

CNRS International Research Project (IRP) PICASSO

hyPerbolic models, numerical Analysis and Scientific cOmputation



2nd Workshop

Book of Abstracts

Coimbra, Portugal

9–11 November 2026

https://indico.math.cnrs.fr/e/2nd_Picasso_Workshop

<https://picasso.math.cnrs.fr>

Welcome

On behalf of the IRP , we are delighted to welcome you to Coimbra for its **2nd Workshop**, held on **9–11 November 2026**.

The workshop gathers researchers from Spain, Portugal and France working on *hyperbolic models, numerical analysis and scientific computation*, the broad theme that gave the IRP its name. It is one of the events supported by PICASSO to create and strengthen scientific relationships across the three countries, following the kick-off meeting held in Málaga in March 2025.

This booklet collects the abstracts of the **25 contributions** presented at the workshop: **6 plenary lectures** and **19 contributed talks**. They cover, among other topics: well-balanced and structure-preserving schemes for balance laws and shallow-water systems (finite volume, WENO, Active Flux and discontinuous Galerkin methods); entropy stability and boundary conditions; non-conservative hyperbolic systems and Riemann problems; relativistic fluid dynamics; IMEX and minimally implicit time integration; reduced-order modelling; spectral and multigrid methods; and models for granular, plastic-laden and compressible depth-averaged flows.

The programme is on page [6](#). The list of abstracts on page [9](#) is hyperlinked: click on a title (or on its page number) to jump directly to the corresponding abstract, and use the link at the top of each abstract to come back to the list.

We wish you a stimulating workshop and a pleasant stay in Coimbra.

The organization team

The IRP PICASSO



What is PICASSO?

PICASSO stands for “hyPerbolic models, numerical Analysis and Scientific cOmputation”. It is an *International Research Project* (IRP) of CNRS Mathématiques (INSMI), created on **1 January 2025** for a period of **five years**. It aims at enhancing the network between Spanish, Portuguese and French applied mathematicians, a community of about **200 researchers** spread over the three countries, and supports any action that creates or fosters their scientific relationships through its own funding, provided by CNRS and other academic institutions.

An international research project

An IRP aims at structuring an international scientific community around shared themes. It promotes the organisation of international workshops, seminars and thematic schools by the partners, in France and abroad, and receives specific CNRS funding for international mobility between laboratories and for the organisation of such events.

A strong relationship, finally formalised

For decades, researchers from France, Spain and Portugal have intensively collaborated, exchanging ideas, co-mentoring students, visiting each other, organising joint events, building projects and, above all, writing influential articles on the development and analysis of models, numerical methods and high-performance simulations for fluid flow problems, notably from geophysical and environmental applications. Until 2025 this relationship had never been formalised into any administrative structure; the creation of an IRP was the natural step.

Why “PICASSO”?

The idea of an IRP called PICASSO emerged during the ice-breaking cocktail of the HYP conference held in Málaga in June 2022. Málaga is the birthplace of Pablo Picasso (1881–1973), the celebrated painter, Francophile and path-finder in art. The acronym was found afterwards.

Three hubs, one wide community

The project is structured around three “hub” laboratories, in **Málaga**, **Coimbra** and **Bordeaux**, and was made possible by the active support of researchers from Sevilla and Granada in Spain, Minho and NOVA Lisboa in Portugal, and Nantes and Versailles Saint-Quentin in France.

Direction

Stéphane Clain (Coimbra, Portugal) · Tomás Morales de Luna (Málaga, Spain) · Raphaël Loubère (Bordeaux, France)

Scientific committee

Duarte Albuquerque (Lisbon), Christophe Berthon (Nantes), Christophe Chalons (Versailles), Stéphane Clain (Coimbra), Enrique Fernández-Nieto (Sevilla), Raphaël Loubère (Bordeaux), Gaspar Machado (Minho), Tomás Morales de Luna (Málaga), Pep Mulet (Valencia).

Funding opportunities

PICASSO regularly issues calls for “one-week research trips” between laboratories of the three countries. Laboratories wishing to host a presentation of the IRP are welcome to get in touch. Few grants may still be available. Contact us!

Contact

IRP PICASSO, IMB, UMR 5251, 351 cours de la Libération, 33405 Talence, France
picasso@math.u-bordeaux.fr · <https://picasso.math.cnrs.fr>

Supporting institutions



Welcome to Coimbra



Coimbra, in central Portugal, rises on the hills above the **Mondego** river, roughly halfway between Lisbon and Porto. It is one of the oldest and most emblematic university cities in Europe, and a natural meeting place for this workshop: the Department of Mathematics of the University of Coimbra (CMUC) is one of the three PICASSO hubs.

A capital and a university

During the twelfth and thirteenth centuries Coimbra was the capital of the young Kingdom of Portugal, and the city still treasures that medieval heritage: the Romanesque *Sé Velha* (Old Cathedral), the Monastery of Santa Cruz, where the first kings of Portugal are buried, and the narrow streets of the lower town. The University, founded in 1290 and established in Coimbra since 1537, is among the oldest in the world.

The University of Coimbra – Alta and Sofia

The historic University buildings on the *Alta* hill, depicted on the workshop banner, together with the *Sofia* street area, were inscribed on the **UNESCO World Heritage List** in 2013. Highlights include the *Paço das Escolas* courtyard dominated by the clock tower (*a Cabra*, the “goat”, as students affectionately call its bell), the Baroque *Biblioteca Joanina*, the *Capela de São Miguel* and the Botanical Garden.

Traditions and Worth a visit if you have a few hours

Coimbra is also the home of the *Fado de Coimbra*, a serenade tradition distinct from the fado of Lisbon, traditionally performed by university students in black academic gowns. Student life shapes the city’s rhythm, culminating in the *Queima das Fitas* festivities every spring.

- The Paço das Escolas and the Joanina Library (advance booking recommended);
- The Old Cathedral (*Sé Velha*) and the Monastery of Santa Cruz;
- The Botanical Garden and the Machado de Castro National Museum;
- The Mosteiro de Santa Clara-a-Velha, a stroll along the Mondego and a *Fado de Coimbra* evening.

Programme

Plenary lectures last 45 minutes, contributed talks 30 minutes; coffee breaks last 15 minutes. Each session opens with a plenary lecture, and a second plenary lecture opens the second part of the Monday and Tuesday afternoon sessions. Click on a title to jump to its abstract. Authors are listed as in the abstracts; the speaker of each talk is to be confirmed.

Monday 9 November 2026

Afternoon session • Entropy, thermodynamic consistency and well-balanced schemes

13:30–14:00 Welcome and opening talk by the organiser

14:00–14:45

PLENARY [An Asymptotic- and Curl-Preserving Finite Volume Scheme for a Novel Hyperbolic Barotropic Navier–Stokes Model](#) (p. 10)

A. Guines, W. Boscheri, S. Gavriluk, B. Re

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15:15–15:45 [Partial Approximate Riemann Solvers for Discrete Dissipative Boundary Conditions of the Shallow-Water Equations](#) (p. 27)

G. Beck, L. Martaud

15:45–16:00 Coffee break

16:00–16:45

PLENARY [Well-Balanced Semi-Implicit Relaxed Finite Volume Schemes for Hyperbolic Balance Laws](#) (p. 12)

I. Gómez-Bueno, M. J. Castro, A. Thomann

16:45–17:15 [Development of Entrovolume Method Techniques on 2D Horizontal Domains for Saint-Venant–Exner Models with Gravitational Effects](#) (p. 22)

T. Morales de Luna

17:15–17:45 [On the Design of Well-Balanced Active Flux Schemes for Balance Laws](#) (p. 19)

W. Barsukow, M. J. Castro, T. Morales de Luna, S. Valiente

End of the day: 17:45

Tuesday 10 November 2026**Morning session • IMEX, all-speed and reduced-order approaches for multiscale flows****9:00–9:45****PLENARY All Mach Number Implicit-Explicit Schemes for Compressible Cahn–Hilliard–Navier–Stokes Equations** (p. 14)*A. Martorell, P. Mulet, D. Yáñez***9:45–10:15****A Second-Order Positivity-Preserving IMEX (PP-IMEX) Finite Volume Scheme for Compressible Flows on Staggered Meshes with Large Time-Steps** (p. 24)*T. Harbreteau, R. Herbin, J.-C. Latché***10:15–10:45****Low-Froude Shallow Water Equations: An Enriched Discretization and Multigrid Approach** (p. 25)*D. Inzunza, V. Perrier***10:45–11:00**

Coffee break

11:00–11:30**Discrete Variational Physics-Informed Neural Networks** (p. 21)*D. Cotrim, J.A. Ferreira, L. Marques and G. Pena.***11:30–12:00****High-Order IMEX Local Discontinuous Galerkin Methods with Sparse Grids for High-Dimensional Nonlinear Parabolic PDEs** (p. 38)*J. P. Villarino, M. J. Castro, J. A. García-Rodríguez, J. G. López-Salas***12:00–13:30**

Lunch

Afternoon session • Multiscale limits, relativistic hyperbolic models and complex flows**13:30–14:15****PLENARY Asymptotic-Preserving Schemes** (p. 15)*M.-H. Vignal***14:15–14:45****An Analysis of the Riemann Problem for the Self-Organized Hydrodynamics Model and Applications** (p. 20)*C. Berthon, M. Compain, C. Parés, E. Pimentel-García***14:45–15:15****Modelling Equations of State for Relativistic Flow: Shock Waves and Phase Transitions in Non-Convex Hyperbolic Systems** (p. 36)*S. Serna***15:15–15:45****Minimally Implicit Runge–Kutta (MIRK) Methods for Multicomponent Relativistic Fluid Dynamics** (p. 29)*I. Martínez-Donato, S. J. Bonanad-Hurtado, I. Cordero-Carrión, M. Perucho, J. M. Martí***15:45–16:00**

Coffee break

16:00–16:45**PLENARY Fractional Derivative Models for Complex Materials: Caputo and Tempered Formulations** (p. 11)*L. L. Ferrás***16:45–17:15****A Multilayer Model with Plastic Transport in Rivers** (p. 30)*J. Moya, R. M. L. Ferreira, E. D. Fernández-Nieto, A. M. Ricardo***17:15–17:45****On Depth-Averaged Compressible Models with Free Surface or in Pressurized Domains** (p. 32)*D. Bresch, E. D. Fernández-Nieto, J. Garres Díaz, P. Le Vourc'h, G. Narbona***17:45–18:15****In-Cell Discontinuous Reconstruction Path-Conservative Methods for Conservative Systems in Primitive Variables: How to Skip Hybrid Approaches** (p. 33)*E. Pimentel-García, M. Ricchiuto, C. Parés*

End of the day: 18:15

20:30–**Gala diner at "Fangas Maior"** fangas.pt/home R. de Fernandes Thomas, nº 29 Coimbra

Wednesday 11 November 2026**Morning session • High-order, well-balanced and Galerkin-type methods; numerical analysis****9:00–9:45**

PLENARY Numerical Approach to the Fractional-Order Neural Field Equation (p. 13)
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9:45–10:15

Theoretical and Numerical Aspects of Pseudo-Parabolic Reaction–Diffusion Problems
 (p. 16)
S. Barbeiro, A. Costa, Á. Durán

10:15–10:45

Active Flux and the Superconvergence of Discontinuous Galerkin Methods (p. 17)
W. Barsukow

10:45–11:00

Coffee break

11:00–11:30

Well-Balanced DG Schemes for Multidimensional Shallow Water Systems with Coriolis Forces: Evolving Fluctuations of Stationarity Preserving Continuous Finite Elements
 (p. 34)
V. G. Tabernero, M. J. Castro, J. A. García-Rodríguez, M. Ricchiuto

11:30–12:00

Well-Balanced High-Order WENO Finite-Difference Methods via a Local Flux Formulation (p. 26)
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12:00–12:30

Modelling Dry Granular Flows over Curved Topographies (p. 37)
S. Valiente, T. Morales de Luna, J. Garres-Díaz, M. J. Castro

12:30–13:00

Numerical Approximation of 2D Models for Internal and Surface Waves with Spectral Methods (p. 23)
Á. Durán

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An Asymptotic- and Curl-Preserving Finite Volume Scheme for a Novel Hyperbolic Barotropic Navier–Stokes Model

A. Guines^a, W. Boscheri^{a,*}, S. Gavriluk^b, B. Re^c

^aLaboratoire de Mathématiques UMR 5127 CNRS, Université Savoie Mont Blanc, 73376 Le Bourget du Lac, France.

^bAix Marseille Université, CNRS, IUSTI, UMR 7343, 13453 Marseille, France.

^cDepartment of Aerospace Science and Technology, Politecnico di Milano, via La Masa 34, Milano, 20156, Italy.

*Correspondence to walter.boscheri@univ-smb.fr

ABSTRACT

We introduce a first-order hyperbolic and thermodynamically compatible formulation of the barotropic Navier–Stokes equations. This new model is asymptotically consistent with the Stokes model for compressible viscous fluids, provided that suitable relaxation terms and evolution equations for two auxiliary vector and tensor fields are added. The tensor field is endowed with a curl-free constraint. A novel second-order finite volume scheme is designed to solve the governing equations: a semi-implicit time discretization allows the scheme to be asymptotic preserving with respect to the low Reynolds and low Mach number limits of the original hyperbolic Navier–Stokes model. Nonlinear equations of state are handled by means of a nested Newton algorithm, which includes the pressure sub-system arising from the implicit treatment of the pressure terms. Convective fluxes are discretized explicitly, ensuring stability and shock-capturing properties in the case of high Mach number flows. The scheme is proven to be curl-free thanks to compatible differential operators defined on a vertex-staggered spatial discretization. A large suite of test cases in multiple space dimensions demonstrates the accuracy, robustness, and effectiveness of the new numerical method across multiple stiffness regimes for both Mach and Reynolds numbers.

Fractional Derivative Models for Complex Materials: Caputo and Tempered Formulations

Luís L. Ferrás^{a,*}

^aCEFT, ALiCE, Faculty of Engineering, University of Porto, Rua Dr. Roberto Frias, Porto, 4200-465, Portugal.

*Correspondence to lferras@fe.up.pt

ABSTRACT

The use of fractional elements, such as springpots, has enhanced the modelling of complex viscoelastic materials [1, 2, 3, 4, 5, 6, 7] by capturing memory effects and power-law responses. However, classical springpot-based models typically exhibit solid-like behaviour and cannot represent fluid-like responses without additional elements.

In this work, we propose a fractional viscoelastic model for small deformations that combines a springpot, defined using a Caputo derivative, with a fractional dashpot [7] based on a tempered Caputo derivative. This configuration enables fluid-like behaviour while preserving the flexibility of fractional formulations. A full mathematical formulation is presented, and analytical expressions are derived for key rheological functions, including the complex modulus, storage and loss moduli, relaxation modulus, creep compliance, and steady shear viscosity. The model is compared with several classical and fractional viscoelastic models, showing that it provides accurate and physically consistent descriptions of experimental data in both frequency and time domains while successfully capturing fluid-like behaviour.

This is a joint work with M. Rebelo (Center for Mathematics and Applications (NOVA Math) and Department of Mathematics, NOVA School of Science and Technology (NOVA FCT)) and M.L. Morgado (Departamento de Matemática, Universidade de Trás-os-Montes e Alto Douro, UTAD).

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Well-Balanced Semi-Implicit Relaxed Finite Volume Schemes for Hyperbolic Balance Laws

I. Gómez-Bueno^{a,*}, M. J. Castro^a, A. Thomann^b

^aUniversidad de Málaga, Spain.

^bInria center University of Lorraine and antennae University Strasbourg, France.

*Correspondence to igomezbueno@uma.es

ABSTRACT

We propose a well-balanced semi-implicit relaxed finite volume approach for hyperbolic systems of balance laws. The motivation is to combine the robustness of relaxation-based discretizations for multi-scale regimes [1, 2] with the preservation of relevant stationary solutions at the discrete level.

The construction follows the spirit of semi-implicit relaxation strategies for hyperbolic systems, where the introduction of relaxation variables leads to a more convenient numerical structure and allows fast or stiff components to be treated implicitly. In this framework, the main difficulty is to incorporate the source term without destroying the steady states of the original balance law.

To achieve the well-balanced property, we follow the strategy introduced in [3]. Once the local steady states have been computed on both the primal and dual meshes, a well-balanced reconstruction operator is built from a standard reconstruction procedure by measuring the reconstructed quantities as fluctuations with respect to the corresponding local equilibrium. In the relaxed semi-implicit setting considered here, this equilibrium information is also used to formulate the relaxation procedure around the steady configuration rather than around the original variables themselves. In order to avoid solving the nonlinear steady-state problem repeatedly during the implicit integration, the local equilibrium is frozen over each time step and the numerical method evolves only the time fluctuation of the cell averages. This provides a natural way to combine the preservation of stationary solutions with the relaxed finite volume structure. The approach is applied to shallow-water type systems, including the classical shallow-water equations with topography and the Ripa model.

As a representative application, we consider shallow-water type models with topography, which provide a standard setting to test the preservation of equilibria and the behaviour of the method in multi-scale regimes. The approach is intended to provide a flexible basis for accurate well-balanced relaxed schemes, while keeping the semi-implicit structure inherited from the relaxation formulation.

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Plenary lecture P4

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Numerical Approach to the Fractional-Order Neural Field Equation

D. Gaspar^a, W. Erhlagen^b, P. Lima^{a,*}, L. Morgado^c

^aInstituto Superior Técnico, Universidade de Lisboa, Portugal.

^bDepartamento de Matemática, Universidade do Minho, Portugal.

^cDepartamento de Matemática, Universidade de Trás-os-Montes e Alto Douro, Portugal.

*Correspondence to plima@math.tecnico.ulisboa.pt

ABSTRACT

The Neural Field Equation (NFE) and Fractional Derivatives are established fields that have recently converged into the study of the Fractional Neural Field Equation (FNFE). In this work we extend the results in [1] by providing a comprehensive analysis and numerical approximation of this equation, starting from the classical Amari neural field model. We consider a generic initial value problem for the FNFE of order $0 < \alpha < 1$ and discuss traveling front solutions, which are well-documented for the first-order case but present new challenges in the fractional context. We use the analytical solution, U_{Gfrac} , which is available in closed form for the classical case ($\alpha = 1$), as an initial guess to approximate the case of fractional order. To solve the resulting integro-differential equations, we derive a semi-discretized scheme using the L1 method [2]. We provide theoretical convergence results for this method on both uniform and graded time meshes, highlighting the impact of the singularity at $t = 0$.

We present numerical results obtained by this algorithm which illustrate how the fractional neural field equation can be used to simulate working memory, which has important applications in robotics.

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Plenary lecture P5

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All Mach Number Implicit-Explicit Schemes for Compressible Cahn–Hilliard–Navier–Stokes Equations

Andreu Martorell^a, Pep Mulet^{a,*} and Dionisio Yáñez^a

^aDepartment of Mathematics, University of Valencia, Spain.

*Correspondence to mulet@uv.es

ABSTRACT

We propose a second-order all Mach number implicit-explicit (IMEX) time-stepping scheme for the isentropic compressible Cahn–Hilliard–Navier–Stokes equations and for the non-barotropic compressible Cahn–Hilliard–Navier–Stokes–Fourier equations discretized on staggered (MAC) grids.

The scheme is based on finite difference approximations that ensure a stable coupling among the velocity and the rest of the state variables, with symmetric operators acting on the discretized viscosity terms.

Standard explicit methods suffer from severe time-step restrictions due to the presence of second to fifth-order terms introduced by the Cahn–Hilliard and Navier–Stokes–Fourier operators.

These challenges are dealt with a second-order IMEX Runge–Kutta scheme that treats only the stiff terms implicitly.

Numerical experiments are performed to verify the stability, accuracy and efficiency of the proposed approach.

Asymptotic-Preserving Schemes

Marie-Hélène Vignal^a

^a Institut Camille Jordan, Université Lyon 1, France.

* Correspondence to marie-helene.vignal@univ-lyon1.fr

ABSTRACT

In many applications, multiple physical scales, both small and large, coexist. This is the case for instance in fluid mechanics and plasma physics. Generally, the smaller scales, generically denoted by ε , have a significant impact on the cost of numerical simulations because, in the absence of specialized schemes, spatial and/or temporal discretizations must resolve the smallest of these scales. As a result, the development of efficient numerical methods for solving such multiscale problems constitutes a major computational challenge.

One way to perform numerical simulations of such models at a reasonable cost is to develop asymptotic-preserving schemes. These schemes are uniformly stable with respect to the parameter ε , making it possible to use meshes that are independent of the smallest scales present. They are said to be asymptotically stable in the limit $\varepsilon \rightarrow 0$. Moreover, in regions where ε is very small, these schemes recover a discretization of the limiting model obtained as ε tends to zero. They are then said to be asymptotically consistent. Schemes that are both asymptotically stable and consistent are called asymptotic-preserving in the limit $\varepsilon \rightarrow 0$, because they preserve this limit.

In this presentation, I will focus on a particular scheme that preserves the quasi-neutral limit for the Vlasov-Poisson equations. This work is a collaboration with Alain Blaustein (INRIA Lille), Giacomo Dimarco (University of Ferrara), and Francis Filbet (University of Toulouse). I will discuss the difficulties associated with this problem, particularly those related to its numerical simulation, and show how asymptotic-preserving schemes make it possible to overcome them.

Contributed talk 1

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Theoretical and Numerical Aspects of Pseudo-Parabolic Reaction–Diffusion Problems

Sílvia Barbeiro^{a,*}, Afonso Costa^a, Ángel Durán^b

^aCMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal.

^bApplied Mathematics Department, University of Valladolid, 47011 Valladolid, Spain.

*Correspondence to silvia@mat.uc.pt

ABSTRACT

This talk addresses several theoretical and numerical aspects of pseudo-parabolic reaction–diffusion initial-boundary-value problems of the form

$$\frac{\partial v}{\partial t} - \nabla \cdot \left(A(v) \nabla \frac{\partial v}{\partial t} \right) = -\nabla \cdot (D(v) \nabla v) + f(v),$$

posed in $\Omega = (-1, 1)^d$, $d = 1, 2, 3$, and supplemented with homogeneous Neumann boundary conditions and suitable initial data.

Pseudo-parabolic equations of this type arise in the modelling of several physical phenomena, either along with hyperbolic effects, [2], or as a way to control the smoothing effect in diffusion problems, [3, 5]. More recently, such models have also been explored in image-processing applications, particularly in texture recognition, [4, 6].

The presentation focuses on three main aspects. First, the well-posedness of the associated initial-boundary-value problem is investigated, following the same approach as that of [1] for the Dirichlet problem. Next, a discontinuous Galerkin (DG) numerical approximation is introduced and analyzed, with emphasis on stability and convergence properties. Finally, computational experiments illustrating the performance of the proposed method are presented, with particular attention to applications in image processing.

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Contributed talk 2

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Active Flux and the Superconvergence of Discontinuous Galerkin Methods

Wasilij Barsukow^{a,*}

^aCNRS, Institut de Mathématiques de Bordeaux (IMB), UMR 5251, 351 Cours de la Libération, 33405 Talence, France.

Present address: Imperial College London, IRL2004, Huxley Building, South Kensington Campus, London, United Kingdom.

*Correspondence to wasilij.barsukow@math.u-bordeaux.fr

ABSTRACT

The Active Flux method evolves averages/moments and, additionally, point values at cell interfaces. This is at once reminiscent of Discontinuous and Continuous Galerkin methods. This talk aims at relating Active Flux to existing families of (stabilized) Finite Element methods.

The point values are shared among adjacent cells, giving rise to a continuous approximation of the solution. The key difference to classical continuous Galerkin methods is that there is no need to invert any global (or even local) mass matrix: Active Flux does not start out from a variational form and, instead, the update of the degrees of freedom is written down directly.

At the same time Active Flux is conservative and can be seen as a high-order extension of Finite Volume methods, while keeping the stencil very compact, similar to Discontinuous Galerkin methods.

In the talk it is shown that Active Flux can be understood as a Petrov–Galerkin method with a biorthogonal family of continuous basis functions and discontinuous test functions. Moreover, an equivalence property to Discontinuous Galerkin methods is uncovered, which offers a new perspective on their well-known superconvergence at Radau points.

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Quantifying Numerical Entropy Loss with a Projection Method

Nina Aguillon^{b,a}, Marie Boussard^{a,b,*}, Julien Salomon^{a,b}

^aInria Paris.

^bLaboratoire Jacques-Louis Lions, Sorbonne Université.

*Correspondence to marie.boussard@inria.fr

ABSTRACT

Finite volume schemes used in ocean circulation codes usually generate a significant loss of numerical entropy, leading to unphysical behaviours such as the mixing of water masses with distinct temperatures and salinities. Localising and quantifying the numerical entropy loss in high-order codes is therefore a crucial issue. A method was proposed by Aguillon, Audusse, Desveaux, and Salomon [1], but is limited to the one-dimensional case.

We propose a method of quantification applicable to two-dimensional high-order finite-volume schemes, suitable for unstructured meshes, in presence of source terms. The main idea is to project a consistent numerical entropy flux onto the set of fluxes satisfying a discrete entropy inequality. The projected flux allows us to show an extended Lax–Wendroff theorem: if convergence occurs, it is towards the entropy solution. Besides, we discuss the computational aspects of the projection, which is carried out by solving a constrained minimisation problem. We implement a semi-smooth Newton algorithm combined with a continuation method. Numerical examples, including numerical entropy loss estimation for shallow water equations, will be provided.

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On the Design of Well-Balanced Active Flux Schemes for Balance Laws

Wasilij Barsukow^a, Manuel J. Castro^{b,*}, Tomás Morales de Luna^b, Sergio Valiente^b

^aInstitut de Mathématiques de Bordeaux (IMB), CNRS UMR 5251, Université de Bordeaux.

^bDpto. Análisis Matemático, Estadística e Investigación Operativa, y Matemática Aplicada, Universidad de Málaga.

*Correspondence to mjcastro@uma.es

ABSTRACT

This work presents preliminary results on the construction of exactly well-balanced numerical schemes for systems of balance laws using the Active Flux technique. Accurately capturing the non-trivial steady states of these systems is a crucial requirement for robust numerical simulations. Building upon the semi-discrete Active Flux method (see, for example, [2, 3]), which was previously adapted for well-balanced properties in specific contexts [4], we propose a framework that integrates the general strategy introduced by Castro and Parés (2020) [1] for the computation of local stationary solutions. This combined approach allows the scheme to maintain its characteristic high-order accuracy while preserving the steady states at the discrete level. Furthermore, this work considers the presence of singular source terms, extending the proposed framework to ensure stability and exact balance in such scenarios. The resulting scheme demonstrates high resolution, offering a highly effective tool for a wide range of physical applications governed by balance laws.

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An Analysis of the Riemann Problem for the Self-Organized Hydrodynamics Model and Applications

Christophe Berthon^a, Marie Compain^{a,*}, Carlos Parés^b, Ernesto Pimentel-García^c

^aLaboratoire de Mathématiques Jean Leray, CNRS UMR 6629, Nantes Université, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France.

^bDepartamento de Análisis Matemático, Estadística e Investigación Operativa, y Matemática Aplicada, Universidad de Málaga, Campus de Teatinos s/n, Málaga 29080, Spain.

^cDepartamento de Matemática Aplicada, Universidad de Málaga, Campus de Teatinos s/n, Málaga 29080, Spain.

*Correspondence to marie.compain@univ-nantes.fr

ABSTRACT

In this work, we numerically approximate the weak solutions of the following 1D non-linear evolution PDE system:

$$\partial_t \rho + \partial_x (\rho u) = 0, \quad (1)$$

$$\rho (\partial_t u + cu \partial_x u) + \lambda (1 - u^2) \partial_x \rho = 0, \quad (2)$$

$$\rho (\partial_t v + cu \partial_x v) - \lambda uv \partial_x \rho = 0, \quad (3)$$

$$\sqrt{u^2 + v^2} = 1. \quad (4)$$

This system depicts a collective self-organized motion where $\rho > 0$ is the density of individuals, $\Omega = (u, v)^T$ the speed of the flow, and $0 < c < 1$ and $\lambda > 0$ two given constants prescribed by the physics. The Self-Organized Hydrodynamics (SOH) model [2] is understood as the macroscopic limit of the Vicsek model [4].

System (1)–(4) is hyperbolic and non-conservative. These non-conservative products are not well-defined for shock wave solutions. To overcome such an issue, different frameworks were introduced in order to define these weak solutions. The path theory [1] encompasses a wide range of results for the resolution of Riemann problems. However, a choice of path needs to be made in order to close the definition of shock waves. Here, following [3], shocks are described as limits of traveling wave solutions of the associated viscous model [2]. Using this definition, a characterization of shock waves is exhibited. In addition, using these generalized Rankine–Hugoniot conditions, we are able to determine the simple waves of the SOH model (1)–(4). It is then possible to solve the Riemann problem of interest within some regions to be specified.

Finally, shock-capturing numerical methods are presented in order to approximate these solutions.

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Discrete Variational Physics-Informed Neural Networks

D. Corim^a, J.A. Ferreira^a, L. Marques^a, G. Pena^a.

^a CMUC Department of Mathematics, University of Coimbra, Coimbra, Portugal.

* Correspondence to cotrimnunes@uc.pt

ABSTRACT

We introduce discrete variational PINNs (dvPINNs), a novel Neural Network (NN) framework that minimizes a discrete energy functional derived from FDM/FEM discretizations. Unlike classical or continuous variational PINNs, this method avoids backpropagating PDE derivatives and bypasses Monte Carlo integration. Assuming a symmetric, coercive, and continuous bilinear form, we prove an error bound governed by optimization, NN approximation, and classical FDM/FEM errors. We demonstrate this framework on the 1D stationary coupled elasticity problem, where we recover the underlying FDM/FEM scheme's second-order convergence in a discrete H^1 norm.

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Development of Entrovolume Method Techniques on 2D Horizontal Domains for Saint-Venant–Exner Models with Gravitational Effects

T. Morales de Luna^{a,*}

^aDpto. de Análisis Matem., Estad. e I.O. y Matemática Aplicada, Universidad de Málaga.

*Correspondence to tmorales@uma.es

ABSTRACT

Developing numerical methods that maintain steady states while ensuring physical correctness of the solution is a major challenge in computational fluid dynamics.

Here we intend to focus and analyze the entropy relation. We present first a versatile and robust methodology to achieve both discrete entropy stability and well-balancing in first-order finite volume solvers applied to balance laws and hyperbolic systems with non-conservative terms. This technique uses an artificial viscosity, by means of a parameter that controls it. The entropy relation is then fulfilled without destroying the well-balance nature of the scheme.

Other similar techniques allow to even exactly preserve the entropy equation for smooth solutions, reducing the numerical dissipation of the approximations.

Finally, approaches based on the use of staggered meshes will be studied, combined with semi-implicit approximations.

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Numerical Approximation of 2D Models for Internal and Surface Waves with Spectral Methods

Ángel Durán^{a,*}^aDepartment of Applied Mathematics, University of Valladolid, P/ Belén 15, 47011 Valladolid, Spain.*Correspondence to angeldm@uva.es

ABSTRACT

The numerical approximation of some asymptotic models for the propagation of internal and surface waves in 2D, proposed in e. g. [1, 2, 3], is studied. The talk will be focused on the semi-discretization in space of several initial-boundary-value problems with spectral Galerkin methods. Some error estimates will be derived (extending the results obtained in [4] for the one-dimensional case) and numerical experiments will illustrate the performance and applicability of the use of spectral methods for the numerical treatment of these models.

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A Second-Order Positivity-Preserving IMEX (PP-IMEX) Finite Volume Scheme for Compressible Flows on Staggered Meshes with Large Time-Steps

Thomas Harbreteau^{a,*}, Raphaële Herbin^{a,†}, Jean-Claude Latché^{b,‡}

^aAix-Marseille Université, I2M.

^bAutorité de Sûreté Nucléaire et Radioprotection (ASNR).

*Correspondence to thomas.harbreteau@univ-amu.fr; [†]raphaele.herbin@univ-amu.fr; [‡]jean-claude.latche@asnr.fr

ABSTRACT

In this work, we present a positivity-preserving quasi-second order in space and time finite volumes scheme on staggered meshes for Navier–Stokes equations. This is an on-going work destined to be used by the LIE team at the Autorité de Sûreté Nucléaire et de Radioprotection (ASNR) in a LES solver for turbulence simulations.

The space discretization is a finite volume method on staggered meshes and the time-stepping strategy is a 4-stage Implicit-Explicit (IMEX) Runge–Kutta (RK) method.

Convection is computed explicitly using a positivity-preserving Explicit RK (PP-ERK) method. It is positivity preserving thanks to a flux limitation in-between intermediate RK steps derived from [2] and the material CFL can be taken up to 4 times the usual CFL. The limiter used is the Algebraic-MUSCL (A-MUSCL) algorithm [1]. It computes quasi-second order positivity preserving inter-cell values by projecting high-order (not positivity preserving) inter-cell approximations on a stability interval given by the values of adjacent cells. It is an easy-to-implement limiter that works well regardless of the space dimension.

Acoustics, heat diffusion and viscous / dissipative effects are coupled and solved implicitly. The implicit time-stepping method is a non-standard diagonally implicit RK method with explicit first step (NSEDIRK) where every stage (including the last one) possesses an implicit part, for stability purpose. Thanks to the implicit computations, no acoustic CFL is required. Moreover, the 4-stage NSEDIRK is actually a 2-stage NSEDIRK rewritten as a 4-stage method, meaning that for a complete 4-stage iteration of the IMEX method, the implicit part is taken into account only twice, leading to computational cost reduction.

The resulting PP-IMEX method is positivity-preserving for the density, second-order in space and time and computations are 4 times faster than traditional IMEX methods with the same convection / acoustics and viscous effects splitting. The method is multi-dimensional, however the work is still in progress so only 1D numerical tests are performed.

Keywords: Compressible flows; Finite volume; Staggered; Segregated Runge–Kutta; Positivity-preserving

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Low-Froude Shallow Water Equations: An Enriched Discretization and Multigrid Approach

Daniel Inzunza^{a,*}, Vincent Perrier^a

^aTeam Cagire, INRIA Bordeaux Sud-Ouest, Laboratoire de Mathématiques et de leurs applications, Université de Pau et des Pays de l'Adour.

*Correspondence to daniel.inzunza-herrera@univ-uppa.fr

ABSTRACT

In the low-Froude regime, the shallow water equations present a well-known numerical challenge due to the separation between fast gravity waves and slower advective dynamics. This difference of scales imposes severe stability restrictions on fully explicit methods and motivates the use of semi-implicit formulations.

In this work, we consider a linear-implicit discretization in which the pressure-driven wave dynamics are treated implicitly, while the transport contribution is handled explicitly. This approach avoids the time-step restriction associated with the fastest waves and leads, at each time step, to the solution of a coupled wave system.

The spatial discretization combines a cellwise constant space for the water height with an enriched cell-centered vector space for the momentum. The enriched space retains a local finite-volume character and provides a suitable representation of the discrete incompressible structure arising in the low-Froude limit. To solve the resulting coupled height-momentum system, we investigate a geometric multigrid method with a relaxation strategy adapted to the discrete spaces and their additional local degrees of freedom. We discuss the shallow water formulation, the structure of the discrete wave problem, and preliminary numerical experiments concerning accuracy and multigrid performance for different meshes and Froude numbers. The development of a robust multigrid method for increasingly stiff regimes is ongoing work.

Well-Balanced High-Order WENO Finite-Difference Methods via a Local Flux Formulation

Maria Kazolea^a, Carlos Parés^b, Baifen Ren^{c,*}

^aINRIA, Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, 200 Avenue de la Vieille Tour, 33405 Talence cedex, France.

^bDepartamento de Análisis Matemático, Estadística e Investigación Operativa, y Matemática Aplicada, Universidad de Málaga, Bulevar Louis Pasteur, 31, 29010 Málaga, Spain.

^cSchool of Mathematics and Information Science, Shandong Technology and Business University, Yantai 264005, China.

*Correspondence to BBB.CCC@univ.gl

ABSTRACT

High-order well-balanced WENO schemes have already been developed for hyperbolic conservation laws (see for example [4]), where the effects of convective fluxes and source terms cancel each other for still water and for moving shallow water (see for example [5]).

Different strategies have been discussed in the literature to extend high-order WENO finite-difference methods to hyperbolic systems with source terms and/or non-conservative products. In [2], a strategy based on the so-called Local Flux approach was introduced. In this framework, source terms and non-conservative products are represented as the derivative of a generalized flux function that, unlike the Global Flux approach [1], is defined locally based on the choice of a family of paths. Two different techniques were proposed in that work to achieve the well-balanced property. First, the technique developed in [3], based on the computation of local equilibria, was extended. Second, a method relying on an appropriate choice of the family of paths was introduced. This strategy allows non-conservative products to be written as the derivative of a generalized flux function that is defined locally on the basis of the selected family of paths. WENO reconstructions are then applied to this generalized flux.

The goal of this work is to further investigate this approach, which in principle allows the design of well-balanced methods without the additional computational cost associated with computing local equilibria. The approach is tested to different hyperbolic systems like the Burgers equations, the nonlinear shallow water equations and the two layer shallow water equations.

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Partial Approximate Riemann Solvers for Discrete Dissipative Boundary Conditions of the Shallow-Water Equations

Geoffrey Beck^{a,*}, Ludovic Martaud^{a,†}

^aUniv Rennes, Inria Bretagne Atlantique.

*Correspondence to geoffrey.beck@inria.fr; [†]ludovic.martaud@inria.fr

ABSTRACT

This work concerns numerical approximations for a class of boundary conditions of the shallow-water equations. The discrete boundary conditions are defined with an extension of standard approximate Riemann solvers to enforce weak-consistency, positivity and discrete entropy stability on the boundary. Such a procedure is applied to the symmetric HLL Finite-Volume scheme and numerical tests show its relevancy.

This work concerns the Finite-Volume approximation of the weak solutions of the shallow-water equations posed in the half-plane $[0, +\infty)$. Denoting $g > 0$ the gravitational constant, $(h, q) \in \mathbb{R}_+^* \times \mathbb{R}$ the height and the discharge of the fluid, these equations write

$$\begin{cases} \partial_t \begin{pmatrix} h \\ q \end{pmatrix} + \partial_x \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix} = 0, & \forall (x, t) \in (\mathbb{R}_+^*)^2, \\ q(0, t) = q_0(t), & \forall t \geq 0, \end{cases} \quad (1)$$

where $q_0(t)$ stands for the given boundary data. Considering the entropy–entropy flux pair $\eta(h, q) = q^2/(2h) + gh^2/2$, $G(h, q) = q^3/(2h^2) + qgh$, it is well known [3] that the admissible solutions of (1) must satisfy $\partial_t \eta(h, q) + \partial_x G(h, q) \leq 0$. Hence, if $q_0(t) \leq 0, \forall t \geq 0$, then the following global entropy inequality occurs:

$$\frac{d}{dt} \int_0^{+\infty} \eta(h, q) dx \leq 0. \quad (2)$$

At a discrete level, the weak solutions $w = (h, q)^\top \in \mathbb{R}^2$ of (1) are approximated with a Finite-Volume scheme defined in a space-time mesh of steps $\Delta x, \Delta t > 0$ and with a numerical flux $\mathcal{F}(\cdot, \cdot)$. Such a scheme reads

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}(w_i^n, w_{i+1}^n) - \mathcal{F}(w_{i-1}^n, w_i^n) \right), \quad \forall i \geq 1, \quad (3)$$

where $(w_i^n)_{i \geq 0} \subset \mathbb{R}^2$ approximates $w(\cdot, t^n)$. The numerical scheme (3) requires an additional numerical procedure to compute the updated discrete boundary conditions $w_0^{n+1} = (h_0^{n+1}, q_0^{n+1})^\top$ but, on the contrary to $q(0, t^{n+1})$, the quantity $h(0, t^{n+1})$ is not explicitly given by (1).

In this regard, this work proposes to compute w_0^{n+1} with the following procedure:

$$\begin{aligned} h_0^{n+1} &= \frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{h}((x - x_{\frac{1}{2}})/\Delta t, w_0^n, w_1^n) dx, \\ q_0^{n+1} &= \mathcal{Q}(w_0^n, w_1^n, q_0), \end{aligned} \quad (4)$$

where $\tilde{h}(\cdot, w_0^n, w_1^n)$ is a partial approximate Riemann solver and $\mathcal{Q}(w_0^n, w_1^n, q_0)$ denotes a consistent approximation of the boundary data $q_0(t^{n+1})$. In the spirit of [1, 2], definitions are proposed to design $\tilde{h}(\cdot, w_0^n, w_1^n)$ in order to guarantee the consistency of h_0^{n+1} with $h(0, t^{n+1})$ and also to enforce a fully discrete version of

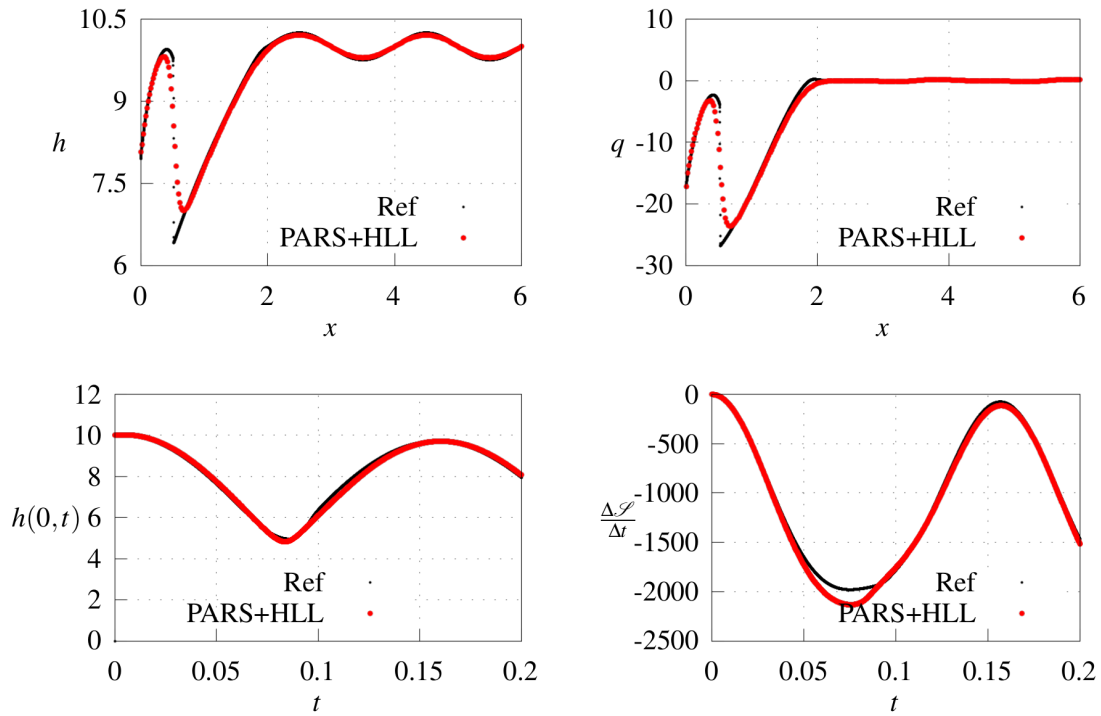


Figure 1: Numerical results for Partial Approximate Riemann Solvers (PARS) (4) endowed with the symmetric HLL scheme (3) in the interior domain. On the top : numerical results for h and q , on the bot : time evolution of $h(0,t)$ and $\Delta \mathcal{S} / \Delta t$ that stands for a discrete version of the left-hand side of (2).

(2). Endowing (3) with the standard symmetric HLL scheme, such a design is applied for several choices of partial approximate Riemann solvers in (4) and Figure 1 displays some numerical results.

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Minimally Implicit Runge–Kutta (MIRK) Methods for Multicomponent Relativistic Fluid Dynamics

I. Martínez-Donato^{a,b,*}, S. J. Bonanad-Hurtado^a, I. Cordero-Carrión^b, M. Perucho^{a,c}, J. M. Martí^{a,c}

^aDepartament d’Astronomia i Astrofísica, Universitat de València.

^bDepartament de Matemàtiques, Universitat de València.

^cObservatori Astronòmic, Universitat de València.

*Correspondence to izarne.martinez@uv.es

ABSTRACT

The numerical solution of hyperbolic balance laws with stiff source terms remains challenging, as standard explicit time integration methods require significantly small time steps due to stability conditions. Minimally Implicit Runge–Kutta (MIRK) schemes provide an attractive alternative by introducing only a minimal level of implicitness while offering good stability properties at the same time. These methods have been successfully applied to several hyperbolic systems, including the resistive relativistic magnetohydrodynamic equations [1], the neutrino transport equations [2], and the one-dimensional shallow water equations with Manning friction [3].

In this work, we investigate the application of MIRK schemes to the numerical solution of the hyperbolic balance laws governing the dynamics of a multicomponent gas with stiff source terms. This is useful in astrophysical scenarios such as the propagation of relativistic jets in active galaxies. The interaction between these jets and the surrounding inhomogeneous interstellar medium naturally gives rise to processes involving strong chemical exchange and radiative losses [4], thereby introducing stiffness to the system of equations. We investigate the construction of high-order MIRK schemes for this scenario and study their convergence and computational efficiency through a series of numerical experiments, including representative shock–cloud interaction simulations, with comparisons to standard explicit Runge–Kutta methods.

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A Multilayer Model with Plastic Transport in Rivers

Jorge Moya^b, Rui M. L. Ferreira^a, Enrique D. Fernández-Nieto^b, Ana M. Ricardo^{a,*}

^aCERIS, Instituto Superior Técnico, Universidade de Lisboa, Lisbon, Portugal.

^bDepartamento de Matemática Aplicada I, ETS Arquitectura, Universidad de Sevilla, Avda. Reina Mercedes 2, 41012 Sevilla, Spain.

*Correspondence to jmoya1@us.es

ABSTRACT

The transport of small plastic particles in river flows is a challenging environmental problem that requires physically consistent and computationally efficient models [2, 3, 4]. This is the main subject of TRAP project [1].

In this work we present a coupled framework combining a multilayer shallow water model [5, 6, 7] with particle dynamics governed by the Maxey–Riley equations [6, 8].

Originally derived for three-dimensional flows, the Maxey–Riley model describes the motion of small spherical particles under drag, added mass, pressure gradients, and Basset history forces [9]. The framework is further extendable to non-spherical (prolate ellipsoidal) particles, where translational dynamics follow generalized Maxey–Riley-type equations with shape-dependent coefficients, while rotational motion is governed by Jeffery’s equation [8], capturing orientation dynamics and Jeffery orbits through coupling with the local velocity-gradient tensor [7].

These force terms are consistently adapted to depth-averaged layered velocity fields, preserving essential vertical structure without resorting to fully three-dimensional simulations. Vertical particle segregation is thus resolved efficiently within the multilayer setting, in line with recent multilayer approaches for stratified and particulate flows [10, 11].

The history (Basset) force is treated via a sum-of-exponentials approximation of the $1/\sqrt{t}$ kernel [8], reducing computational complexity from $\mathcal{O}(N_t^2)$ to $\mathcal{O}(MN_t)$. Overall, the method provides an efficient and physically grounded tool for microplastic transport in riverine flows, capturing both vertical structure and anisotropic particle dynamics [5, 2].

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On Depth-Averaged Compressible Models with Free Surface or in Pressurized Domains

D. Bresch^a, E. D. Fernández-Nieto^{b,*}, J. Garres Díaz^c, P. Le Vourc'h^d, G. Narbona^b

^aLAMA UMR5127 CNRS, Univ. Savoie Mont-Blanc, France.

^bDepartamento de Matemática Aplicada I, Universidad de Sevilla, Spain.

^cDepartamento de Matemática Aplicada II, Universidad de Sevilla, Spain. ^dIMAG – UMR 5149 – Université de Montpellier, France.

*Correspondence to edofner@us.es

ABSTRACT

In this talk we present an ongoing project about the development of depth-averaged compressible models (see [1, 2, 3]). They are deduced by a depth-averaging of the compressible barotropic Navier–Stokes system, with the following equation of state:

$$p(\rho) = a\rho^\gamma,$$

with $a > 0$ and $\gamma > 1$, where ρ denotes the density of the fluid, and p its pressure. First, the case corresponding to $\gamma = 2$ will be considered for the case of a domain with a free surface. The case $\gamma \neq 2$ deserves special attention, because several of the computations and simplifications done in the previous case are not yet possible. The energy balance and comparison with the depth-averaged energy of the 3D system is presented. The following part of the talk follows an analogous structure but for the case of pressurized domains. Finally, we study the limit case when γ tends to infinity, allowing us to recover the incompressible shallow water equations.

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In-Cell Discontinuous Reconstruction Path-Conservative Methods for Conservative Systems in Primitive Variables: How to Skip Hybrid Approaches

Ernesto Pimentel-García^a, Mario Ricchiuto^b, Carlos Parés^{c,*}

^aDepartamento Matemática Aplicada, Universidad de Málaga, C/ Dr. Ortiz Ramos s/n, Campus de Teatinos, 29071, Málaga, Spain.

^bInria, Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, 200 Avenue de la Vieille Tour, 33405 Talence cedex, France.

^cDepartamento de Análisis Matemático, Estadística e Investigación Operativa, y Matemática Aplicada, Universidad de Málaga, Bulevar Louis Pasteur, 31, 29010, Málaga, Spain.

*Correspondence to erpigar@uma.es

ABSTRACT

We propose a numerical methodology for the treatment of conservative systems formulated exclusively in primitive variables, building upon the framework introduced in [1, 2, 3]. The main motivation is to design numerical methods capable of solving fluid models in primitive variables without altering shock waves. In several applications, the use of primitive variables is not only advantageous but mandatory, since formulations based on conserved variables may produce spurious oscillations near contact discontinuities. However, working solely with primitive variables is known to introduce convergence difficulties in the presence of shocks due to nonconservative products.

These difficulties are addressed here through the identification of a suitable family of paths and the incorporation of in-cell discontinuous reconstructions within a path-conservative framework. In smooth regions and across contact discontinuities, a standard path-conservative method is employed, while in the presence of shocks, an in-cell discontinuous reconstruction is locally activated to ensure the correct resolution of discontinuities. The approach is further extended to achieve high-order accuracy.

Unlike previous hybrid strategies, where primitive variables are used in smooth regions and conservative variables are required near shocks, the present methodology eliminates the need for conservative formulations. Numerical experiments for the Euler multimaterial and shear interface models demonstrate that the proposed scheme converges to the correct weak solution even in the presence of shocks and effectively suppresses spurious oscillations near contact discontinuities, all while working exclusively with primitive variables.

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Well-Balanced DG Schemes for Multidimensional Shallow Water Systems with Coriolis Forces: Evolving Fluctuations of Stationarity Preserving Continuous Finite Elements

Víctor G. Tabernero^a, Manuel J. Castro^a, J.A. García-Rodríguez^{b,*}, Mario Ricchiuto^c

^aDepartamento de Análisis Matemático, Estadística e Investigación Operativa y Matemática Aplicada, Universidad de Málaga, Campus de Teatinos s/n, 29080 Málaga, Spain.

^bCITIC and Department of Mathematics, Universidade da Coruña, Campus de Elviña s/n, 15071 A Coruña, Spain.

^cTeam CARDAMOM, Inria Bordeaux Sud-Ouest, Talence, France.

*Correspondence to jose.garcia.rodriguez@udc.es

ABSTRACT

This contribution presents a high-order well-balanced Discontinuous Galerkin (DG) framework for two-dimensional balance laws, with emphasis on rotating shallow-water systems with bathymetry and Coriolis forcing. The method preserves nontrivial steady states by computing fully local discrete equilibria on small cell patches, rather than solving global stationary problems. These local states are built with operators compatible with stationarity-preserving continuous finite element discretizations based on Global Flux Quadrature (GFq), and are then used in an equilibrium-plus-fluctuation DG update. The resulting scheme is high-order accurate away from equilibrium, exactly well-balanced for the targeted discrete stationary states, and naturally suited to parallel implementation.

Keywords: well-balanced schemes; discontinuous Galerkin methods; shallow-water equations; Coriolis forces; stationary solutions.

1 INTRODUCTION

We consider two-dimensional systems of balance laws

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{S}(\mathbf{u}, \mathbf{x}), \quad (\mathbf{x}, t) \in \Omega \times (0, T], \quad (1)$$

for which rotating shallow-water equations with bathymetry and Coriolis forcing provide a representative two-dimensional example. In this model $\mathbf{u} = (h, hu, hv)^T$, where h is the water height and hu, hv are the two discharge components, while the source term couples bed-slope effects with Coriolis acceleration. In geophysical regimes the relevant dynamics often consist of small perturbations of nontrivial equilibria, such as lake-at-rest, vortical, or geostrophic states. Standard high-order discretizations of (1) generally do not preserve the exact balance between flux gradients and source terms, and the resulting spurious imbalance can dominate the physically meaningful signal. The difficulty is especially severe in several space dimensions, where arbitrary stationary states are not known explicitly and global equilibrium solves are too costly for large simulations.

2 NUMERICAL SCHEME

We start from high-order DG schemes on structured meshes and aim to turn this underlying discretization into a well-balanced one. The DG approximation uses tensor-product nodal bases and quadrature-consistent volume and face integration. The well-balanced correction follows the Castro–Parés fluctuation strategy [2]: for each target cell, a stationary reference state $\mathbf{u}^*$ is computed on a small stencil, the DG polynomial is decomposed as $\mathbf{u} = \mathbf{u}^* + \bar{\mathbf{u}}$, and this decomposition is used to build the well-balanced residual. The discrete flux and source contributions of $\mathbf{u}^*$ are then subtracted and added consistently, so that the complete DG residual vanishes when the numerical solution coincides with the local stationary state.

The main contribution is a well-balanced DG scheme based on a local process, namely the fully local construction of the required stationary profiles. These local equilibria are represented in a continuous tensor-product finite element space using constrained operators inherited from the GFq stationarity-preserving framework [3]. In this formulation, the discrete divergence/source balance has a rich kernel that contains the stationary manifolds relevant to rotating shallow-water flows. On each patch, the stationary state is obtained by minimizing the discrete residual with analytic gradients and an efficient L-BFGS procedure. Since each patch problem is local, the preprocessing has low memory requirements and is well suited for parallel execution, while remaining compatible with the DG update.

3 NUMERICAL EXPERIMENTS

The numerical validation covers linear and nonlinear shallow-water systems with Coriolis terms, including analytic stationary vortices, perturbed equilibria, bottom-source configurations, rational vortices and low-Froude-number regimes. The well-balanced scheme is compared with the underlying non well-balanced DG method. In stationary tests, initializing with the compatible discrete equilibrium gives machine-precision preservation. For continuous analytic equilibria and perturbations, the proposed method strongly reduces spurious imbalance errors, keeps divergence diagnostics closer to zero, prevents long-time drift, and resolves the physically relevant waves more accurately. The same behavior is observed for higher-order configurations, and the low-Froude tests show a clear accuracy advantage as the dynamics approach the incompressible limit. Figure 1 reports representative long-time linear and nonlinear vortex tests initialized from analytic stationary states. The local WB technique computes the compatible discrete stationary profile on local stencils; after the adjustment perturbations leave the domain, the time evolution shows that this discrete equilibrium is preserved, whereas the underlying DG scheme accumulates imbalance errors and destroys the discrete stationary state.

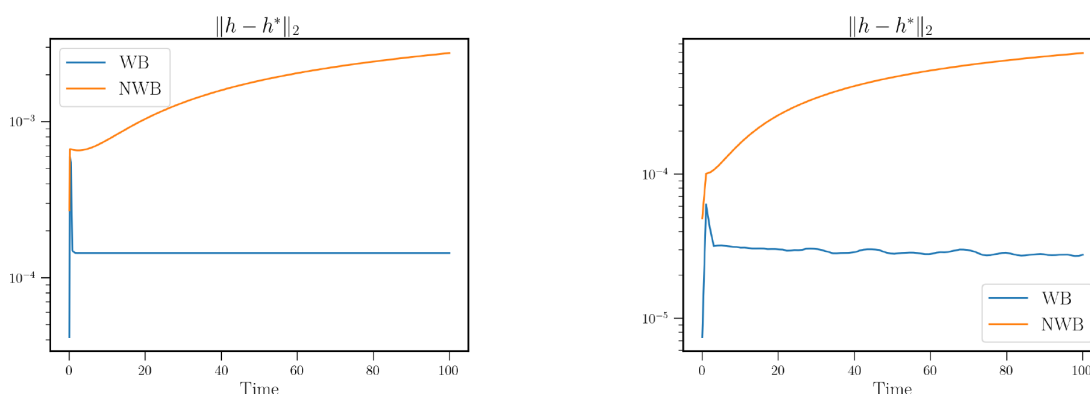


FIGURE 1: Long-time L^2 errors for the linear test (left) and the nonlinear stationary vortex (right): well-

Figure 1: Long-time L^2 errors for the linear test (left) and the nonlinear stationary vortex (right): well-balanced (WB) and non well-balanced (NWB) DG schemes.

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Modelling Equations of State for Relativistic Flow: Shock Waves and Phase Transitions in Non-Convex Hyperbolic Systems

Susana Serna^{a,*}^aDepartament de Matemàtiques, Universitat Autònoma de Barcelona.*Correspondence to susana.serna@uab.cat

ABSTRACT

This work surveys the modelling of Equations of State (EoS), which describe the thermodynamic properties of fluids, within the framework of relativistic fluid dynamics to investigate shock wave propagation and phase transition phenomena. Particular attention is given to regimes in which the classical convexity assumptions on the EoS break down, leading to non-convex hyperbolic systems of conservation laws. In this setting, the Riemann problem admits non-unique and non-classical solutions. In the relativistic regime, these include composite waves, undercompressive shocks, and phase boundaries that are not captured by classical theory. By combining thermodynamic consistency requirements, including causality, the Liu entropy condition, and the Rankine–Hugoniot jump conditions governing discontinuities, with tailored EoS models, we develop a unified analytical framework. This approach identifies admissible wave configurations and characterizes phase transition dynamics under extreme conditions relevant to astrophysical flows. The results demonstrate that incorporating non-convex EoS into relativistic hyperbolic systems provides a robust mathematical framework for describing complex shock structures and phase transitions beyond the scope of classical convex theory.

Modelling Dry Granular Flows over Curved Topographies

Sergio Valiente^{a,*}, Tomás Morales de Luna^a, José Garres-Díaz^b, Manuel Jesús Castro^a

^aUniversidad de Málaga, Spain.

^bUniversidad de Sevilla, Spain.

*Correspondence to sergiovaliente@uma.es

ABSTRACT

Depth-integrated models provide a fundamental tool for the description of gravity-driven flows in which one spatial dimension is much smaller than the others, and they are widely used in geophysical and industrial applications, including dry granular flows and avalanches. When such flows evolve over curved topographies, intrinsic or curvilinear coordinate systems are required to properly account for geometric effects arising from the dimensional reduction of the Navier–Stokes equations.

In this work, we propose a new geometric depth-integrated model for dry granular flows defined along a regular reference curve, in which the dependence of the geometric terms on both the curve parameter and the normal direction is explicitly retained. This leads to a more general formulation than existing intrinsic shallow models and introduces additional conservative variables that play a key role in ensuring mass conservation.

A second major contribution concerns the modelling of basal Coulomb friction in curvilinear coordinates. Extending recent results on corrected bed pressure formulations, we derive a generalized static equilibrium criterion consistent with the local geometry of arbitrary curves, thus avoiding incorrect predictions of the motion threshold. The resulting system can be written as a balance law with source terms and admits physically relevant stationary solutions corresponding to material at rest.

To accurately capture these equilibria, we develop a well-balanced finite volume scheme based on the Polynomial Viscosity Matrix framework combined with a hydrostatic reconstruction adapted to the proposed friction correction. Numerical experiments demonstrate the robustness of the method and highlight the importance of the geometric formulation and the corrected friction criterion for flows over curved geometries.

High-Order IMEX Local Discontinuous Galerkin Methods with Sparse Grids for High-Dimensional Nonlinear Parabolic PDEs

Joel P. Villarino^{a,*}, Manuel J. Castro^b, J.A. García-Rodríguez^a, J.G. López-Salas^a

^aCITIC and Department of Mathematics, Universidade da Coruña, Campus de Elviña s/n, 15071 A Coruña, Spain.

^bDepartamento de Análisis Matemático, Estadística e Investigación Operativa y Matemática Aplicada, Universidad de Málaga, Campus de Teatinos s/n, 29080 Málaga, Spain.

*Correspondence to joel.perez.villarino@udc.es

ABSTRACT

This contribution presents a high-order implicit–explicit local discontinuous Galerkin (IMEX–LDG) framework on sparse grids for scalar nonlinear parabolic problems in moderate and high spatial dimensions. The method combines robust LDG discretizations for convection–diffusion operators, IMEX Runge–Kutta time integrators that treat stiff diffusion implicitly and nonlinear transport and reaction terms explicitly, and sparse-grid combination techniques that reduce the dimensional complexity of tensor-product meshes. The resulting scheme is implemented in a dimension-independent MPI code.

Keywords: local discontinuous Galerkin methods; IMEX Runge–Kutta methods; sparse grids; nonlinear parabolic PDEs; high-dimensional parabolic PDEs.

1 INTRODUCTION

We consider d -dimensional scalar convection–diffusion–reaction problems of the form

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \nabla_{\mathbf{x}} \cdot (A \nabla_{\mathbf{x}} u) + s(u), \quad (\mathbf{x}, t) \in \Omega \times (0, T),$$

where A is symmetric positive semi-definite and the flux and source terms may be nonlinear. Such models arise, for example, in phase-field dynamics, transport-diffusion systems and quantitative finance. The development of numerical schemes for such nonlinear problems in high dimension remains a major mathematical and computational challenge. Their numerical approximation involves several coupled difficulties. First, convection-dominated regimes require robust upwinding without destroying high-order accuracy. Second, nonlinear fluxes and stiff source terms should be handled without turning every time step into a large nonlinear solve. Third, after spatial discretization the parabolic operator imposes severe explicit time-step restrictions of diffusive type. Finally, in moderate and high dimensions, tensor-product grids make both the number of degrees of freedom and the associated implicit linear systems grow very rapidly. The goal is therefore to combine high-order accuracy, robustness, stiff stability and dimensional efficiency in a single computational framework.

2 NUMERICAL SCHEME

The spatial semidiscretization is addressed with the LDG method [1]. The second-order equation is rewritten as a first-order system by introducing auxiliary gradient variables. On each structured tensor-product cell we use discontinuous polynomial bases, which preserve locality and lead to diagonal or block-local mass operators. Convective terms are treated with upwind numerical fluxes, such as Rusanov fluxes, providing the robustness needed in convection-dominated regimes. Diffusion is handled through alternating LDG fluxes for the primal and gradient variables, giving a stable high-order discretization while keeping the algebraic structure sparse and parallelizable. The stiffness and nonlinearity are handled by IMEX Runge–Kutta time marching [2, 3]. The diffusive LDG operator is advanced implicitly, which removes the prohibitive

parabolic restriction $\Delta t = O(h^2)$, whereas the nonlinear convection and source terms are evaluated explicitly. This separation avoids fully implicit nonlinear systems and leaves, at each implicit stage, an elliptic-type linear problem that can be solved with Krylov methods. Depending on the size of the run, the solver uses assembled preconditioned operators or matrix-free products to reduce memory consumption. The dimensional bottleneck is tackled with the sparse-grid combination technique [4]. Rather than solving the LDG–IMEX scheme on a single full tensor grid at the finest resolution, the method solves it on a family of semi-coarsened anisotropic grids and recombines the partial solutions. For smooth enough solutions, this retains the convergence trends of full grids while reducing the effective number of degrees of freedom by orders of magnitude. Since the component-grid solves are naturally independent, the full framework is implemented in a dimension-independent C++/MPI code.

3 NUMERICAL EXPERIMENTS

First, the method is validated on academic linear transport–diffusion tests, including Gaussian-pulse benchmarks, and nonlinear Burgers equations. It is then assessed on Allen–Cahn phase-field models and a five-dimensional Black–Scholes equation with credit valuation adjustments, with tests performed up to spatial dimension five. The 3D linear benchmark reports clean third-order convergence for quadratic LDG polynomials and highlights the accuracy-versus-time gain obtained with the sparse-grid combination technique. As can be seen from Table 1, the expected convergence order is obtained for polynomial degree $\nu = 2$, comparing the reference full-grid refinement with two sparse-grid strategies: refinement of the base mesh for a fixed maximum level, and refinement in the sparse-grid levels for a fixed base mesh. Efficiency for both second- and third-order discretizations is shown in Figure 1. As expected, the most efficient option is the level refinement in the combination technique.

Table 1: 3D linear convergence, polynomial degree $\nu = 2$.

Case	Config.	$\ e\ _1$	EOC	$\ e\ _2$	EOC	Time (ms)
Full	$\alpha = 40$	1.24e-3	–	1.01e-3	–	2.84e4
	$\alpha = 80$	1.55e-4	3.00	1.26e-4	3.00	5.24e5
	$\alpha = 160$	1.93e-5	3.00	1.58e-5	3.00	4.41e6
	$\alpha = 320$	2.43e-6	2.99	1.97e-6	3.00	2.08e8
CT, $N_L = 3$	$N_B = 10$	2.56e-4	–	1.88e-4	–	2.25e5
	$N_B = 20$	2.37e-5	3.43	1.96e-5	3.26	3.10e6
	$N_B = 40$	2.98e-6	2.99	2.47e-6	2.99	3.71e7
	$N_B = 80$	3.84e-7	2.96	3.11e-7	2.99	1.69e8
CT, $N_B = 10$	$N_L = 3$	2.56e-4	–	1.88e-4	–	2.22e5
	$N_L = 4$	3.44e-5	2.89	2.58e-5	2.87	1.21e6
	$N_L = 5$	4.60e-6	2.90	3.47e-6	2.89	4.92e6
	$N_L = 6$	6.09e-7	2.92	4.62e-7	2.91	2.58e7

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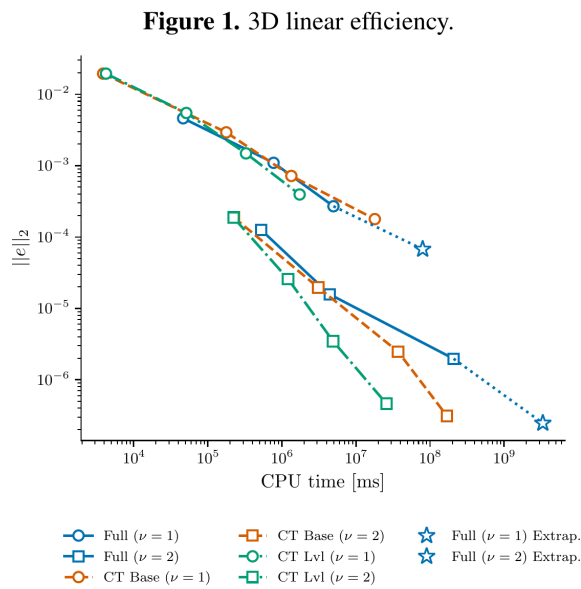


Figure 1: 3D linear efficiency.

List of Participants

39 participants from 3 countries (Portugal: 11, Spain: 16, France: 12), sorted alphabetically by surname. Participants marked with † are speakers who do not appear in the registration list of 7 October 2026; their affiliation is taken from their abstract.

#	Surname	First name	Affiliation	Country
1	Barbeiro	Sílvia	CMUC, Department of Mathematics, University of Coimbra	Portugal
2	Barsukow	Wasilij	CNRS, Institut de Mathématiques de Bordeaux (IMB)	France
3	Berthon	Christophe	Nantes Université	France
4	Boscheri	Walter	CNRS, Laboratoire de Mathématiques (LAMA), Université Savoie Mont Blanc	France
5	Boussard	Marie	Inria Paris; Laboratoire Jacques-Louis Lions, Sorbonne Université	France
6	Branco	João	Polytechnic University of Coimbra (ISEC)	Portugal
7	Castro Díaz	Manuel J.	University of Málaga	Spain
8	Clain	Stéphane	University of Coimbra	Portugal
9	Compain	Marie	Laboratoire de Mathématiques Jean Leray (LMJL), Nantes Université	France
10	Cordero-Carrión	Isabel	University of Valencia	Spain
11	Cotrim	Diogo	University of Coimbra	Portugal
12	Durán	Ángel	University of Valladolid	Spain
13	Fernández-Nieto	Enrique D.	Departamento de Matemática Aplicada I, University of Seville	Spain
14	Ferrás	Luís L.	CEFT, ALiCE, Faculty of Engineering, University of Porto	Portugal
15	Figueiredo	Jorge	University of Minho	Portugal
16	García Rodríguez	José Antonio	University of A Coruña	Spain
17	Gómez-Bueno	Irene	University of Málaga	Spain
18	Harbreteau	Thomas	Aix-Marseille Université, I2M	France
19	Inzunza	Daniel	Université de Pau et des Pays de l'Adour (UPPA)	France
20	Kazolea	Maria	Inria center at the University of Bordeaux	France
21	Lima	Pedro	Instituto Superior Técnico, University of Lisbon	Portugal
22	Loubère	Raphaël	Institut de Mathématiques de Bordeaux (IMB)	France
23	Machado	Gaspar	Centre of Mathematics, University of Minho	Portugal
24	Malheiro	M. Teresa	Department of Mathematics, Centre of Mathematics, University of Minho	Portugal
25	Martaud	Ludovic	Inria, Univ Rennes	France
26	Martínez-Donato	Izarne	University of Valencia	Spain
27	Morales de Luna	Tomás	University of Málaga	Spain
28	Moya	Jorge	University of Seville	Spain
29	Mulet	Pep	University of Valencia	Spain
30	Núñez Fernández	Carlos	University of Seville	Spain
31	Parisot	Martin	Inria center at the University of Bordeaux	France
32	Pena	Gonçalo	University of Coimbra	Portugal
33	Pereira	Paulo	Department of Mathematics, Centre of Mathematics, University of Minho	Portugal

#	Surname	First name	Affiliation	Country
34	Pérez Villarino	Joel	University of A Coruña	Spain
35	Pimentel-García	Ernesto	University of Málaga	Spain
36	Pla-Martos	Francisco	University of Castilla-La Mancha (UCLM)	Spain
37	Serna	Susana	Universitat Autònoma de Barcelona	Spain
38	Valiente Ávila	Sergio	University of Málaga	Spain
39	Vignal	Marie-Hélène	Institut Camille Jordan, Université Lyon 1	France
40	Ramírez Fuentes	José Ignacio	University of Málaga	Spain