$ pressure

prediction step: use skew-symmetric trilinear form b to find $\tilde{\boldsymbol{u}}_N^{n+1} \in \boldsymbol{X}_N$:

$$\frac{\tilde{\boldsymbol{u}}_N^{n+1} - \boldsymbol{u}_N^n}{\delta t_N} + b(\tilde{\boldsymbol{u}}_N^n, \tilde{\boldsymbol{u}}_N^{n+1}, \cdot\,) + a(\tilde{\boldsymbol{u}}_N^{n+1}, \cdot\,) + (\boldsymbol{\nabla} p_N^n, \cdot\,) = \langle \boldsymbol{f}_N^{n+1}, \cdot \rangle$$

correction step: find $p_N^{n+1} \in M_N$:

$$\big(\boldsymbol{\nabla}(p_N^{n+1} - p_N^n), \boldsymbol{\nabla} q\big) = \frac{1}{\delta t_N}\big(\tilde{\boldsymbol{u}}_N^{n+1}, \boldsymbol{\nabla} q\big) \quad \forall q \in M_N$$

and set:

$$\boldsymbol{u}_N^{n+1} = \tilde{\boldsymbol{u}}_N^{n+1} - \delta t_N\, \boldsymbol{\nabla}(p_N^{n+1} - p_N^n)$$

Assumptions on spaces M_N and $\boldsymbol{X}_N$

- uniform inf-sup condition replaced by weaker condition: existence of family (Π_N) of Fortin-type interpolators such that

$$\Pi_N : C_c^\infty(\Omega)^d \cap \boldsymbol{V}(\Omega) \longrightarrow \boldsymbol{E}_N = \boldsymbol{X}_N \cap \boldsymbol{V}_N \cap L^\infty(\Omega)^d$$

  where discrete divergence-free space $\boldsymbol{V}_N$ defined by:

$$\boldsymbol{V}_N = \left\{ \boldsymbol{v} \in L^2(\Omega)^d : (\boldsymbol{v}, \boldsymbol{\nabla} q) = 0 \ \forall q \in M_N \right\}$$

- for all $\boldsymbol{\varphi} \in C_c^\infty(\Omega)^d \cap \boldsymbol{V}(\Omega)$, $\Pi_N \boldsymbol{\varphi} \xrightarrow{H_0^1} \boldsymbol{\varphi}$ and $\|\Pi_N \boldsymbol{\varphi}\|_{\boldsymbol{E}} := \|\boldsymbol{\nabla} \Pi_N \boldsymbol{\varphi}\|_{L^2} + \|\Pi_N \boldsymbol{\varphi}\|_{L^\infty}$ bounded
- pressure approximation converges:

$$\mathcal{P}_{M_N}(p) \xrightarrow{H^1} p \quad \forall p \in H^1(\Omega) \cap L_0^2(\Omega)$$

example: the standard Taylor–Hood pair

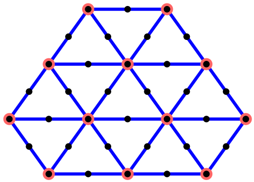
on simplicial meshes $\mathcal{T}_N$:

$$\boldsymbol{X}_N = \mathbb{P}^2(\mathcal{T}_N)^d \cap H^1_0(\Omega)^d$$

$$M_N = \mathbb{P}^1(\mathcal{T}_N) \cap H^1(\Omega) \cap L^2_0(\Omega)$$

degrees of freedom:

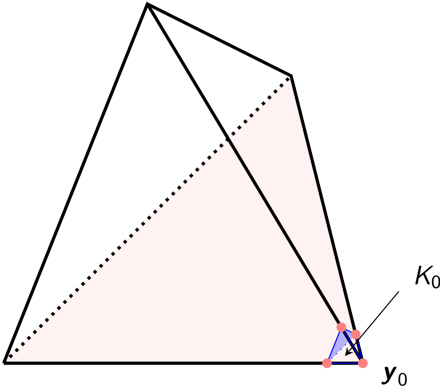
- velocity ($\mathbb{P}^2$):
  vertices + edge midpoints
- pressure ($\mathbb{P}^1$):
  vertices only



pathological tetrahedron K_0:
all edges on $\partial\Omega$, vertex $\boldsymbol{y}_0$ only in K_0

- $\boldsymbol{v} \in \boldsymbol{X}_N \Rightarrow \boldsymbol{v} = 0$ on K_0
- $q \in M_N$: $q(\boldsymbol{y}_0) = 1 + c$,
  $q(\boldsymbol{y}_i) = c$ elsewhere
- $(\boldsymbol{\nabla} q, \boldsymbol{v}) = 0 \Rightarrow$ inf-sup $= 0$

standard theory **fails**
$\Rightarrow$ this paper's Π_N **applies**



construction of Π_N for Taylor–Hood

two-step operator inspired by: L. Diening, J. Storn, and T. Tscherpel
Fortin operator for the Taylor–Hood element.
Numerische Mathematik, 150:671–689, 2021

$$\Pi_N \boldsymbol{v} = \Pi_L \boldsymbol{v} + \Pi_D(\boldsymbol{v} - \Pi_L \boldsymbol{v})$$

- Π_L: nodal $\mathbb{P}^2$ Lagrange interpolator
- Π_D: divergence correction via $\mathbb{P}^2$ bubble functions
  $\mathrm{div}(\Pi_D \boldsymbol{w})$ = projection of $\mathrm{div}\, \boldsymbol{w}$ on $\mathbb{P}^1$
- compact support of $\boldsymbol{v} \Rightarrow \Pi_N \boldsymbol{v} \in \boldsymbol{E}_N$

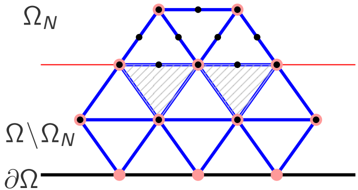
Assumptions on discrete spaces satisfied

extension: pair without inf-sup condition

near $\partial\Omega$ (strip $\Omega \setminus \Omega_N$):
use $\mathbb{P}^1$ or *transition elements*
(hatched triangles: 4 velocity dofs)

interior Ω_N: standard $\mathbb{P}^2/\mathbb{P}^1$

Π_N still satisfies assumptions
even without inf-sup condition



main convergence theorem (E.M.)

Theorem

there exists a subsequence and

$$\bar{\boldsymbol{u}} \in L^2\big(0, T; H_0^1(\Omega)^d \cap \boldsymbol{V}(\Omega)\big) \cap L^\infty\big(0, T; L^2(\Omega)^d\big)$$

such that $\tilde{\boldsymbol{u}}_N, \boldsymbol{u}_N \to \bar{\boldsymbol{u}}$ *strongly in* $L^2(0, T; L^2(\Omega)^d)$*, and* $\bar{\boldsymbol{u}}$ *is a* weak solution *to the Navier–Stokes equations.*

merci Jean-Claude