On the convoy of the ASEP speed process

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Plan of the talk

Background and definition

The main results

Proof ingredients

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Proof ingredients

ASEP

ASEP (asymmetric simple exclusion process) is a continuous Markov process, whose transition rates depend on an asymmetry parameter q. The model is called T(totally)ASEP when $q = 0$.

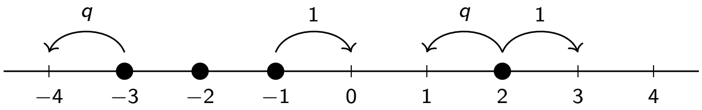

Figure 1: The ASEP on $\mathbb{Z}$ with jumping rate

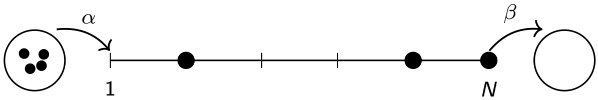

Figure 2: Open ASEP with injection rate α and extraction rate β

The ASEP with step initial condition

The step initial condition:

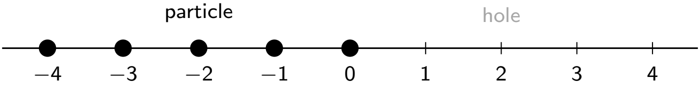

Figure 3: The ASEP on $\mathbb{Z}$ with step initial condition

Theorem (Rost '81)

The TASEP configuration σ_t^0 with step initial condition converges to Burgers equation

$$\lim_{t\to\infty}\frac{1}{t}\sum_{ut<j<vt}\sigma_t^0(j)=\int_u^v h(x)\,dx,$$

$$h(x)=\begin{cases}1, & \text{if } x<-1,\\ \frac{1-x}{2}, & \text{if } -1\le x\le 1,\\ 0, & \text{if } x>1.\end{cases}$$

ASEP with a second class particle

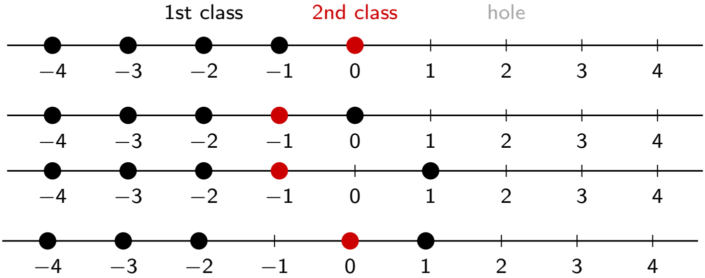

Figure 4: The ASEP on $\mathbb{Z}$ with a second class particle

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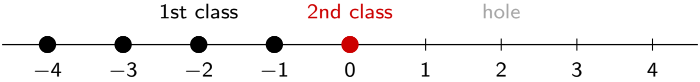


Figure 4: The ASEP on $\mathbb{Z}$ with a second class particle

Theorem (Mountford-Guiol '05; Ferrari-Pimentel '05; Aggarwal-Corwin-Ghosal '23)

Let $Y_0(t)$ be the position of the second class particle at time t, then

$$\lim_{t\to\infty} \frac{Y_0(t)}{(1-q)t} \overset{a.s.}{=} \mathcal{U}[-1,1] := U.$$

U is called the speed of the second class particle.

ASEP with a second class particle

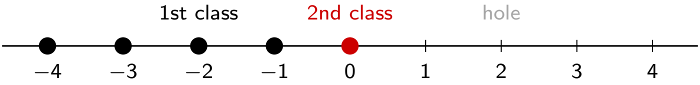


Figure 4: The ASEP on $\mathbb{Z}$ with a second class particle

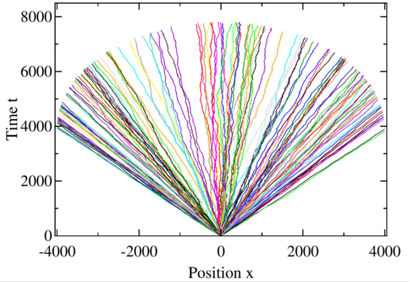
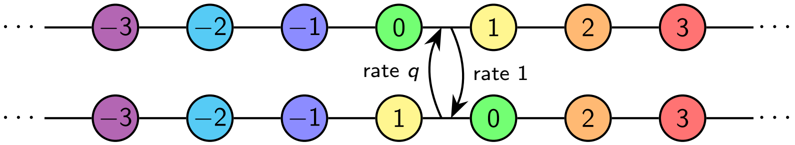


Figure 5: The speed of the second class particle

The Multi species ASEP



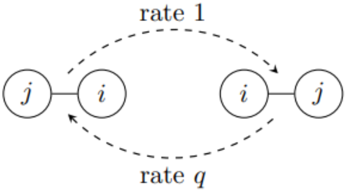


Figure 6: The multi-species ASEP with swap rate ($j < i$)

Color projection

Denote the position of particle 0 at time t as $X_t(0)$, consider the following color projection:

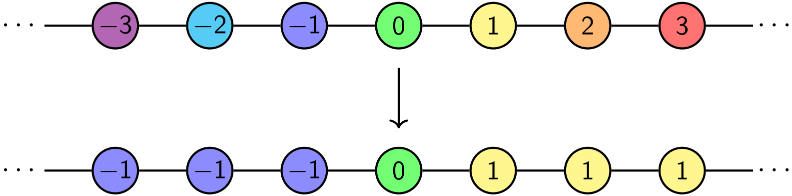


Figure 8: Projection regarding 0 as the second class particle

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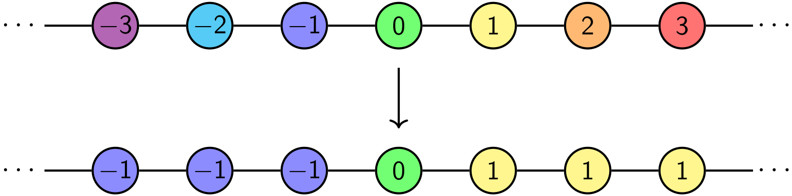

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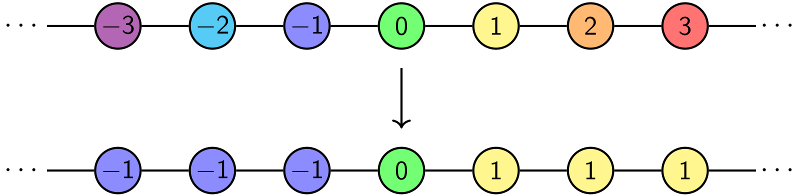

Figure 8: Projection regarding 0 as the second class particle

As a corollary of the speed of the second class particle,

$$\lim_{t\to\infty} \frac{X_0(t)}{(1-q)\,t} \overset{\text{a.s.}}{:=} \mathcal{U}_0.$$

Color projection

More generally, denote the position of particle i at time t as $X_t(i)$, consider the following color projection:

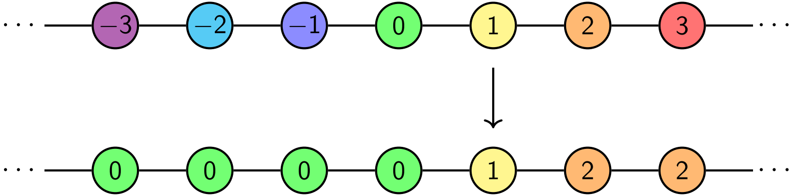

Figure 9: Projection regarding i ($i = 1$) as the second class particle

Similarly,

$$\lim_{t\to\infty} \frac{X_i(t) - i}{(1-q)\,t} \overset{\text{a.s.}}{:=} \mathcal{U}_i.$$

The ASEP speed process

Definition (Amir-Angel-Valkó '11)

Considering the color projection at each site, we get a family of uniform random variables $\{\mathcal{U}_n, n \in \mathbb{Z}\}$, which is called the **ASEP speed process** .

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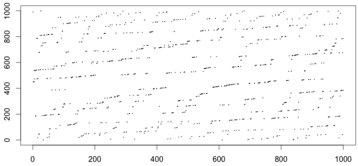

Figure 10: TASEP

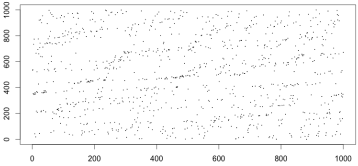

Figure 11: ASEP ($q = 0.8$)

The (T)ASEP speed process, images from [Martin '20].

The convoy of the ASEP speed process

Definition (Amir-Angel-Valkó '11)

The **convoy** of particle 0 is the random set of indices corresponding to particles that have the same speed as particle 0. Formally,

$$\mathcal{C} = \{j \in \mathbb{Z} : \mathcal{U}_j = \mathcal{U}_0\}.$$

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- To study the asymptotics, we also define

$$\mathcal{C}^n = \{j \in [1, n] : \mathcal{U}_j = \mathcal{U}_0\} \qquad \mathcal{C}^{\mathbb{Z}^+} = \{j \in \mathbb{Z}^+ : \mathcal{U}_j = \mathcal{U}_0\}.$$

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Question: How large is the set $\mathcal{C}^n$ for large n ?

Digression: the mixing time for open TASEP

It was shown in [Schmid '23] that the mixing time for open TASEP on a segment of size N at maximal current $(\alpha, \beta > 1/2)$ is of order $N^{\frac{3}{2}}$. The convoy was used to explain the appearance of $N^{\frac{3}{2}}$.

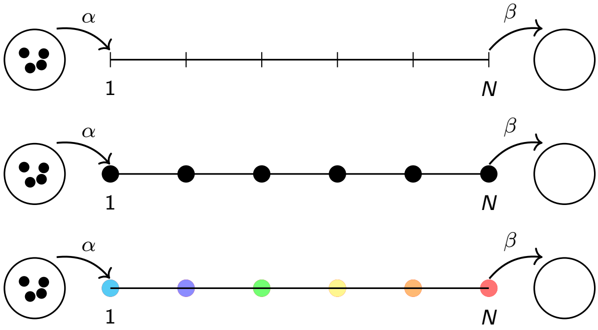

Figure 12: The disagreement process of two open TASEP

Plan of the talk

Background and definition

The main results

Proof ingredients

The q-Genocchi number

Definition (Han-Zeng '99)

The q-Gandhi polynomials $B_n(x,q)$ are defined by the following recurrence, with $B_1(x,q) = 1$, and for $n \geq 2$,

$$B_n(x,q) := \Delta_q \left(x^2 B_{n-1}(x,q)\right),$$

where

$$\Delta_q f(x) := \frac{f(1+qx) - f(x)}{(1+qx) - x}.$$

The q-Genocchi number is defined as the q-Gandhi polynomials at $x = 1$.

Example

$$B_1(1,q) = 1,\ B_2(1,q) = 2 + q,\ B_3(1,q) = 5 + 7q + 4q^2 + q^3,$$
$$B_4(1,q) = 14 + 36q + 45q^2 + 35q^3 + 18q^4 + 6q^5 + q^6.$$

Main results: An exact formula

Notation: u is the speed of particle 0, $x = \frac{1+u}{2}$, $c = x(1-x)$
$\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot | x = \frac{1+U_0}{2}]$.

Theorem (T. '25)

The expected size of the convoy can be expressed explicitly.

$$\mathbb{E}_x[\#\mathcal{C}^n] = \sum_{k=0}^{n} (1-q)^{2k+1} (-x(1-x))^{k+1} \binom{n+1}{k+1} B_k(1,q). \quad (1)$$

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Idea 2: Using the generating function [Han-Zeng '99] of $B_k(1,q)$ to write $\mathbb{E}_x[\#\mathcal{C}^{n,(q)}]$ as a complex integral.

$$\sum_{n\geq 1} B_n(1,q)\, t^n = \sum_{n\geq 1} \frac{([n]_q!)^2\, q^n t^n}{\left(q+[1]_q^2\, t\right)\left(q^2+[2]_q^2\, t\right)\dots\left(q^n+[n]_q^2\, t\right)},$$

where $[n]_q := 1 + q + \dots q^{n-1}$, $[n]_q! := \prod_{i=1}^{n} [i]_q$.

Main results: The asymptotic size of the convoy

Theorem (T. '25)

The asymptotic expected size of the ASEP convoy of particle 0 *is of order* $\sqrt{n}$. *More precisely, for any fixed* $q \in [0,1)$ *and fixed* $x \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{E}_x[\#\mathcal{C}^n] = \sqrt{\frac{4x(1-x)}{\pi}}. \qquad (2)$$

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Conjecture

Under the same assumption,

$$\lim_{n\to\infty} \frac{\#\mathcal{C}^n}{\sqrt{n}} \overset{(d)}{=} |\mathcal{N}(0, 2x(1-x))|.$$

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Theorem (T. '25)

The conjecture holds when $q = 0$.

Main results: The $q \to 1^-$ limit

Theorem (T. '25)

Consider $q_n = e^{-\frac{\gamma}{\sqrt{n}}}$, *we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\#\mathcal{C}^n}{\sqrt{n}} \mid U_0 = u\right] = C^{\gamma,c}.$$

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- Explicit formula for C^{γ}.

Explicit formula for C^{γ}

$$C^{\gamma,c} = \mathbb{E}\left[Y^{\gamma} - X^{\gamma}\right],$$

(X^{γ}, Y^{γ}) is a random vector with joint density

$$f^{\gamma}_{X,Y}(x,y) = \frac{e^{-\gamma x}}{2\pi} \int_0^{\infty} e^{-cw^2} \frac{|\Gamma(iw/\gamma)|^2}{|\Gamma(2iw/\gamma)|^2} \mathcal{J}(x,w)\,\mathcal{J}(y,w)\,dw,$$

and the function $\mathcal{J}(z,u)$ is

$$\mathcal{J}(z,w) = 2\Re\left[\left(\gamma e^{\gamma z}\right)^{\frac{iw}{\gamma}} \frac{\Gamma\left(\frac{2iw}{\gamma}\right)}{\Gamma\left(\frac{iw}{\gamma}\right)} {}_1F_1\left(1 - \tfrac{iw}{\gamma}; 1 - \tfrac{2iw}{\gamma}; -\tfrac{1}{\gamma} e^{-\gamma z}\right)\right].$$

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- $\lim_{\gamma\to\infty} C^\gamma = \sqrt{\frac{4c}{\pi}} = \mathbb{E}[|\mathcal{N}(0,2x(1-x))|]$,
  $\gamma \to \infty \iff 1-q_n \to 0$ slow.

Plan of the talk

Background and definition

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Proof ingredients

Characteristic theorem for the convoy set: Intuition

Theorem (Angel-Holroyd-Romik '09; Bufetov '20)

Let X_t be the random permutation of the configuration at time t. Then

$$X_t \overset{(d)}{=} X_t^{-1}.$$

$$\begin{aligned}
\mathbb{P}\big(\mathcal{U}_0 \in [x, x+\varepsilon]\big) &= \mathbb{P}\Big(\lim_{t\to\infty} \frac{X_t(0)}{t} \in [x, x+\varepsilon]\Big) \\
\text{(a.s. convergence)} \quad &= \lim_{t\to\infty} \mathbb{P}\big(X_t(0) \in [xt, (x+\varepsilon)t]\big) \\
\text{(symmetry)} \quad &= \lim_{t\to\infty} \mathbb{P}\big(X_t^{-1}(0) \in [xt, (x+\varepsilon)t]\big).
\end{aligned}$$

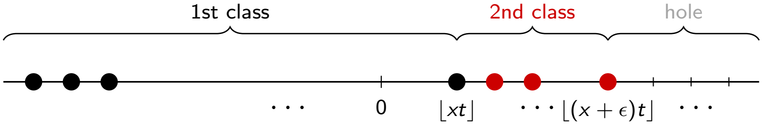

Figure 13: Color projection for computing $\mathbb{P}\big(X_t^{-1}(0) \in [xt, (x+\varepsilon)t]\big)$

Characteristic theorem for the convoy set

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Definition (Martin '20)

- $Q_1 \sim \frac{(q;q)_\infty}{(q;q)_k} q^k$, $k \geq 0$, $x \in [0,1]$, and $\mathcal{U} = \emptyset$,
- with probability $x(1-x)$, $Q_{i+1} = Q_i + 1$, $i \notin \mathcal{U}$,
- with probability $x^2 + (1-x)^2$, $Q_{i+1} = Q_i$, $i \notin \mathcal{U}$,
- with probability $(1-x)xq^{Q_i}$, $Q_{i+1} = Q_i$, $i \in \mathcal{U}$,
- with probability $(1-x)x\left(1-q^{Q_i}\right)$, $Q_{i+1} = Q_i - 1$, $i \notin \mathcal{U}$.

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Theorem (Martin '20; T. '25)

The random set characterizes the convoy set of particle 0. *That is,*

$$\mathcal{U} \overset{(d)}{=} \mathcal{C}^{\mathbb{Z}^+}. \tag{3}$$

Note that when $q = 0$, the Martin's construction is much simpler!

A coupling argument

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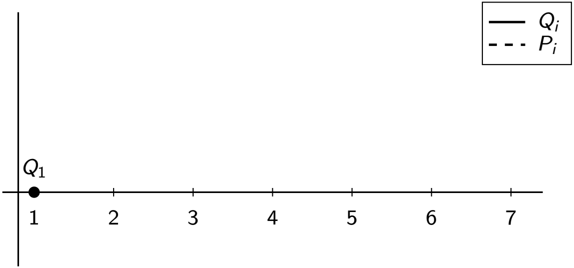

Figure 14: The coupling of Q_i and P_i

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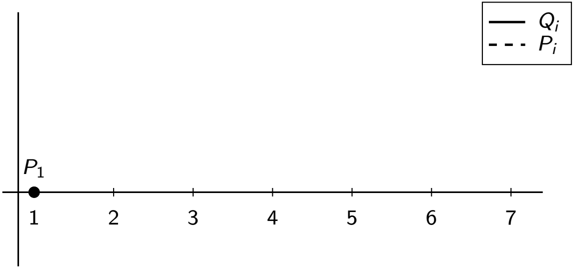

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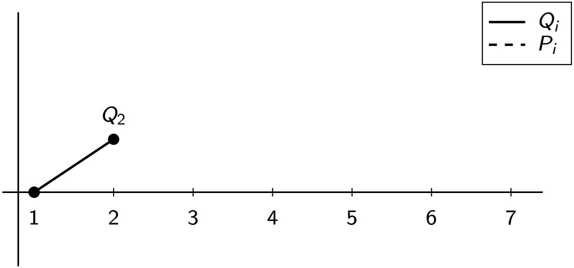

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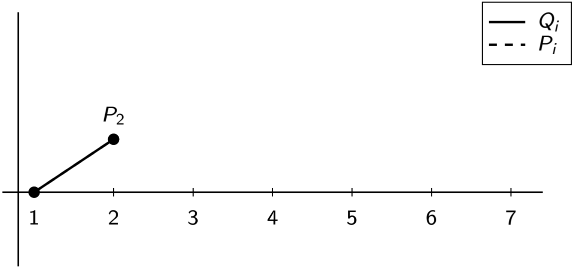

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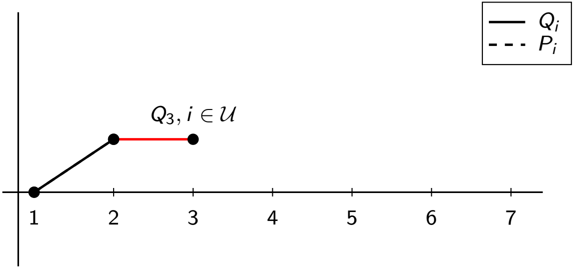

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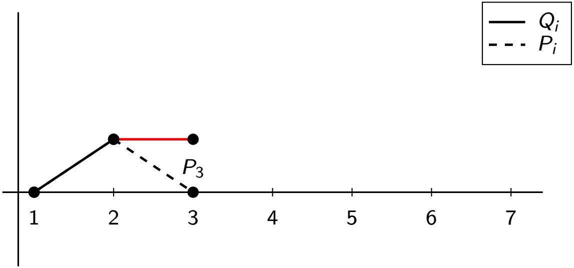

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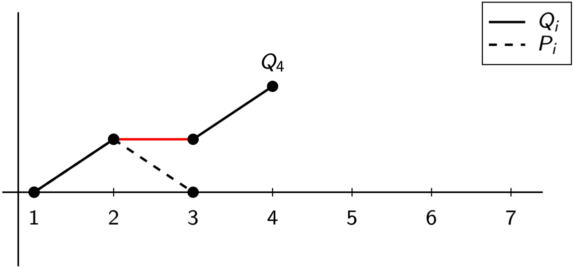

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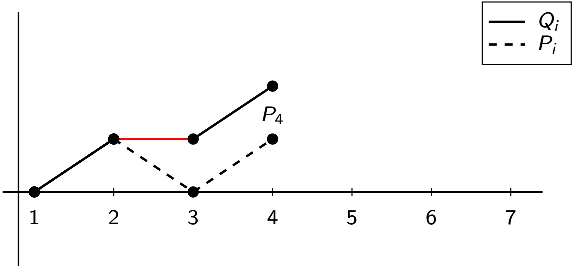

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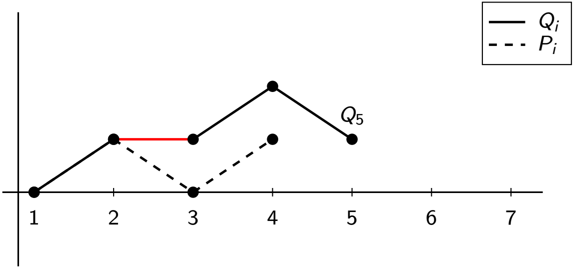

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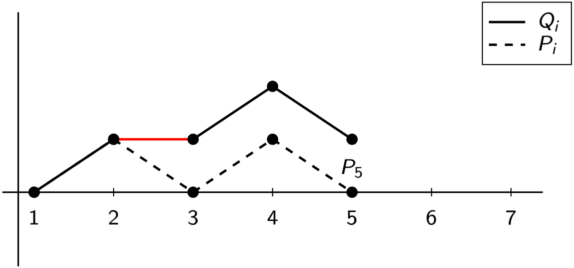

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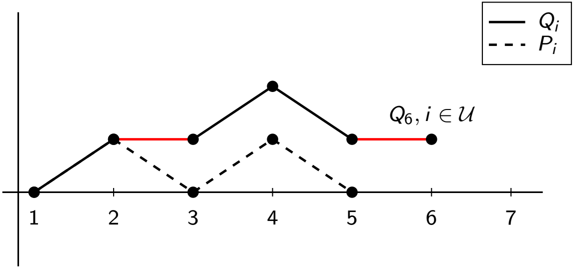

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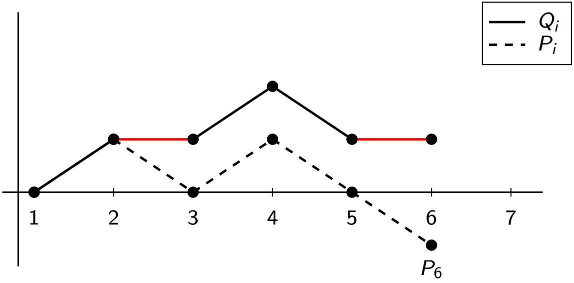

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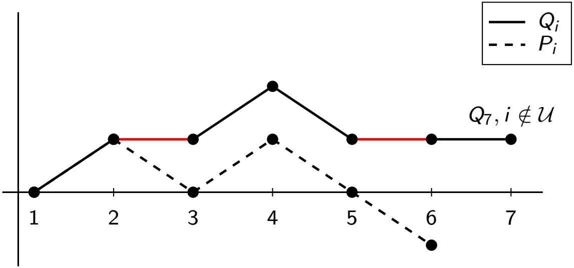

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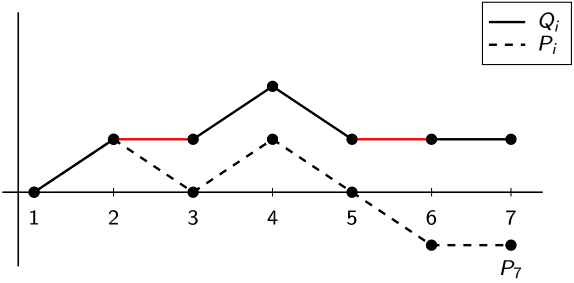

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Theorem

P_n is a symmetric homogeneous random walk. Under the coupling, for all $n \geq 1$,

$$\mathcal{U}_n = Q_{n+1} - P_{n+1} \quad \textit{almost surely.}$$

The Karlin-McGregor theory

A markov chain X_n is a birth-death chain if its transition probabilities $\mathbb{P}_{ij}$ is tridiagonal; that is,
$P_{i,i} = r_i, P_{i,i+1} = p_i, P_{i,i-1} = q_i$ and $P_{ij} = 0$, if $|i - j| > 1$.

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Theorem (Karlin-McGregor '59)

For a birth-death chain X_n, we have

$$\mathbb{P}_{ij}^m := \mathbb{P}\left(X_{n+m} = j | X_n = i\right) = \theta_j \int_{-1}^{1} x^n R_i\left(x\right) R_j\left(x\right) \psi\left(x\right) dx,$$

where $\theta_n = \frac{\prod_{i=0}^{n-1} p_i}{\prod_{i=1}^{n} q_i}$ and R_j determined by the three-term recurrence

$$xR_n\left(x\right) = p_n R_{n+1}\left(x\right) + r_n R_n\left(x\right) + q_n R_{n-1}\left(x\right),$$

and $\psi\left(x\right)$ is the unique reference measure of R_n.

The continuous big q-Hermite orthogonal polynomials

Definition (Askey-Wilson '85)

Let $H_0(x;a) = 1, H_{-1}(x;a) = 0$. $|q| < 1$ and $a \in [0,1)$. The continuous big q-Hermite polynomials $\{H_n(x;a)\}_{n=0}^{\infty}$ are defined by

$$2xH_n(x;a) = H_{n+1}(x;a) + aq^n H_n(x;a) + (1-q^n) H_{n-1}(x;a).$$

The reference measure of H_n is supported on $(-1,1)$ with density

$$w(x) := \frac{(q;q)_\infty}{2\pi\sqrt{1-x^2}} \frac{\left|\left(e^{2i\theta};q\right)_\infty\right|^2}{\left|\left(ae^{i\theta};q\right)_\infty\right|^2}, \quad x = \cos\theta.$$

$H_n(x;a)$ is a special case of Askey-wilson polynomials, which have been used for the study of open ASEP [Bryc-Wesołowski '10; Bryc-Kuznetsov-Wesołowski '25].

Outline of our proof

- Step 1: Use the coupling argument

$$\mathbb{E}[\mathcal{U}_n] = \mathbb{E}[Q_{n+1} - P_{n+1}] = \mathbb{E}[Q_{n+1} - Q_1].$$

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- Step 2: Express the transition probability

$$\mathbb{P}(Q_{n+1} = j | Q_1 = i) = \frac{1}{(q;q)_j} \int_{1-4c}^{1} x^n P_i(x) P_j(x) \psi(x)\, dx.$$

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$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\left(Q_{n+1} = \lfloor y\sqrt{n} \rfloor | Q_1 = 0\right) = \frac{1}{\sqrt{c\pi}} e^{-\frac{y^2}{4c}}.$$

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$$\lim_{n\to\infty} \frac{Q_{n+1}}{\sqrt{n}} \overset{(d)}{=} \left|\mathcal{N}\left(0, 2x(1-x)\right)\right|.$$

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$$\mathbb{P}\left(Q_{n+1} = j | Q_1 = i\right) = \frac{1}{(q;q)_j} \int_{1-4c}^{1} x^n P_i(x) P_j(x) \psi(x)\, dx.$$

- Step 3: Establish the local limit theorem for $y \in \mathbb{R}$

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\left(Q_{n+1} = \lfloor y\sqrt{n} \rfloor | Q_1 = 0\right) = \frac{1}{\sqrt{c\pi}} e^{-\frac{y^2}{4c}}.$$

- Step 4: Upgrade it to convergence in distribution

$$\lim_{n\to\infty} \frac{Q_{n+1}}{\sqrt{n}} \overset{(d)}{=} \left|\mathcal{N}\left(0, 2x(1-x)\right)\right|.$$

Thanks for your attention!