Limit shapes and harmonic tricks

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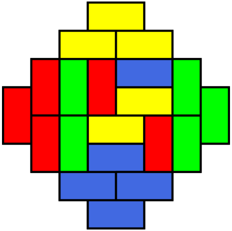
Outline

1. Introduction to the random tilings
2. The Tangent plane method and conformal coordinates
3. The Tangent plane method: Aztec diamond with a hole



1. Model of 3d crystal
2. Arrangement of cubes after projection becomes lozenge tiling of hexagon (i.e., covering by three types of lozenges without gaps or overlaps)
3. The coordinate lost under π is the height function
4. we study uniform measure on lozenge tilings

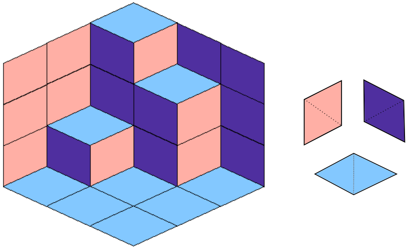


Figure 1: Crystal surface as lozenge tiling of hexagon

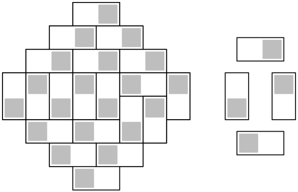


Figure 2: Domino tiling of AD_4

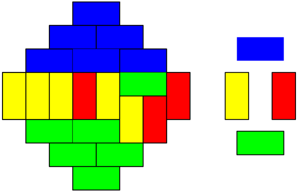


Figure 3: Color representation

- Fix a chessboard coloring and distinguish 4 types of dominoes.
- Consider the uniform measure on the set of domino tilings.
- We encode domino tilings by the height function

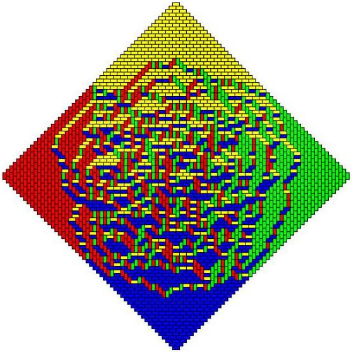


Figure 4: Aztec diamond AD_{50}

Theorem 1 (Jockusch, Propp, Shor, '98)

Typical domino tiling of AD_N outside the inscribed circle is deterministic as $N \to \infty$, the circle is the arctic curve $\partial\mathcal{L}$, boundary of the liquid region $\mathcal{L}$

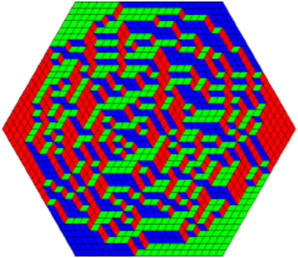


Figure 5: Computer simulation for random tilings of the hexagon

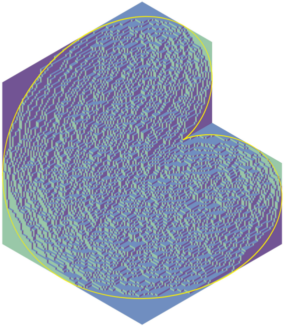


Figure 6: The cardiod curve, *Kenyon,Okounkov 05'*

Critical points of $\lim_{N\to\infty} N^{-1} \log Z_N$, Z_N is a partition function enumerating tilings. Also Tangent method [1]

[1] *Arctic curves of the six-vertex model on generic domains: the Tangent Method*, Colomo, Sportiello

I.Variational principle for domino tilings (Cohn-Kenyon-Propp 00')

Theorem 2

Suppose $\Gamma_N \subset \frac{1}{N}\mathbb{Z}^2$ *and it approximates* Ω. *Also* $\mathcal{H}(\Omega,\chi) := \{\mathfrak{h} : \Omega \to \mathbb{R} : \mathfrak{h}\big|_{\partial\Omega} = \chi$ *and* $\partial_x\mathfrak{h} + \partial_y\mathfrak{h} \le 2\}(\nabla\mathfrak{h} \in \mathcal{N})$

$$\frac{1}{N^2}\log Z(\Gamma_N) \to \min_{\mathfrak{h}\in\mathcal{H}} \int_\Omega \sigma(\partial_x\mathfrak{h}, \partial_y\mathfrak{h})dxdy \qquad (1)$$

- Minimizer $\mathfrak{h}^\star$ is the limit of height functions on Γ_N
- $\sigma(s,t) = -1/\pi\,(L(\pi p_a) + L(\pi p_b) + L(\pi p_c) + L(\pi p_d))$, where $L(z) = \int_0^z \log|2\sin t|dt$

There are 3 phases [2],

- Frozen region, where fluctuations tend to 0, $\nabla\mathfrak{h} \in \partial\mathcal{N}$
- Rough(liquid) region, where they are governed by Gaussian Free field $\nabla\mathfrak{h} \in \mathcal{N}^\circ$
- Gas region, fluctuations are exponentialy suppresend, $\nabla\mathfrak{h} = q \in \mathcal{N}^\circ$

[2] *Dimers and amoebae 03'* Kenyon-Okounkov-Sheffild

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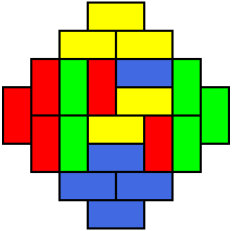

The Euler-Lagrange equation in $\mathcal{L}$

$$\operatorname{div}(\nabla\sigma \circ \nabla\mathfrak{h}) = 0. \tag{2}$$

- Is extremely difficult to analyze directly.
- $\nabla\sigma$ is singular

Yet, no need to jump out of the window

The gradient $\nabla \mathfrak{h} = (s, t)$ can be parametrized by one complex z, the so-called complex gradient map [3].

Theorem 3

There are defined two holomorphic maps $z(x, y), w(x, y)$ on $\mathcal{L}$ such that

- $(s, t) = (\arg w, -\arg z)$
- $\frac{z_x}{z} + \frac{w_y}{w} = 0$
- *The spectral curve* $\mathcal{C} := \{z, w \in \mathbb{C}^\star | P(z, w) = z + w - 1 = 0\}$

Also they showed the local existance of the arctic curve.

[3] *Limit shapes and the complex Burgers equation, Kenyon-Okounkov 05*

It was shown[4] for a wide family of polygonal domains with natural boundary conditions and any periodic dimer model i.e. $P(z, w)$,

Theorem 4

- *Global existance of the arctic curve, it is an algebraic curve*
- *Classification of singularities of $\mathcal{L}$*
- *Harmonic parametrization of $\nabla\mathfrak{h}$*

[4] *Dimer Models and Conformal Structures* 22, Kari Astala, Erik Duse, István Prause, Xiao Zhong.

- Legendre transform of $\mathfrak{h}$ is c, the intercept $c = \mathfrak{h} - sx - ty$.
- Plane tangent to graph of $\mathfrak{h}$ at $(x_0, y_0) \in \mathcal{L}(\Omega)$

$$\mathcal{P}_{x_0,y_0} = \{(x, y, z) \in \mathbb{R}^3 | \{s(x_0, y_0)x + t(x_0, y_0)y + c(x_0, y_0) = z\}. \tag{3}$$

- s,t and c have piecewise constant boundary conditions

- $Hess(\sigma)$ defines a metric on $\mathcal{N}, g = \sigma_{ss} ds^2 + 2\sigma_{st} ds dt + \sigma_{tt} dt^2$
- Conformal coordinate u on $\mathcal{N} : g(u) \approx e^{\rho} dv_1 dv_2 (u = v1 + \sqrt{-1} v2)$, $\rho > 0$ is the potential (For our case $\rho = 2\pi$.)
- Tangency condition $xs_u(u) + yt_u(u) + c_u(u) = 0$ (real system for Im and Re)
- Matching of critical points of s_u, t_u and c_u (the hard part).

Theorem 5 (Kenyon, Prause, '22)

Intercept c, s and t are harmonic functions in the liquid region $\mathcal{L}(\Omega)$ in z-variable.
Connected boundary components of $\mathcal{L}(\Omega)$ are isomorphic to $\mathbb{P}^1(\mathbb{R})$.

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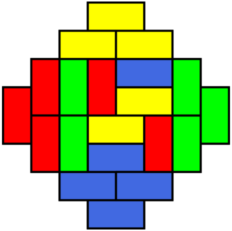


From the table of boundary conditions of s, t and c for $\mathcal{AD}$

	$z < -1$	$-1 < z < 0$	$0 < z < 1$	$1 < z$
s	0	2	0	-2
t	2	0	-2	0
c	1	-1	1	-1

One sees for $z \in \mathbb{H}^+$

$$s = \frac{2}{\pi}(-\frac{\pi}{2} + \arg(z-1) + \arg(z) - \arg(z+1)) \tag{4}$$

$$t = \frac{2}{\pi}(-\frac{\pi}{2} - \arg(z-1) + \arg(z) + \arg(z+1)) \tag{5}$$

$$c = \frac{2}{\pi}(-\frac{\pi}{2} + \arg(z-1) - \arg(z) + \arg(z+1)). \tag{6}$$

The arctic curve determined from the following complex equation (property of Legendre dual function),

$$s_z x + t_z y + c_z = 0. \tag{7}$$

- $x(u)$ and $y(u)$ are found from the linear system for real and imaginary parts.
- Arctic curve is the image of $\partial\pm$ under map $z \mapsto (x(z), y(z))$.
- For the Aztec diamond, and map $u(x, y)$ can be found explicitly (it is the hard part),

$$z(x, y) = \frac{y - x + i\sqrt{1 - 2(x^2 + y^2)}}{1 + x + y}. \tag{8}$$

III. Computer simulation of Aztec diamond with a hole

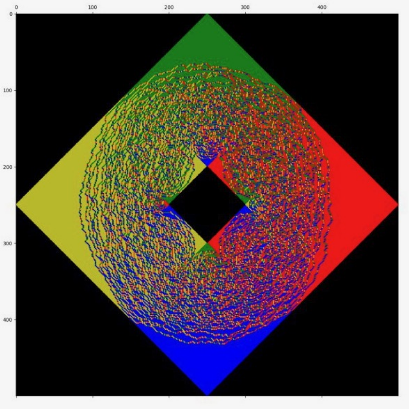


Figure 7: Aztec diamond with a hole [5]

[5] N.K. arXiv:2603.21255

- Harmonic extensions are given by argument rational function in terms of σ-Weierstrass functions(analog of u on elliptic curve)
- These rational functions are elliptic functions on conformal model of $\mathcal{L}$, $\Sigma = [0,\tau] \times [-1,1]$. Conformal coordinate u is a ramified cover of z.
- We obtain 1-parametric family of limit shapes.

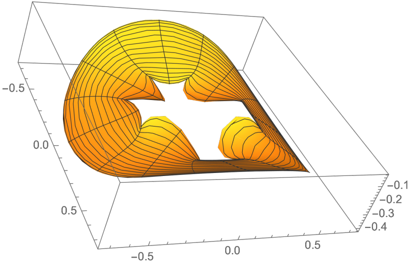


Figure 8: Graph of the limiting height function [6]

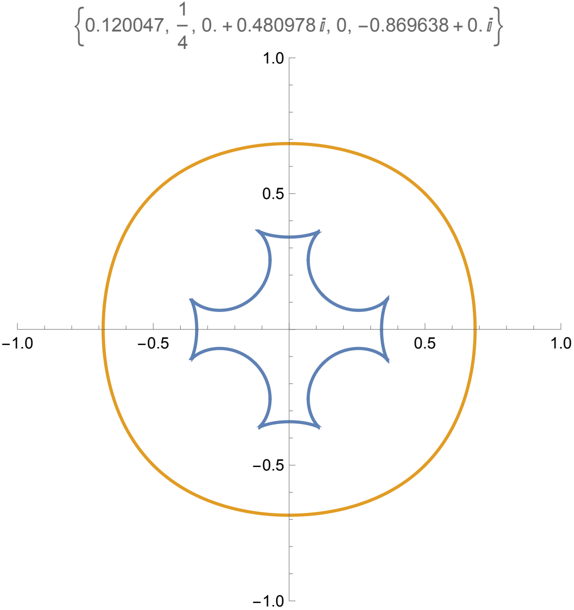
[6] N.K. arXiv:2603.21255

One could summarize all the geometrical objects in a communative diagram as follows [7]

$$
\begin{array}{ccccc}
\mathcal{L} & \xrightarrow{\nabla \mathfrak{h}} & \mathcal{N} & \xrightarrow{\nabla \sigma} & \mathcal{A} \\
\big\downarrow{\scriptstyle 1:1} & & \big\downarrow{\scriptstyle 1:1} & \nearrow{\scriptstyle (\log|z|, \log|w|)} & \\
\Sigma^{+} & \xrightarrow{\deg d} & \mathcal{C}^{+} & & \\
\cap & & \cap & & \\
\Sigma & \xrightarrow{\pi} & \mathcal{C} & &
\end{array}
$$

As $\tau \to \infty$ harmonic extensions s, t, c converge to the ones of usual Aztec diamond.

[7] N.K. arXiv:2603.21255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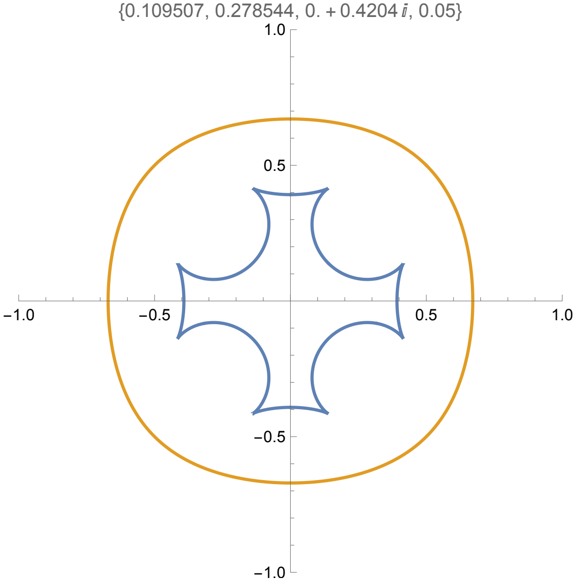
−1.0
−1.0
−0.5
0.5
1.0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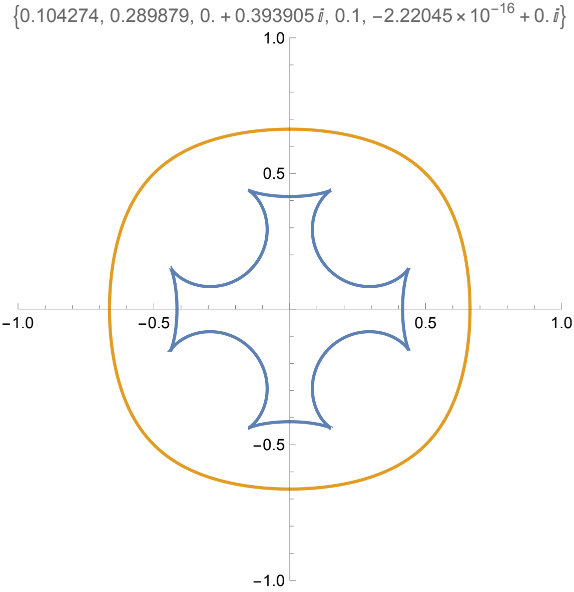
0.5
1.0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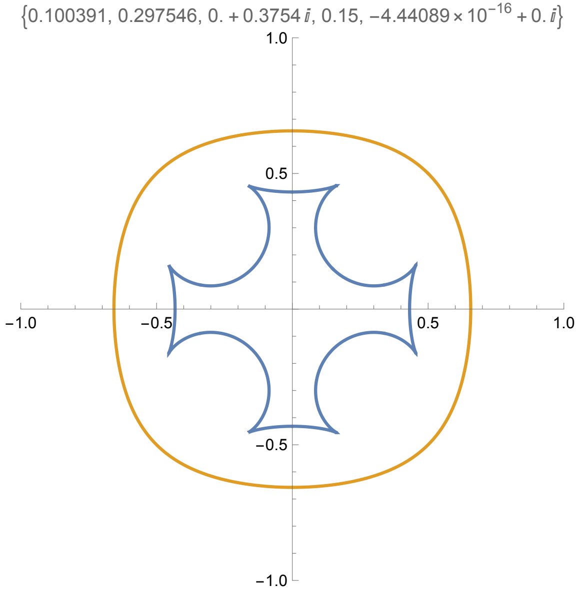
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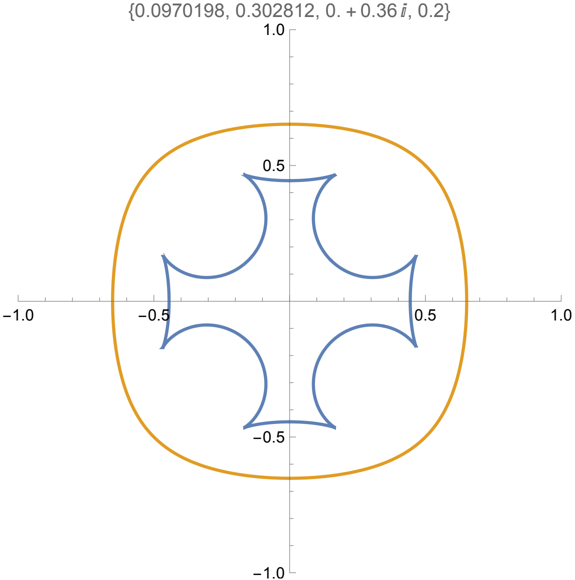
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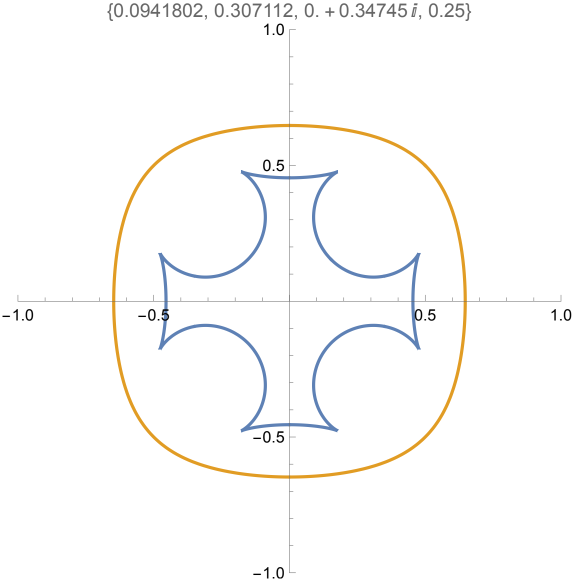
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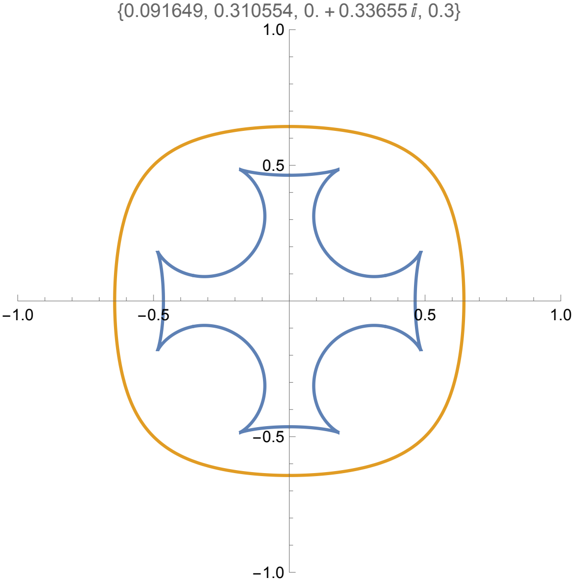
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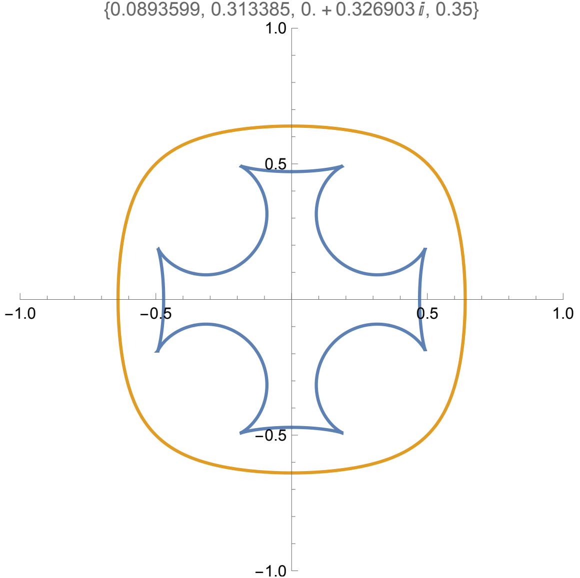
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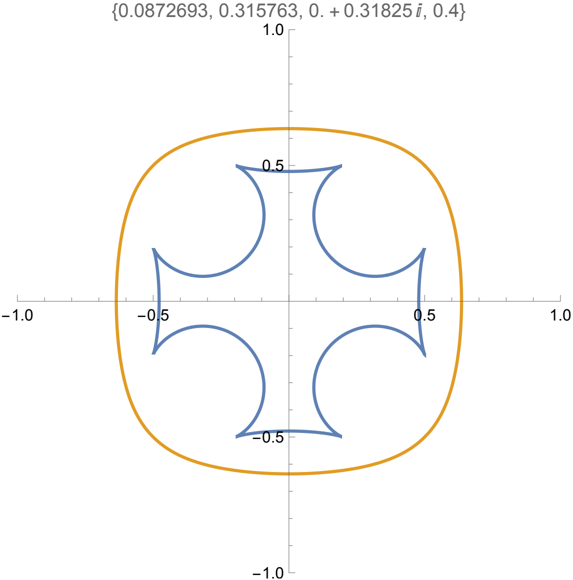
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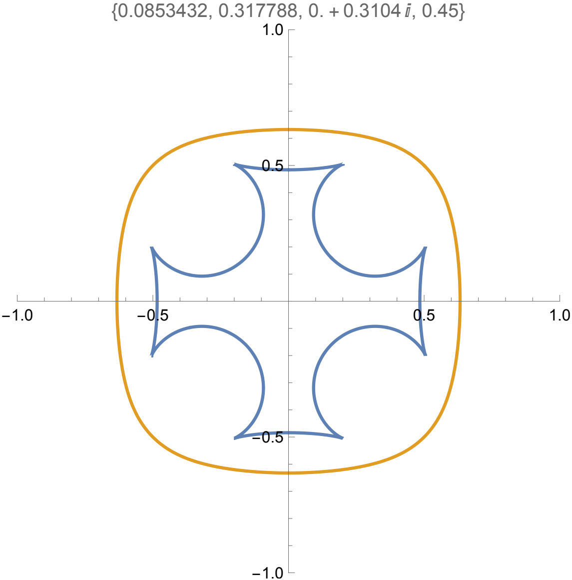
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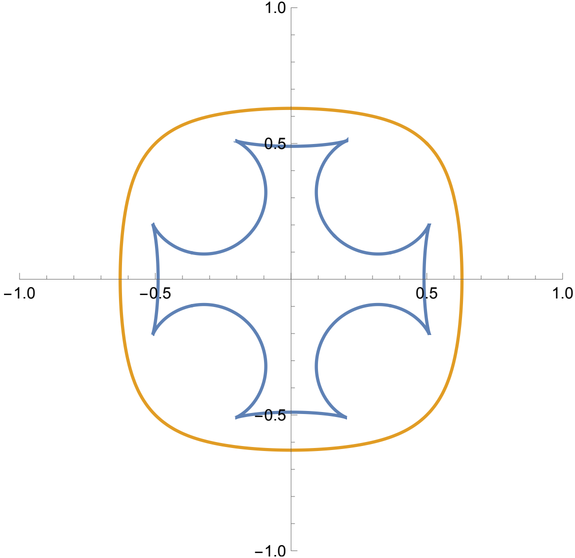

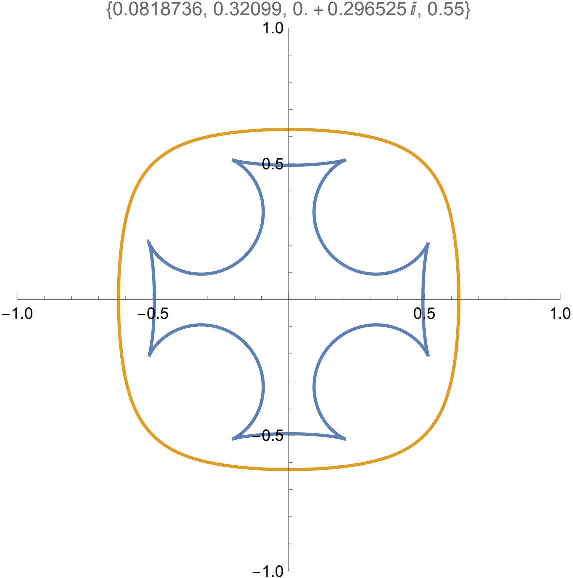
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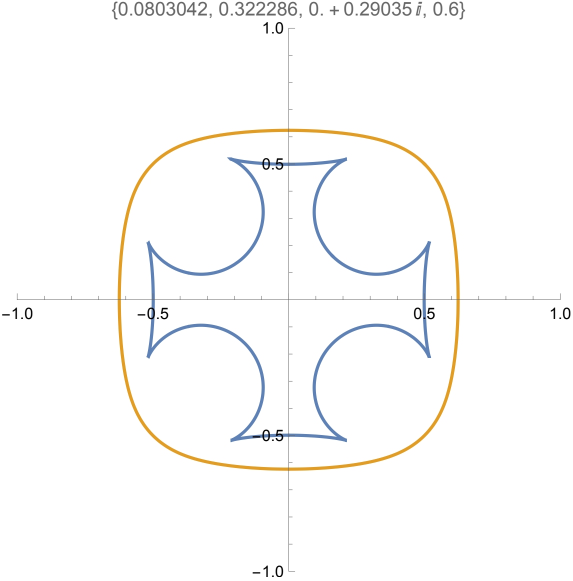
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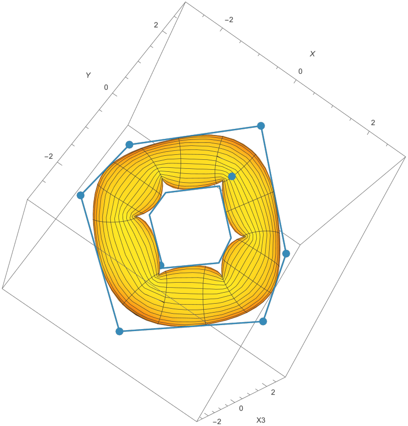
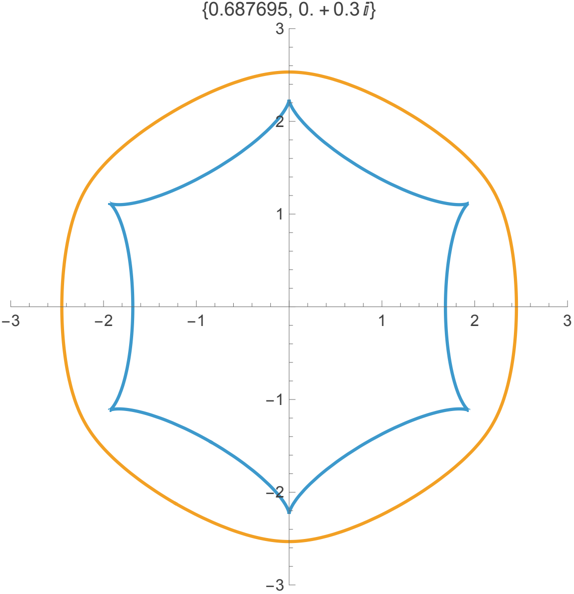


Figure 9: Graph of the limit height function

III. Hexagon with a hexagonal hole (joint with M.Eriksson)



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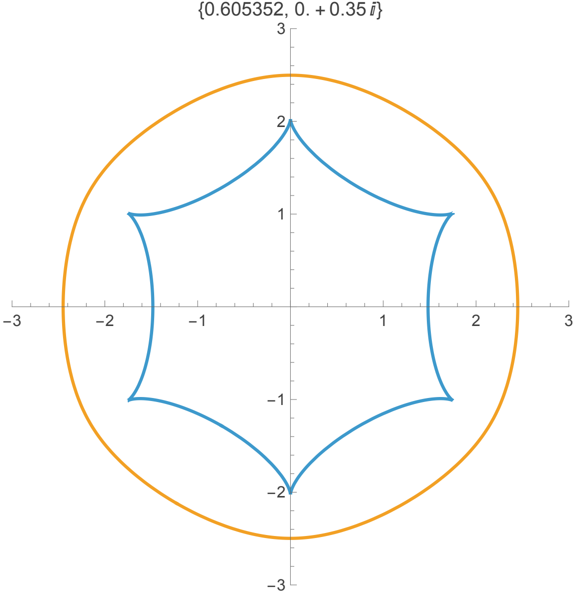

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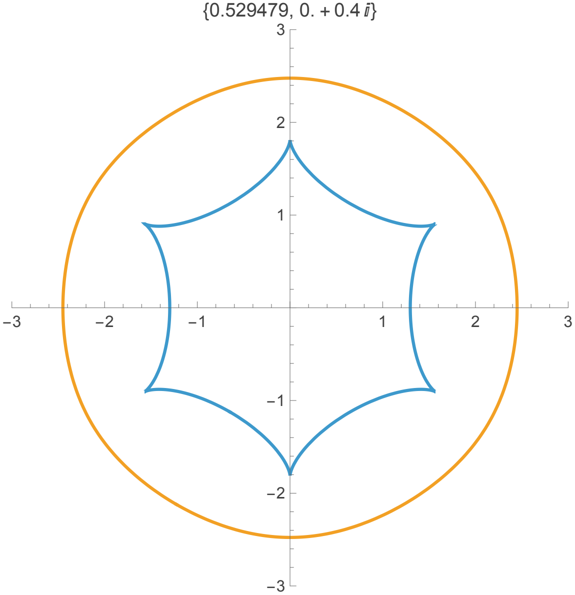

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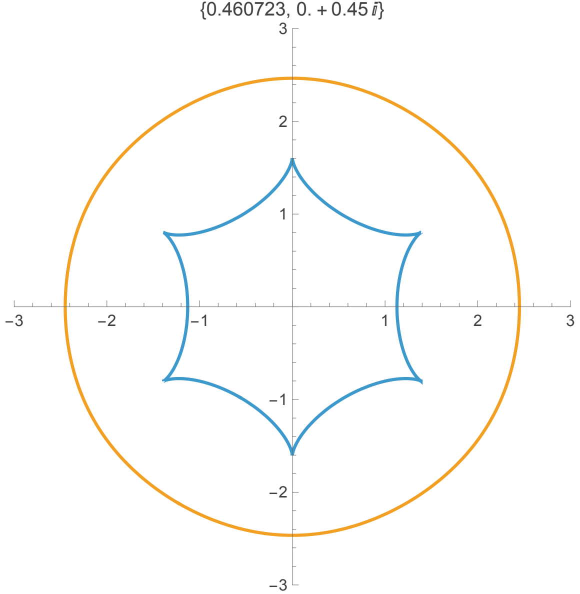

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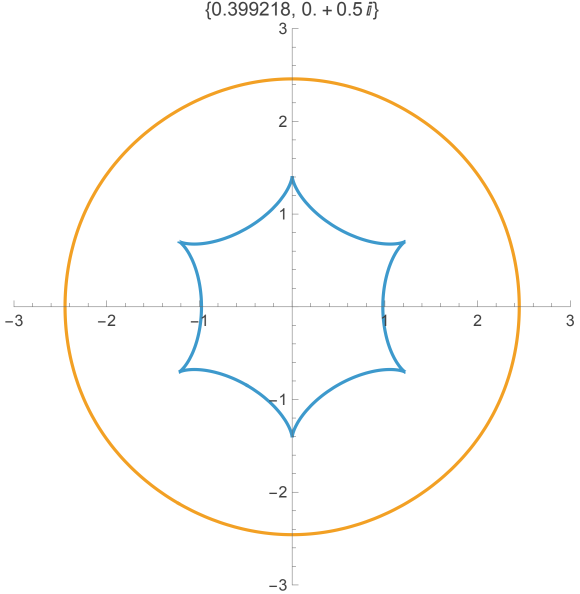

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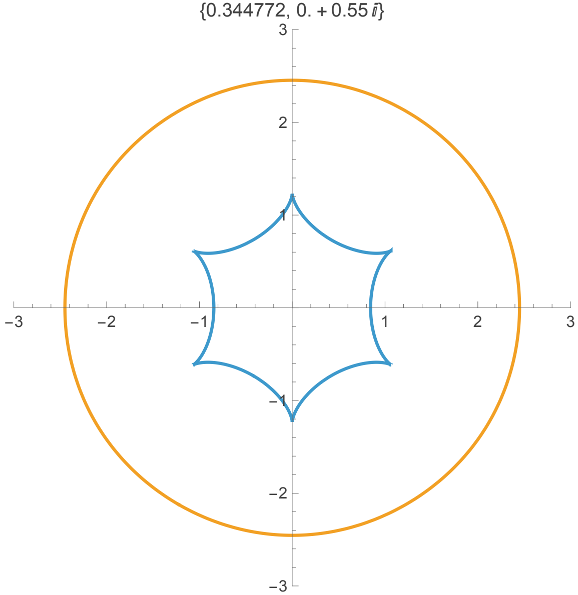

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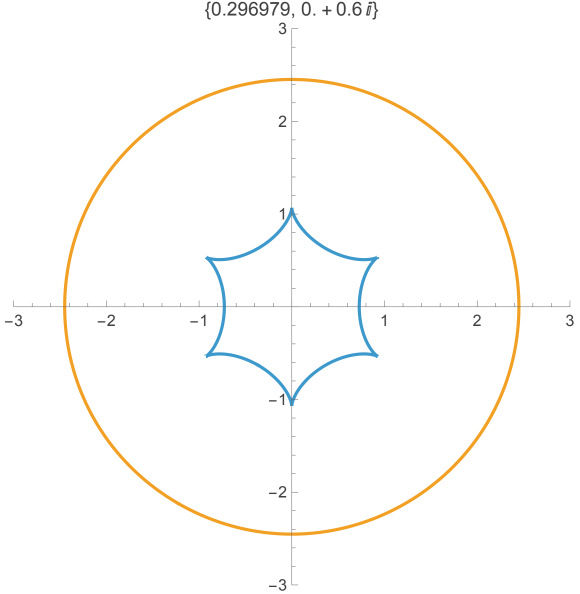

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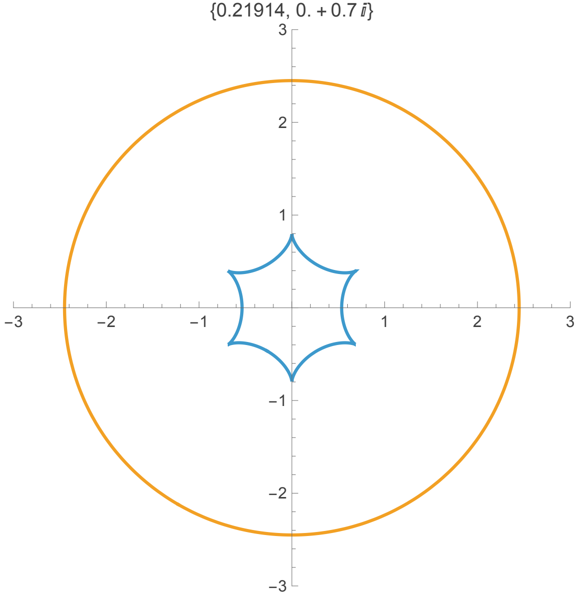

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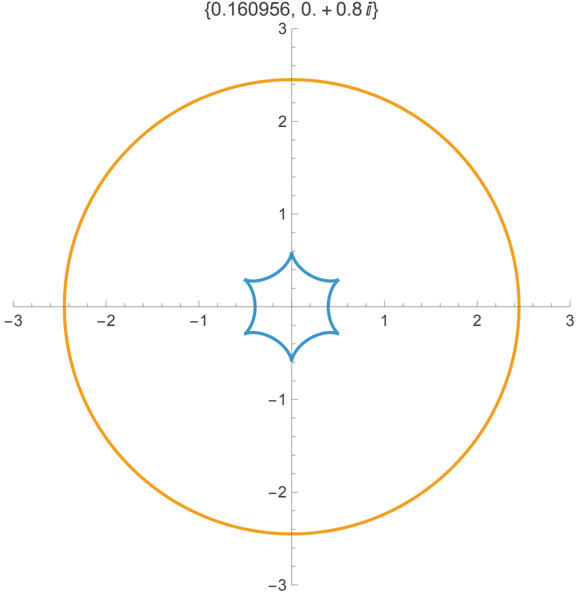

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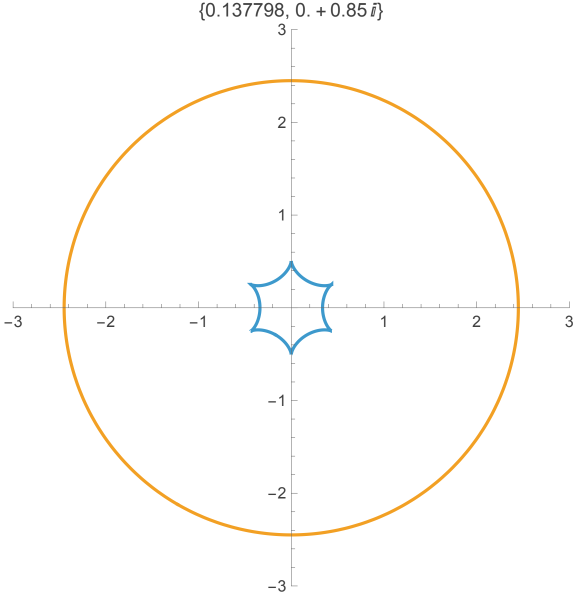

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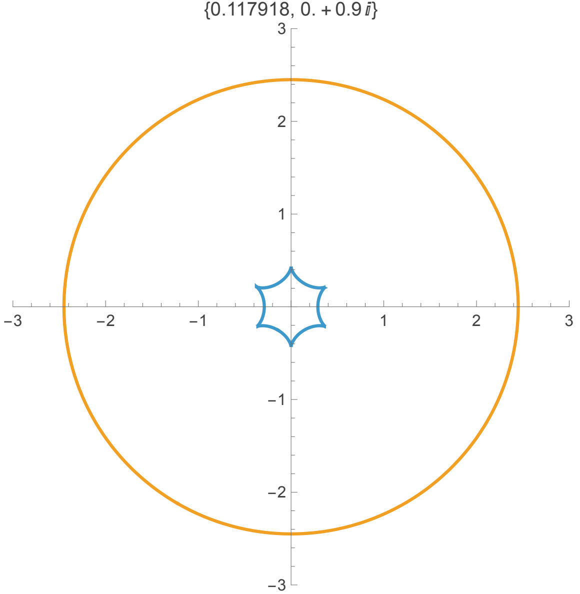

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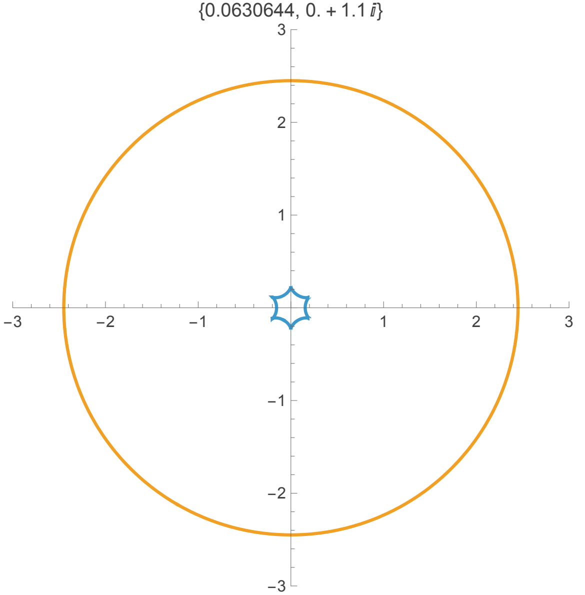

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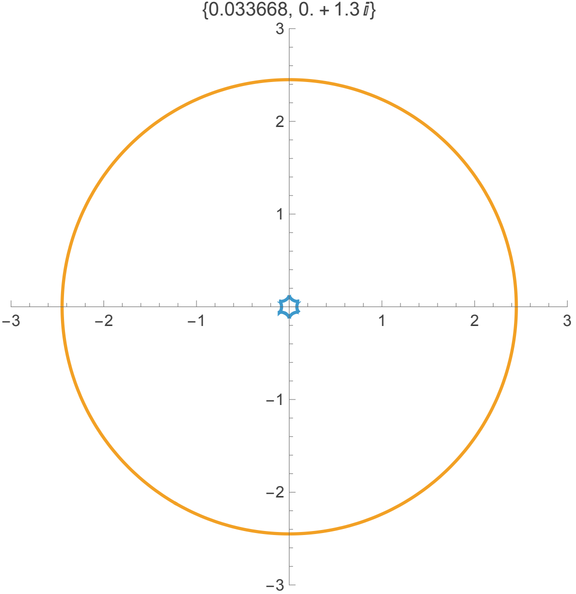

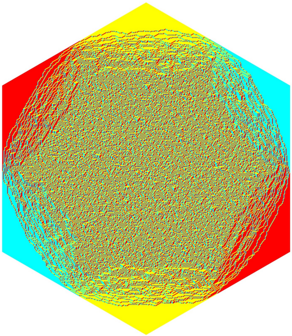

Figure 10: Computer simulation for 3 by 3 periodic weights [a]

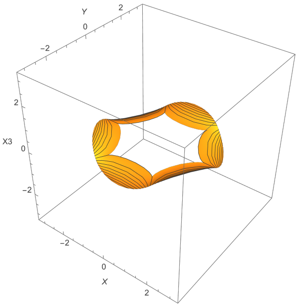

Figure 11: Our 3D limit shape for it.

[a] *Arno Kuijlaars 2025'*

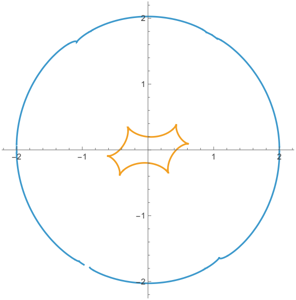

Figure 12: The arctic curve for the gas bubble with Fock's weights

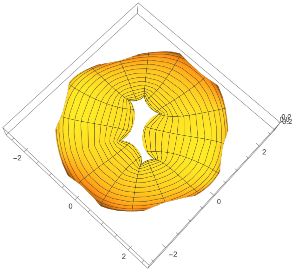

Figure 13: 3D limit shape for the bubble with Fock's weights

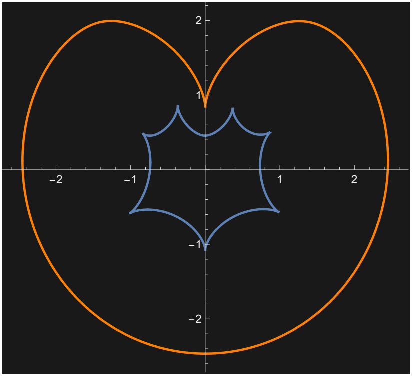

Figure 14: Arctic curve with the defects, 3d shape is work in progress

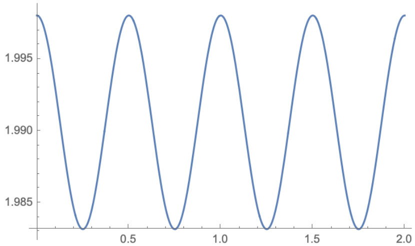

Figure 15: Equation $x^2 + y^2$ evaluated on the arctic curve for doubly-periodic Aztec diamond.

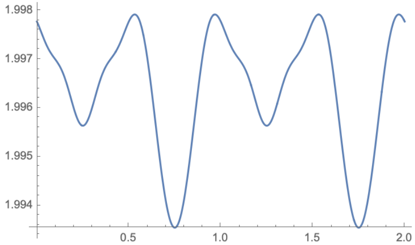

Figure 16: Equation $x^2 + y^2$ evaluated on the curve of multiply-connected Aztec diamond.

Therefore, with 3σ(99.7%)-precision it is the Arctic circle, but not with 4σ (99.9937%).

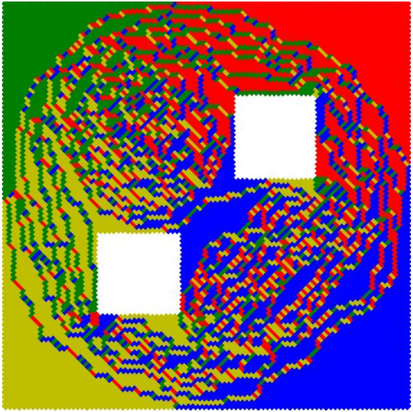

Figure 17: Computer simulation of an Aztec diamond with two holes