Scalar Quantum Field Theory on AdS spacetimes: Boundary conditions and Hadamard states

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Tours – Institut Denis Poisson – 05/02/2026

Dipartimento di Fisica – Università di Pavia

Outline of the Talk

- Scalar Fields on AdS Spacetimes
- Boundary Conditions and Quantization: Mode Expansion
- Unruh-de Witt Detector and anti-Hawking effect
- The Hadamard expansion

Based on

- C. D. and H. R. C. Ferreira, Phys. Rev. D **94** (2016) no.12, 125016 & Rev. Math. Phys. **30** (2017) no.02, 1850004
- C. D., N. Drago and H. Ferreira, Lett. Math. Phys. **109** (2019) no.10, 2157–2186
- L. De Souza Campos and C. D., Phys. Lett. B **816** (2021), 136198 & Phys. Rev. D **103** (2021) no.2, 025021
- B. Costeri, C. D., B. A. Juárez-Aubry and R. D. Singh, [arXiv:2509.26035 [math-ph]].

The Poincaré patch of AdS_{d+1}

In the Poincaré chart, PAdS_{d+1}, the metric reads

$$ds^2 = \frac{\ell^2}{z^2}[-dt^2 + dz^2 + \delta^{ij} dx_i dx_j], \quad i, j = 1, ..., d-1$$

where $z > 0$. Observe that

1. PAdS_{d+1} is conformally related to the upper half plane $(\mathring{\mathbb{H}}^{d+1}, \eta)$ with conformal factor $\Omega = \frac{z}{\ell}$.
2. The hyperplane $z = 0$ in $\mathbb{R}^{d+1}$ is the conformal boundary of PAdS_{d+1}.

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AQFT - I : Scalar field

Goal: Outline AQFT via a good example!

Basic assumptions:

- (M, g) is an arbitrary $4D$ globally hyperbolic spacetime. Hence, up to an isometry, $M \simeq \mathbb{R} \times \Sigma$

$$ds^2 = -\beta dt^2 + h_t, \quad \beta \in C^\infty(M; \mathbb{R}^+) \text{ and } h_t \in Riem(\Sigma),\ \forall t \in \mathbb{R}$$

- $\phi : M \to \mathbb{R}$ is a *conformally coupled real scalar field*

$$P\phi = \left(-\Box + \frac{R}{6}\right)\phi = 0,$$

The space of all smooth solutions is

$$\mathcal{S}_P(M) = \{\phi \in C^\infty(M) \mid \exists f \in C^\infty_{tc}(M) \text{ and } \phi = G(f)\},$$

where $G = G^+ - G^-$ is the causal propagator.

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AQFT - II : Useful facts

It will be useful later to keep in mind that

- $f \in C^\infty(M)$ is **timelike compact (tc)** iff

$$\text{supp}(f) \cap J^+(p) \text{ and } \text{supp}(f) \cap J^-(p) \text{ is compact } \forall p \in M.$$

- $G^\pm : C^\infty_{tc}(M) \to C^\infty(M)$ are the advanced $(+)$ and retarded $(-)$ fundamental solutions for P such that
  1. $P \circ G^\pm = G^\pm \circ P = id_{C^\infty_{tc}(M)}$
  2. $\forall f \in C^\infty_{tc}(M)$, it holds $\text{supp}(G^\pm(f)) \subseteq J^\mp(\text{supp}(f))$

- All *dynamical configurations* of a real scalar field are

$$\mathcal{S}_P(M) \simeq \frac{C^\infty_{tc}(M)}{P[C^\infty_{tc}(M)]}.$$

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AQFT - III : Classical Observables

Paradigm: A *classical observable* is an assignment of a real number to each dynamical configuration.

- for every $\alpha \in C_0^\infty(M)$, define $F_\alpha : C^\infty(M) \to \mathbb{R}$

$$F_\alpha(\phi) = (\alpha, \phi) \doteq \int_M d\mu_g \, \phi(x)\alpha(x),$$

- Implement *dynamics* at a dual level: For all $\phi \in \mathcal{S}_P(M)$

$$0 = F_\alpha(P\phi) = (\alpha, P\phi) = (P^*\alpha, \phi) = (P\alpha, \phi).$$

- We have built the following

$$\forall [\alpha] \in \frac{C_0^\infty(M)}{P[C_0^\infty(M)]} \mapsto \mathcal{F}_{[\alpha]} : \mathcal{S}_P(M) \to \mathbb{R}$$

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AQFT - IV : Algebra of Observables

1. Construct the unital *Borchers-Uhlmann* $*$-algebra

$$\mathcal{T}(M) = \bigoplus_{n=0}^{\infty} \mathcal{E}^{obs}(M)^{\otimes n},$$

where $\mathcal{E}^{obs}(M)^0 \doteq \mathbb{C}$ and the $*$-operation is complex conjugation.

2. Construct the ideal $\mathcal{I}(M) \subset \mathcal{T}(M)$ generated by elements of the form

$$[\alpha] \otimes [\alpha'] - [\alpha'] \otimes [\alpha] - iG([\alpha], [\alpha'])\mathbb{I} \quad \text{(CCR)}$$

where $\mathbb{I}$ is the unit in $\mathcal{T}(M)$ and

$$G([\alpha], [\alpha']) \doteq (\alpha, G(\alpha')) = \int_M d\mu_g(x)\, \alpha(x) G(\alpha')(x).$$

3. Define the **Algebra of Fields**

$$\mathcal{F}(M) \doteq \frac{\mathcal{T}(M)}{\mathcal{I}(M)}.$$

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AQFT - V : States

We need still the notion of an algebraic quantum state

For any unital $*$-algebra $\mathcal{A}$, a state is a linear map $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(\mathbb{I}) = 1\ (normal.) \quad \text{and} \quad \omega(a^* a) \geq 0, \ \forall a \in \mathcal{A}\ (posit.),$$

where e is the unit in $\mathcal{A}$.

We recover the probabilistic interpretation of quantum theories via:

- **GNS theorem**: $(\omega, \mathcal{A}) \mapsto (\mathcal{D}_\omega, \pi_\omega, \Omega_\omega)$
  - $\mathcal{D}_\omega$ is a dense subspace of a Hilbert space $\mathcal{H}_\omega$.
  - $\pi_\omega : \mathcal{A} \to \mathcal{L}(\mathcal{D}_\omega)$ is a representation
  - $\Omega_\omega \in \mathcal{D}_\omega$ such that $\|\Omega_\omega\| = 1$ and $\mathcal{H}_\omega = \overline{\pi_\omega(\mathcal{A})\Omega}$.

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega)_{\mathcal{H}_\omega}\,.$$

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AQFT - VI : Hadamard States - why?

Are all states physically acceptable?

Not in the slightest

- Minimal requirements are:
    - existence of a mathematically well-behaved, covariant notion of Wick polynomials to deal with interactions,
    - same UV behaviour of the Minkowski vacuum ,
    - quantum fluctuations of all observables finite.

Answer: **Hadamard States**

- Add-on: Invariance under the action of all isometries.

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AQFT - VII : Hadamard States - how?

How do we characterize Hadamard states?

- Notice that choosing a state $\omega : \mathcal{F}(M) \to \mathbb{C}$ is equivalent to assigning

$$\omega_n(\alpha_1, ..., \alpha_n), \forall n \in \mathbb{N} \text{ and } \forall \alpha_i \in C_0^\infty(M)$$

so that ω is normalized, positive and accounts for the CCRs and dynamics.

- A special class of states are **quasi-free/Gaussian**, *i.e.*, for all $n \in \mathbb{N}$

$$\omega_{2n+1} = 0, \quad \omega_{2n}(\alpha_1, ..., \alpha_n) = \sum_{\pi_{2n} \in S'_{2n}} \prod_{i=1}^{n} \omega_2\left(\alpha_{\pi_{2n}(i-1)}, \alpha_{\pi_{2n}(i)}\right).$$

ω is constructed out of the two-point function $\omega_2 \in \mathcal{D}'(M \times M)$.

Definition

A quasi-free state ω is called **Hadamard** (Radzikowski (1996)) iff

$$WF(\omega_2) = \left\{(x, y, k_x, -k_y) \in T^*M^2 \setminus \{\mathbf{0}\} \mid (x, k_x) \sim (y, k_y),\ k_x \triangleright 0\right\}.$$

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Building Hadamard states

How many Hadamard states do we know?

- Existence: Known for free fields via deformation arguments (S. A. Fulling, F. J. Narcowich and R. M. Wald, (1981)) or on static spacetimes (Sahlmann & Verch - 2000)
- Explicit construction: Known in highly symmetric spacetimes, *e.g.*, Bunch Davies state in de Sitter or states for scalar fields on FRW (Olbermann - (2007), Them & Brum (2013), Degner (2013), Brum & Fredenhagen (2014))
- Functional Analytic method: Exploit pseudodifferential calculus:
  - Gérard & Wrochna - Comm. Math. Phys. **325** (2014), 713
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Exploit **the asymptotic structure**.

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Exploit **the asymptotic structure**.

Klein-Gordon field in PAdS_{d+1}

Consider $\phi : \mathrm{PAdS}_{d+1} \to \mathbb{R}$

$$P\phi = (\Box_{\mathrm{PAdS}} - m_0^2 - \xi R)\phi = 0 \quad \xi \in \mathbb{R} \text{ and } R = -d(d+1)$$

- *Conformal rescaling* $\longrightarrow \Phi \doteq \Omega^{\frac{1-d}{2}}\phi : \mathring{\mathbb{H}}^{d+1} \to \mathbb{R}$ obeys

$$P_\eta \Phi \doteq \left(\Box_\eta - \frac{m^2}{z^2}\right)\Phi = 0, \quad m^2 = m_0^2 + (\xi - \frac{d-1}{4d}R).$$

- We can expand

$$\Phi(\underline{x}, z) = \int_{\mathbb{R}^d} \frac{d^d \underline{k}}{(2\pi)^{\frac{d}{2}}} e^{i\underline{k}\cdot\underline{x}} \widehat{\Phi}_{\underline{k}}(z)$$

yields ($\lambda \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2$)

$$P_\eta \Phi = 0 \iff L\widehat{\Phi}_{\underline{k}} = \left(-\frac{d^2}{dz^2} + \frac{m^2}{z^2} - \lambda\right)\widehat{\Phi}_{\underline{k}} = 0.$$

This is a **singular Sturm-Liouville equation on** $(0, \infty)$.

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- *Conformal rescaling* $\longrightarrow \Phi \doteq \Omega^{\frac{1-d}{2}}\phi : \mathring{\mathbb{H}}^{d+1} \to \mathbb{R}$ obeys

$$P_\eta \Phi \doteq \left(\Box_\eta - \frac{m^2}{z^2}\right)\Phi = 0, \quad m^2 = m_0^2 + (\xi - \frac{d-1}{4d}R).$$

- We can expand

$$\Phi(\underline{x}, z) = \int_{\mathbb{R}^d} \frac{d^d \underline{k}}{(2\pi)^{\frac{d}{2}}} e^{i\underline{k}\cdot\underline{x}} \widehat{\Phi}_{\underline{k}}(z)$$

yields ($\lambda \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2$)

$$P_\eta \Phi = 0 \Longleftrightarrow L\widehat{\Phi}_{\underline{k}} = \left(-\frac{d^2}{dz^2} + \frac{m^2}{z^2} - \lambda\right)\widehat{\Phi}_{\underline{k}} = 0.$$

This is a **singular Sturm-Liouville equation on** $(0, \infty)$.

Klein-Gordon field in PAdS_{d+1}

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The Endpoint Classification - I

The most general solution of $L\widehat{\Phi}_{\underline{k}} = \lambda\widehat{\Phi}_{\underline{k}}$ is for $\lambda > 0$

$$\widehat{\Phi}_{\underline{k}}(z) = a(\underline{k})\sqrt{z}J_\nu(\sqrt{\lambda}z) + b(\underline{k})\sqrt{z}Y_\nu(\sqrt{\lambda}z),$$

where $\nu = \frac{1}{2}\sqrt{1+4m^2} \geq 0$.... $m^2 \in [-\frac{1}{4}, \infty)$, the BF bound.

Which boundary conditions are allowed? How do we implement them?

- $\sqrt{z}J_\nu(\sqrt{\lambda}z) \propto_{z\to 0} z^{\nu+\frac{1}{2}}$ and $\sqrt{z}Y_\nu(\sqrt{\lambda}z) \propto_{z\to 0} z^{-\nu+\frac{1}{2}}$

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Singular Sturm-Liouville theory is the answer

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Goal: Implement boundary conditions at $z = 0$

1. Choose $\Phi_1(z)$ the *principal solution* at $z = 0$, i.e. the unique one (up to scalar multiples)

$$\lim_{z \to 0} \frac{\Phi_1(z)}{\Phi(z)} = 0 \quad \forall \Phi(z) \mid L\Phi = \lambda\Phi,\ \lambda \in \mathbb{C}.$$

2. Pick a *second L^2-solution* $\Phi_2(z)$, linearly indep. from Φ_1 (non unique)
3. Observe that, up to a scalar multiple, $\exists \alpha \in [0, \pi)$ such that

$$\Phi(z) = \cos\alpha\, \Phi_1(z) + \sin\alpha\, \Phi_2(z),$$

and that, for a regular endpoint at $z = 0$,

$$\cos\alpha\, \Phi(0) + \sin\alpha\, \Phi'(0) = 0 \Longleftrightarrow \cos\alpha\, W_z[\Phi, \Phi_1] + \sin\alpha\, W_z[\Phi, \Phi_2] = 0$$

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Recall that we consider $L = -\frac{d^2}{dz^2} + \frac{m^2}{z^2}$ and $\lambda = q^2 \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2$.

The fundamental pair of solutions $(\widehat{\Phi}^1_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}})$ is

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$\nu = \frac{1}{2}\sqrt{1+4m^2}$	Classification of $z=0$	Boundary condition at $z=0$
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$\nu \in [0,1),\, \nu \neq \frac{1}{2}$	Limit-circle (LC)	$-\cot(\alpha)\, W_z\left[\widehat{\Phi}_{\underline{k}}, \widehat{\Phi}^1_{\underline{k}}\right] + W_z\left[\widehat{\Phi}_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}}\right] = 0$
$\nu \in [1,\infty)$	Limit-point (LP)	Not required

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Ground States - Mode Expansion I

1. Let $\omega_{2,\mathbb{H}} \doteq (zz')^{\frac{1-d}{2}}\omega_2 \in \mathcal{D}'(\mathring{\mathbb{H}}^{d+1} \times \mathring{\mathbb{H}}^{d+1})$. It holds

$$(P_\eta \otimes \mathbb{I})\omega_{2,\mathbb{H}} = (\mathbb{I} \otimes P_\eta)\omega_{2,\mathbb{H}} = 0.$$

2. Consider the Fourier transform along $\mathbb{R}^d \ni \underline{x}$. Integral kernel:

$$\omega_{2,\mathbb{H}}(x,x') = \lim_{\epsilon\to 0^+}\int_0^\infty \frac{d\omega}{\sqrt{2\pi}} e^{i\omega(t-t'-i\epsilon)}\int_0^\infty dk\left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr)\widehat{\omega}_{2,\underline{k}}(z,z').$$

with

$$(L\otimes\mathbb{I})\widehat{\omega}_{2,\underline{k}} = (\mathbb{I}\otimes L)\widehat{\omega}_{2,\underline{k}} = \lambda\widehat{\omega}_{2,\underline{k}}, \quad L = -\frac{d^2}{dz^2} + \frac{m^2}{z^2}.$$

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The case with $\nu \in (0,1)$

In order to construct $\widehat{\omega}_{2,\underline{k}}$, we need that $iG(x,x') \doteq \omega_{2,\mathbb{H}}(x,x') - \omega_{2,\mathbb{H}}(x',x)$,

$$G(x,x')|_{t=t'} = 0, \quad \partial_t G(x,x') = -\partial_{t'} G(x,x')|_{t=t'} = \delta(x,x').$$

Setting $c = \cot\alpha$

$$\widehat{\omega}_{2,\underline{k}} = \sqrt{zz'} \int_0^\infty dq\, q \frac{[cJ_\nu(qz) - q^{2\nu} J_{-\nu}(qz)][cJ_\nu(qz') - q^{2\nu} J_{-\nu}(qz')]}{c^2 - 2cq^{2\nu}\cos(\pi\nu) + q^{4\nu}}, \quad c \leq 0$$

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Open Questions

Unknown Properties:

1. What is the most general class of allowed boundary conditions?
2. Setting $G^+ = \Theta(t-t')G$ and $G^- = -\Theta(t'-t)G$ is it true that

$$\mathrm{supp}(G^\pm(f)) \subseteq J^\pm(\mathrm{supp}(f)) \quad \forall f \in C_0^\infty(\mathring{\mathbb{H}}^{d+1}).$$

3. What are the singularities of $\omega_{2,\mathbb{H}}$:

$$\mathrm{WF}(\omega_{2,\mathbb{H}}) = ?$$

4. Can we construct a local form for $\omega_{2,\mathbb{H}}$ and for $G^\pm$?

Unruh-De Witt Detectors

Given $\Phi : M \to \mathbb{R}$ a real scalar field

Detector $\longleftrightarrow$ 2 level system $\{|0\rangle, |\Omega\rangle\}$

$$H_{int} = c\chi(\tau)\Phi(x(\tau)) \otimes \mu(\tau), \quad c \in \mathbb{R} \text{ and } \chi \in C_0^\infty(\mathbb{R}),$$

- $x(\tau)$ is the detector **worldline** (stationary),
- $\mu(\tau) = |\Omega\rangle\langle 0| e^{i\Omega\tau} + |0\rangle\langle\Omega| e^{-i\Omega\tau}$ – **monopole-moment operator**

$$H = H_\Phi \otimes \mathbb{I} + \mathbb{I} \otimes H_D + H_{int}.$$

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Quantities of Interest

We are interested in the *detector response function*

$$\mathcal{F} = \int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau_f} d\tau' e^{-i\Omega(\tau-\tau')}\chi(\tau)\chi(\tau')\omega_2(x(\tau)x'(\tau')),$$

where ω_2 is the two-point function of Φ.

Assuming infinite interaction time, *i.e.*, $\chi(\tau) \to 1$ we look at

- detector response function

$$\int_{\mathbb{R}^2} d\tau d\tau' e^{-i\Omega(\tau-\tau')}\omega_2(x(\tau)x'(\tau')),$$

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$$\dot{\mathcal{F}} = \int_{\mathbb{R}} ds e^{-i\Omega s}\omega_2(x(s)).$$

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Anti-Hawking effect[1]

Consider a static black-hole spacetime $M = \mathbb{R} \times I \times \Sigma_j$

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Sigma_j(\theta_1, \dots, \theta_{n-2})$$

1. M has a *bifurcate horizon* with **surface gravity** κ_h,
2. The detector measures a local Hawking temperature $T_D = \frac{\kappa_h}{2\pi\sqrt{f(r)}}$.
3. The response function of the detector satisfies *detailed balance* at T_D^{-1}

$$\frac{\dot{\mathcal{F}}(\Omega)}{\dot{\mathcal{F}}(-\Omega)} = e^{-\beta\sqrt{f(r)}\Omega}$$

Anti-Hawking Effect $\Longleftrightarrow \frac{\partial \mathcal{F}(T_D)}{\partial T_D} < 0$

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Static BTZ - Geometry

It is a *3D black hole* spacetime:

$$ds^2 = -\left(\frac{r^2}{L^2} - M\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} - M} + r^2 d\theta^2.$$

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- It has a conformal *timelike boundary* at $r \to \infty$

Two-point functions can be constructed as in $PAdS_{d+1}$.

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Static BTZ - Two-point Function - I

Consider a massless, conformally coupled, real scalar field

$$\Phi : \mathrm{BTZ} \to \mathbb{R}, \quad \left(\Box_g - \frac{3}{4}\right)\Phi = 0.$$

We consider $z = \frac{r^2 - L^2 M}{L^2 M} \in (0,1)$ and

$$\Phi(t,z,\theta) = \int_{\mathbb{R}} d\omega \sum_{l \in \mathbb{Z}} e^{i(\omega t + l\theta)} R_{\omega,l,\gamma}(z)$$

where

$$R_{\omega,l,\gamma}(z) = z^{\frac{i\omega}{2r_H}}(1-z)^{\frac{3}{4}}\left(\cos\gamma F_1(1-z) + \sin\gamma(1-z)^{-\frac{1}{2}} F_2(1-z)\right)$$

where F_1, F_2 are hypergeometric functions depending on ω, M, l.

Static BTZ - Two-point Function - II

Ground and thermal states with **Robin boundary conditions** at $T = \frac{\sqrt{M}}{2\pi L}$

$$\omega_2(x,x') = \lim_{\epsilon \to 0^+} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \Theta(\omega) e^{-i\omega(t-t'-i\epsilon)} \mathcal{N}_l R_{\omega,l,\gamma}(z) R_{\omega,l,\gamma}(z') e^{il(\theta-\theta')}$$

$$\omega_{2,T}(x,x') = \lim_{\epsilon \to 0^+} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \Theta(\omega) \left[\frac{e^{\frac{\omega}{T}} e^{-i\omega(t-t'-i\epsilon)} + e^{i\omega(t-t'+i\epsilon)}}{e^{\frac{\omega}{T}} - 1} \right]$$

$$\mathcal{N}_l R_{\omega,l,\gamma}(z) R_{\omega,l,\gamma}(z') e^{il(\theta-\theta')}$$

On a static trajectory $x(\tau) = (\tau, z_D, \theta_D)$ the response function reads for $\Omega < 0$

$$\dot{\mathcal{F}}_0 = \sum_{l \in \mathbb{Z}} \frac{\mathcal{N}_l}{2\pi} R^2_{\omega,l,\gamma}(z_D) \Big|_{\omega = -\sqrt{|g_{00}(z_D)|}\Omega},$$

and

$$\dot{\mathcal{F}} = \frac{\mathrm{sign}(\Omega)}{e^{\mathrm{sign}(\Omega)\frac{\omega}{T}} - 1} \Big|_{\omega = \sqrt{|g_{00}(z_D)|}|\Omega|} \dot{\mathcal{F}}_0.$$

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Static BTZ - Anti-Hawking Effect - I[2]

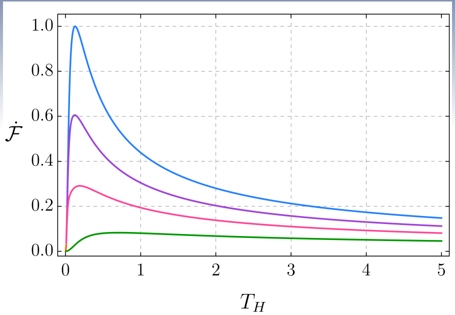
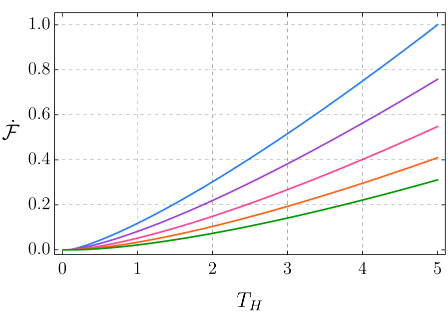


Figure: $l = 0$ contribution to $\dot{\mathcal{F}}_0$ as a function of T_H for $r_h = 1$, $\Omega = 0.1$ and different boundary conditions; from top to bottom, respectively, $\gamma = (0.50, 0.47, 0.40, 0.25, 0)\pi$.

[2]L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith and J. Zhang, Phys. Lett. B **809** (2020), 135732

Static BTZ - Anti-Hawking Effect - II



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Massless Hyperbolic Black Holes

These are n-dimensional spacetimes $M = \mathbb{R} \times I \times \Sigma_{n-2}$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{n-2}^2, \quad f(r) = -1 - \frac{2M}{r^{n-3}} + \frac{r^2}{L^2},$$

where $d\Sigma_{n-2}^2 = d\theta^2 + \sinh^2\theta d\mathbb{S}_{n-3}^2(\varphi_1, \ldots, \varphi_{n-3})$.

- It has a bifurcate Killing horizon at r_H such that $f(r_H) = 0$,
- it is a solution of vacuum Einstein's equations with $\Lambda = -\frac{(n-1)(n-2)}{2L^2}$.
- It has a conformal *timelike boundary* at $r \to \infty$

Two-point functions can be constructed as in $PAdS_{d+1}$ and in BTZ.

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Anti-Hawking Effect - I

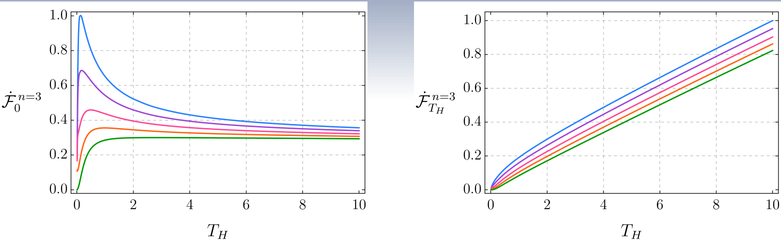

Figure: Transition rate, integrated up to $\ell = 100$, as a function of the local Hawking temperature on the three-dimensional hyperbolic black hole for $\Omega = -0.1$, $\theta = \pi^{-1}$ and for different boundary conditions. From top to bottom $\gamma = (0.50, 0.47, 0.40, 0.25, 0)\pi$. On the left, for the ground state; on the right, for the KMS state.

Anti-Hawking Effect - II

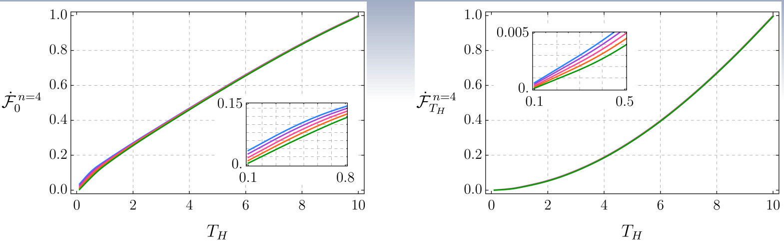

Figure: Transition rate as a function of the local Hawking temperature, summed up to $m_1 = 100$ and integrated up to $\ell = 100$, on the four-dimensional hyperbolic black hole for $\Omega = -0.1$, at $\theta_D = \pi^{-1}$, $\varphi_{1,D} = 0$ and for different boundary conditions: from top to bottom $\gamma = (0.50, 0.47, 0.40, 0.25, 0)\pi$. On the left, for the ground state; on the right, for the KMS state.

The Bondurant-Fulling map

Simplified setting

- $(M,g) \longrightarrow (\mathbb{H}^d, \eta)$ with $\mathbb{H}^d = \{(t, x_1, \dots, x_{d-2}, z) \mid z \geq 0\}$

- Massive scalar field $\Phi : \mathbb{H}^d \to \mathbb{R}$

$$P\Phi = (\Box_\eta - m^2)\Phi = 0,$$

- **Robin boundary conditions**

$$\partial_z \Phi|_{z=0} = \alpha \Phi|_{z=0}, \quad \alpha \in \mathbb{R}.$$

- Relevant Spaces:

$$C^\infty_D(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid f|_{z=0} = 0\},$$

$$C^\infty_\alpha(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}) \mid \partial_z f|_{z=0} = \alpha f|_{z=0}\}.$$

$$T_\alpha : C^\infty_\alpha(\mathbb{R}^d) \to C^\infty_D(\mathbb{R}^d) \quad f \mapsto T_\alpha(f) \doteq \partial_z f - \alpha f.$$

Constructing the propagators

Method of images

- Calling $G^{\pm}(\underline{x}-\underline{x}', z-z')$ the propagators on $(\mathbb{R}^d, \eta)$

$$G^{\pm}_{D/N}(\underline{x}-\underline{x}', z, z') = G^{\pm}(\underline{x}-\underline{x}', z-z') \mp G^{\pm}(\underline{x}-\underline{x}', z+z')$$

Notice

$$\mathrm{WF}(G^{\pm}_{D/N}) =$$
$$\{(x, k_x, x', k_{x'}) \in T^*(\mathcal{M}\times\mathcal{M}) \setminus \{0\} \,|\, (x, k_x) \sim_{\pm} (x', -k_{x'}), k_x \neq 0\} \cup \mathrm{WF}(\delta_2),$$

Constructing the propagators - II

Denote by $\mathcal{L}_\alpha(z) = \Theta(z)e^{-\alpha z}$

$$G_\alpha^\pm := (\mathcal{L}_\alpha \otimes \delta) \star G_D^\pm \circ (\mathbb{I} \otimes T_\alpha) = G_N + 2\kappa \mathcal{L}_\alpha \star G(\underline{x} - \underline{x}', z + z'),$$

- $G_\alpha^\pm$ coincide with the operator counterparts,
- $\mathrm{supp}(G_\alpha^\pm(f)) \subseteq J^\mp(\mathrm{supp}(f))$ for all $f \in C_0^\infty(\mathbb{H}^d)$.
- the wavefront set is

$$\mathrm{WF}(G_\alpha^\pm) =$$
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Propagators - Local Form

$$G^+_\alpha(x,x') = \Theta(t-t')\Big(U(x,x')\delta^{\frac{d-2}{2}}(\sigma) + \delta_d V(x,x')\Theta(\sigma) +$$
$$+U'(x,x')\delta^{\frac{d-2}{2}}(\sigma_-) + \delta_d V'(x,x')\Theta(\sigma_-)\Big)$$

where

$$\delta_d = \begin{cases} 1 \text{ if } d \text{ is even} \\ 0 \text{ if } d \text{ is odd} \end{cases} .$$

Propagators - Recursion Relations

In the even-dimensional case:

$$\begin{cases} U(x,x') = \sum_{j=0}^{\frac{d-4}{2}} u_j(\underline{x}, z, \underline{x}', z')\sigma^j \\ V(x,x') = \sum_{j=0}^{\infty} v_j(\underline{x}, z, \underline{x}', z')\sigma^j \\ U'(x,x') = \sum_{j=0}^{\frac{d-4}{2}} u'_j(\underline{x}, z, \underline{x}', z')\sigma_-^j \\ V'(x,x') = \sum_{j=0}^{\infty} v'_j(\underline{x}, z, \underline{x}', z')\sigma_-^j \end{cases},$$

where, imposing the defining equations for G_α^+

$$\begin{cases} (2-d)\sigma^\mu \partial_\mu u_0 = 0, \\ [u_0] = 1, \\ P u_j + (2j+4-d)\sigma^\mu \partial_\mu u_{j+1} + (j+1)(2j+4-d)u_{j+1} = 0, \\ [u_{j+1}] = -\frac{[Pu_j]}{2j+4-d}, \quad 0 \le j \le \frac{d}{2} - 3, \end{cases}$$

Propagators - Recursion Relations 2

The reflected components abide by

$$\begin{cases} (2-d)\sigma_-^{\mu}\partial_\mu u_0' = 0, \\ u_0'|_{z=0} = u_0|_{z=0}, \\ Pu_j' + (2j+4-d)\sigma_-^{\mu}\partial_\mu u_{j+1}' + (j+1)(2j+4-d)u_{j+1}' = 0, \\ (\partial_z+\kappa)(u_j+u_j')|_{z=0} + \frac{1}{2}(2j+4-d)\partial_z\sigma(u_{j+1}-u_{j+1}')|_{z=0} = 0, \quad 0 \le j \le \frac{d}{2}-3. \end{cases}$$

- Similar transport equations for v_j and v_j',
- The series are converging uniformly on compact subsets.

Global-to-Local Theorem

Theorem (Global-to-Local)

Given $(\mathbb{H}^d, \eta)$ *and* $P := \Box_\eta + m^2$, *consider* $\omega_{2,\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ *abiding by Robin boundary conditions with* $\kappa \geq 0$. *The following statements are equivalent:*

1. $\omega_{2,\kappa}$ *reads*

$$\omega_{2,\kappa}(x,x') = \lim_{\epsilon \to 0^+} U(x,x')\sigma_\epsilon^{\frac{2-d}{2}} + \delta_d V(x,x') \ln\left(\frac{\sigma_\epsilon}{\lambda^2}\right) + U'(x,x')\sigma_{-,\epsilon}^{\frac{2-d}{2}} + \delta_d V'(x,x') \ln\left(\frac{\sigma_{-,\epsilon}}{\lambda^2}\right) + W(x,x'),$$

2. $\omega_{2,\kappa}$ *has the following singular structure:*

$$WF(\omega_{2,\kappa}) = \{(x,k,x',-k') \in T^*(\mathring{\mathbb{H}}^d \times \mathring{\mathbb{H}}^d) \setminus \{\mathbf{0}\} \mid (x,k) \dot{\sim} (x',k') \text{ and } k \rhd 0\},$$

Conclusions

Conclusions:

- Construction of fundamental solutions and states in PAdS_{d+1},
- Identification of admissible boundary conditions,
- Application to black-hole physics,
- Characterization of the local form.

Open Questions:

- Generalization of the local form to PAdS_{d+1},
- Application of the framework to interacting field theories.