Discrete N-particle ensembles at high temperature through Jack polynomials

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Based on joint works with **Florent Benaych-Georges**, **Vadim Gorin** and **Maciej Dolega**.

Outline of the talk

The Gaussian β-ensemble and semicircle distribution

LLN for random β-partitions at high temperature

The limiting measure: moment problem and Jacobi operators

A remark on a deformation of free convolution

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Gaussian Unitary Ensemble

The probability measure on the Weyl chamber

$$\mathcal{W}_N := \Big\{(a_1 \geqslant \cdots \geqslant a_N) \in \mathbb{R}^N\Big\}.$$

with density

$$\boxed{\mathbb{P}_N(a_1, \ldots, a_N) \propto \prod_{1 \leqslant i < j \leqslant N} (a_i - a_j)^2 \prod_{k=1}^{N} e^{-\frac{1}{2} a_k^2}}$$

is called the Gaussian Unitary Ensemble (GUE).

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is called the Gaussian Unitary Ensemble (GUE).

It is the distribution of eigenvalues of the $N \times N$ complex Hermitian random matrix

$$A_N = \frac{M_N + M_N^*}{2}, \quad M_N = [m_{ij}]_1^N, \quad m_{ij} = \mathcal{N}(0,1) + \sqrt{-1} \cdot \mathcal{N}(0,1).$$

Law of Large Numbers for GUE

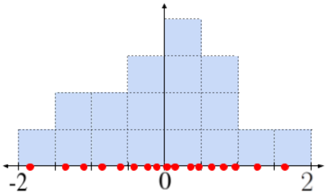
The empirical measures are

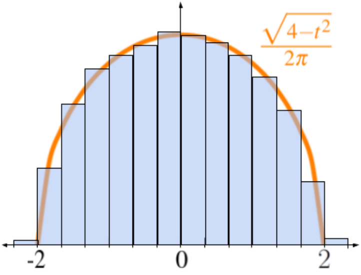
$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\frac{a_i}{\sqrt{N}}}, \text{ where } (a_1 \geqslant \cdots \geqslant a_N) \text{ is } \mathbb{P}_N\text{-distributed.}$$

Theorem (Wigner '55)

The empirical measures μ_N converge weakly, in probability, to the semicircle distribution, with density

$$s(t) := \mathbf{1}_{\{-2\leqslant t\leqslant 2\}}\cdot\frac{\sqrt{4-t^2}}{2\pi}.$$





Moment method and multivariate Bessel functions

A standard proof of Wigner's theorem employs the moment method and matrix representation to find the limits

$$\lim_{N\to\infty} \int_{\mathbb{R}} x^k \mu_N(dx) = \lim_{N\to\infty} \frac{1}{N^{\frac{k}{2}+1}} \cdot \mathbb{E}\Big[\mathrm{Tr}\big(A_N^k\big)\Big], \text{ for all } k \geqslant 1.$$

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New ideas [Bufetov–Gorin '15] use the multivariate Bessel functions

$$B_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N) := \prod_{j=1}^{N-1} j! \cdot \frac{\det\left[e^{a_i x_j}\right]_{i,j=1}^{N}}{\prod_{i<j}(x_i-x_j)(a_i-a_j)},$$

s.t. $B_{(a_1,\ldots,a_N)}(0^N) = 1$ and are eigenfunctions of differential operators

$$\mathcal{D}_k := \frac{1}{\prod_{i<j}(x_i-x_j)} \circ \sum_{i=1}^{N} \frac{\partial^k}{\partial x_i^k} \circ \prod_{i<j}(x_i-x_j), \quad k \geqslant 1.$$

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$$\boxed{\mathcal{D}_k\Big(B_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)\Big) = \sum_{i=1}^N (a_i)^k \cdot B_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)}$$

Bessel generating function = kind of Fourier transform

The idea is to associate, $\mathbb{P}_N(a_1,\ldots,a_N) \mapsto F_N(x_1,\ldots,x_N)$,
to the GUE its Bessel generating function:

$$F_N(x_1,\ldots,x_N) := \int B_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)\mathbb{P}_N(a_1,\ldots,a_N)\,da_1\cdots da_N.$$

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The moments of empirical measures are exactly the "Taylor coeffs",
i.e. first apply $\mathcal{D}_k$, then find the constant term (set $x_i = 0$, $\forall i$):

$$\mathcal{D}_k F_N\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[\sum_{i=1}^{N} (a_i)^k\right].$$

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Upshot: The moments of μ_N can be accessed without matrices!

The difficulty now is to study limits of $\mathcal{D}_k$ applied to the BGF $F_N(x_1, \dots, x_N)$ $\big($which BTW equals $= e^{(x_1^2+\cdots+x_N^2)/2}\big)$, then set $x_i = 0$, $\forall i$, and take the limit, as $N \to \infty$.

Dunkl operators

A new approach was started by [Benaych-Georges–C.–Gorin '22], who used instead the Dunkl differential-difference operators

$$\xi_i := \frac{\partial}{\partial x_i} + \sum_{j:\, j\neq i} \frac{1}{x_i - x_j}(1 - s_{i,j}),$$

$$\mathcal{P}_k := (\xi_1)^k + \cdots + (\xi_N)^k, \quad k \geqslant 1.$$

They also satisfy the key identity:

$$\boxed{\mathcal{P}_k\Big(B_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)\Big) = \sum_{i=1}^{N} (a_i)^k \cdot B_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)}$$

The advantage: they admit a 1-parameter "θ-generalization" $\cdots$

Gaussian Beta Ensemble

For any $\theta > 0$, the following distribution on $\mathcal{W}_N$ is the Gaussian Beta Ensemble (usual parameter is $\beta = 2\theta$):

$$\mathbb{P}_N^{(\theta)}(a_1, \ldots, a_N) \propto \prod_{1 \leqslant i < j \leqslant N} (a_i - a_j)^{2\theta} \prod_{k=1}^{N} e^{-\frac{1}{2} a_k^2}$$

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Motivations:

1. For $\theta = \frac{1}{2}, 2$: $\mathbb{P}_N^{(\theta)}$ = eigenvalue density of Gaussian Orthogonal Ensemble (GOE) and Gaussian Symplectic Ensemble (GSE).

2. $\mathbb{P}_N^{(\theta)}$ is a Boltzmann distribution with logarithmic repulsion and inverse temperature $\boldsymbol{\beta = 2\theta}$.

3. It is related to the Selberg integral (normalization constant) and Jack symmetric polynomials.

Bessel generating functions

The relevant multivariate Bessel functions $B^{(\boldsymbol{\theta})}_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)$ are now defined abstractly from the θ-Dunkl operators

$$\xi_i^{(\boldsymbol{\theta})} := \frac{\partial}{\partial x_i} + \boldsymbol{\theta} \cdot \sum_{j\,:\,j\neq i} \frac{1}{x_i - x_j}(1 - s_{ij}),$$

$$\mathcal{P}_k^{(\boldsymbol{\theta})} := \left(\xi_1^{(\boldsymbol{\theta})}\right)^k + \cdots + \left(\xi_N^{(\boldsymbol{\theta})}\right)^k, \quad k \geqslant 1,$$

and eigenfunction relations

$$\boxed{\mathcal{P}_k^{(\boldsymbol{\theta})}\left(B^{(\boldsymbol{\theta})}_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)\right) = \sum_{i=1}^{N} (a_i)^k \cdot B^{(\boldsymbol{\theta})}_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)}$$

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The relevant Fourier-like transform is now

$$F_N^{(\boldsymbol{\theta})}(x_1,\dots,x_N) := \int B^{(\boldsymbol{\theta})}_{(a_1,\dots,a_N)}(x_1,\dots,x_N)\mathbb{P}_N^{(\boldsymbol{\theta})}(a_1,\dots,a_N)da_1\dots da_N$$

and still satisfies: $\quad \mathcal{P}_k^{(\boldsymbol{\theta})} F_N^{(\boldsymbol{\theta})}\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[\sum_{i=1}^N a_i^k\right].$

LLN for GβE eigenvalues at fixed temperature

Nothing changes if $\theta > 0$ is fixed: as $N \to \infty$, then

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{a_i}{\sqrt{N}}}, \text{ where } (a_1 \geqslant \cdots \geqslant a_N) \text{ is } \mathbb{P}_N^{(\theta)}\text{-distributed},$$

converge weakly, in probability, to a semicircle distribution.

In the limiting $\theta = 0$ case, the interaction $\prod_{i<j}(a_i - a_j)^{2\theta}$ vanishes and we get Gaussian distribution as the limit of empirical measures.

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We were interested in the crossover **high temperature regime**:

$$N \to \infty, \quad \theta \to 0^+, \quad N\theta \to \gamma \in (0, \infty),$$

expecting that: when $\gamma \to \infty$, get semicircle distribution;
when $\gamma \to 0^+$, get Gaussian distribution.

LLN for GβE eigenvalues at high temperature

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

Consider the empirical measures

$$\mu_{N,\theta} := \frac{1}{N}\sum_{i=1}^{N}\delta_{a_i}, \quad \textit{where } (a_1 \geqslant \cdots \geqslant a_N) \textit{ is } \mathbb{P}_N^{(\theta)}\textit{-distributed.}$$

In the regime: $\quad N \to \infty, \quad \theta \to 0^+, \quad N\theta \to \gamma \in (0,\infty),$

the measures $\mu_{N,\theta}$ *converge weakly, in probability, to certain* $\mu^{(\gamma)}$.

The density of $\mu^{(\gamma)}$ is explicit, but complicated: [Allez–Bouchaud–Guionnet '12].

Global asymptotics of GβE eigenvalues at high temp

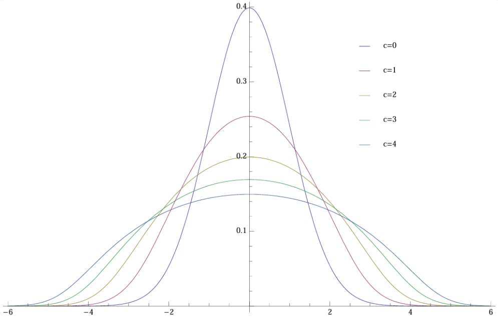
Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

Consider the empirical measures

$$\mu_{N,\theta} := \frac{1}{N}\sum_{i=1}^{N}\delta_{a_i}, \quad \textit{where } (a_1 \geqslant \cdots \geqslant a_N) \textit{ is } \mathbb{P}_N^{(\theta)}\textit{-distributed.}$$

In the limit: $\quad N \to \infty, \quad \theta \to 0^+, \quad N\theta \to \gamma \in (0,\infty),$

we have $\mu_{N,\theta} \to \mu^{(\gamma)}$ *weakly, in probability.*



Density of $\mu^{(\gamma)}$ for various γ (courtesy of [Allez-Bouchaud-Guionnet '12])

Moments of the limiting measure $\mu^{(\gamma)}$

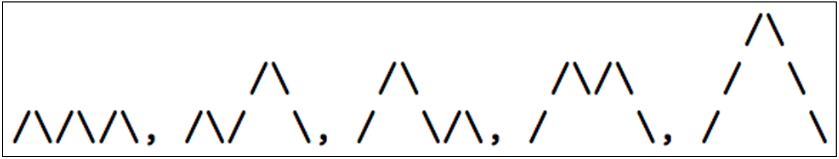
The moment method naturally yielded new moment formulas:

Theorem (Benaych-Georges – Cuenca – Gorin '22)

The limiting measure $\mu^{(\gamma)}$ is uniquely determined by its moments:

$$\int_{-\infty}^{\infty} x^k \mu^{(\gamma)}(dx) \quad = \sum_{\text{Dyck paths } P \text{ of length } k} \text{weight}(P),$$

where: $\text{weight}(P) := \prod_{j \geqslant 1} (j + \gamma)^{\#\text{down steps from height } j}.$



$(1+\gamma)^3, \ (1+\gamma)^2(2+\gamma), \ (1+\gamma)^2(2+\gamma), \ (1+\gamma)(2+\gamma)^2, \ (1+\gamma)(2+\gamma)(3+\gamma)$

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Discrete Beta Ensembles

We switch gears to discrete random partitions.

We are motivated by the **discrete β-ensembles**, due to [Borodin–Gorin–Guionnet '17], on $\mathcal{W}_{N,\mathbb{Z}} := \{(\lambda_1 \geqslant \cdots \geqslant \lambda_N) \in \mathbb{Z}^N\}$:

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- it is convenient to shift the coordinates $\boxed{\ell_i := \lambda_i - (i-1)\theta}$ so that measures will be on N-tuples $(\ell_1 > \cdots > \ell_N)$;
- [BGG '17] considered probability distributions of the form

$$\mathbb{P}_N^{(\theta)}\big(\ell_1 > \cdots > \ell_N\big) \propto \prod_{1 \leqslant i < j \leqslant N} \frac{\Gamma(\ell_i - \ell_j + 1)\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j)\Gamma(\ell_i - \ell_j + 1 - \theta)} \prod_{k=1}^{N} w_N(\ell_k).$$

The interaction factor comes from Jack symmetric polynomials (spherical functions on compact symmetric spaces).

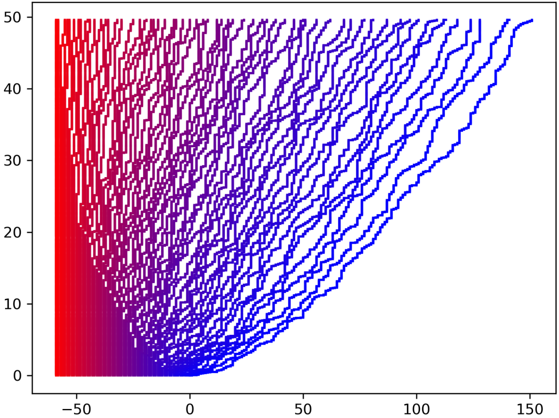
- They proved the LLN for $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i}$ (variational principle).

Discrete β-Dyson Brownian Motion

Another motivation, due to [Gorin–Shkolnikov '15], is a (growing) continuous-time, discrete-space, θ-deformed Markov chain

$$\big(\ell_1(t) > \cdots > \ell_N(t)\big), \quad t \geqslant 0,$$

that is regarded as the discrete β-Dyson Brownian motion.



Initial condition: $\lambda(0) = (0, \ldots, 0) \Leftrightarrow \ell_i(0) = -(i-1)\theta.$

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• [discrete $\to$ contin. space limit] gives β-Dyson Brownian motion;

• random evolution of N non-intersecting particles (not a Doob h-transform, unless $\theta = 1$);

• if started at $\lambda(0) = (0, \cdots, 0) \Leftrightarrow \ell_i(0) = -(i-1)\theta$, then the fixed-time distribution at T is the discrete β-ensemble with

$$w_N(\ell) = \frac{T^\ell}{\Gamma\big(\ell + (N-1)\theta + 1\big)}, \quad \ell \geqslant 0.$$

Cherednik operators, Jack polys & generating functions

The Jack symmetric polys. $J_\lambda^{(\theta)}(x_1,\ldots,x_N) = J_{(\lambda_1,\ldots,\lambda_N)}^{(\theta)}(x_1,\ldots,x_N)$ are defined from the eigenfunction relations

$$\left(\sum_{i=1}^{N}\big(\widetilde{\xi}_i^{(\theta)}\big)^k\right) J_\lambda^{(\theta)}(x_1,\ldots,x_N) = \sum_{i=1}^{N}(\ell_i)^k \cdot J_\lambda^{(\theta)}(x_1,\ldots,x_N), \;\; k \geqslant 1,$$

where the Cherednik operators $\widetilde{\xi}_1^{(\theta)},\ldots,\widetilde{\xi}_N^{(\theta)}$ are

$$\boxed{\widetilde{\xi}_i^{(\theta)} := \theta(1-i) + x_i\frac{\partial}{\partial x_i} + \theta\sum_{j=1}^{i-1}\frac{x_i}{x_i-x_j}(1-s_{i,j}) + \theta\sum_{j=i+1}^{N}\frac{x_j}{x_i-x_j}(1-s_{i,j})}$$

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Our results will apply to measures with "nice" (analytic near 1^N) Jack generating functions:

$$F_N^{(\theta)}(x_1,\dots,x_N) := \sum_\lambda \mathbb{P}_N^{(\theta)}(\lambda)\frac{J_\lambda^{(\theta)}(x_1,\dots,x_N)}{J_\lambda^{(\theta)}(1^N)}.$$

A variation was used for LLN/CLT at fixed temperature [Huang '20].
For **high temperature**, we proved $\cdots$

LLN for random θ-partitions at high temperature

Theorem (C.–Dolega '25; part #1)

Let $\{\mathbb{P}_N\}_{N\geqslant 1}$ *be measures on partitions* $(\lambda_1 \geqslant \cdots \geqslant \lambda_N)$*, with empirical measures* $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{\ell_i}$*, where* $\ell_i = \lambda_i + \theta(i-1)$*.*

<u>If</u> *the JGF's* $\left\{F_N^{(\theta)}(x_1,\ldots,x_N)\right\}_{N\geqslant 1}$ *satisfy:*

- $\displaystyle\lim_{\substack{N\to\infty \\ \boldsymbol{N\theta\to\gamma}}} \frac{1}{(\ell-1)!}\frac{\partial^\ell}{\partial x_1^\ell}\ln\left(F_N^{(\theta)}\right)\Big|_{\boldsymbol{x_1=\cdots=x_N=1}} = \kappa_\ell^{(\gamma)}$*, for all* $\ell \geqslant 1$*.*
- $\displaystyle\lim_{\substack{N\to\infty \\ \boldsymbol{N\theta\to\gamma}}} \frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}}\ln\left(F_N^{(\theta)}\right)\Big|_{\boldsymbol{x_1=\cdots=x_N=1}} = 0$*, for all mixed derivs.*

<u>Then</u> *there is a prob. measure* $\mu^{(\gamma)}$ *with finite moments* $m_1, m_2, \ldots$

s.t. $\boxed{\lim_{N\to\infty}\mu_N = \mu^{(\gamma)}}$ *in the sense of moments, in probability.*

LLN for random θ-partitions at high temperature

Theorem (C.–Dolega '25; part #1). LLN for empirical measures if

- $\displaystyle \lim_{\substack{N\to\infty \\ N\theta\to\gamma}} \frac{1}{(\ell-1)!}\frac{\partial^\ell}{\partial x_1^\ell} \ln\left(F_N^{(\theta)}\right)\Big|_{x_1=\cdots=x_N=1} = \kappa_\ell^{(\gamma)},$ for all $\ell \geqslant 1$.
- $\displaystyle \lim_{\substack{N\to\infty \\ N\theta\to\gamma}} \frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}} \ln\left(F_N^{(\theta)}\right)\Big|_{x_1=\cdots=x_N=1} = 0,$ for all mixed derivs.

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Theorem (C.–Dolega '25; part #1). LLN for empirical measures if

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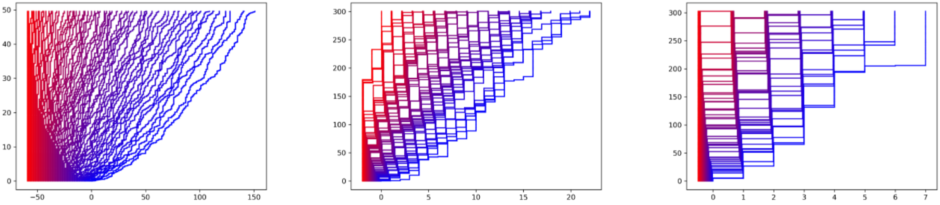
- $$\lim_{\substack{N\to\infty \\ N\theta\to\gamma}} \frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}}\ln\left(F_N^{(\theta)}\right)\Big|_{x_1=\cdots=x_N=1} = 0, \text{ for all mixed derivs.}$$

Example: For fixed-time T distribution of discrete β-DBM from (0^N):

$$\boxed{F_N^{(\theta)}(x_1,\ldots,x_N) = e^{T\cdot\sum_{i=1}^N (x_i-1)}} \Rightarrow \boxed{\kappa_\ell = \delta_{\ell,1}\cdot T}$$



(A) $\theta = 1, N = 60$ (B) $\theta = \frac{2}{N}, N = 60$ (C) $\theta = \frac{1}{2N}, N = 60$

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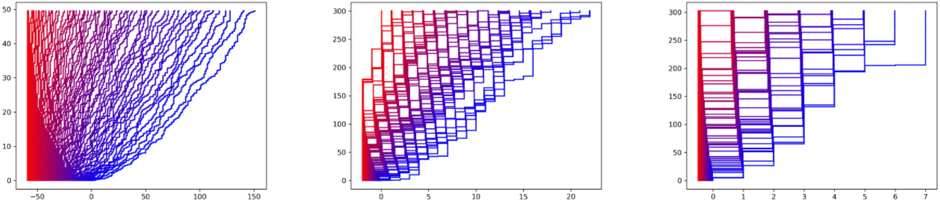
Fixed-time distribution of the discrete β-DBM

Fixed-time T distribution of discrete β-DBM, started at (0^N), is:

$$\mathbb{P}_N^{(\theta;T)}(\ell_1 > \cdots > \ell_N) \propto \prod_{1 \leqslant i < j \leqslant N} \frac{\Gamma(\ell_i - \ell_j + 1)\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j)\Gamma(\ell_i - \ell_j + 1 - \theta)} \prod_{k=1}^{N} \frac{T^{\ell_k}}{\Gamma\big(\ell_k + (N-1)\theta + 1\big)}$$

Theorem (C.–Dolega '25; part #1). The empirical measures $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i}$ where $(\ell_1 > \cdots > \ell_N)$ is $\mathbb{P}_N^{(\theta;T)}$-distributed, converge as $N \to \infty$, $N\theta \to \gamma$, to some prob. measure $\mu^{(\gamma;T)}$.



(A) $\theta = 1, N = 60$ (B) $\theta = \frac{2}{N}, N = 60$ (C) $\theta = \frac{1}{2N}, N = 60$

Law of Large Numbers for random θ-partitions

$$\mathbb{P}_N^{(\theta;T)}\big(\ell_1 > \cdots > \ell_N\big) \propto \prod_{1\leqslant i<j\leqslant N} \frac{\Gamma(\ell_i-\ell_j+1)\Gamma(\ell_i-\ell_j+\theta)}{\Gamma(\ell_i-\ell_j)\Gamma(\ell_i-\ell_j+1-\theta)} \prod_{k=1}^{N} \frac{T^{\ell_k}}{\Gamma\big(\ell_k+(N-1)\theta+1\big)}$$

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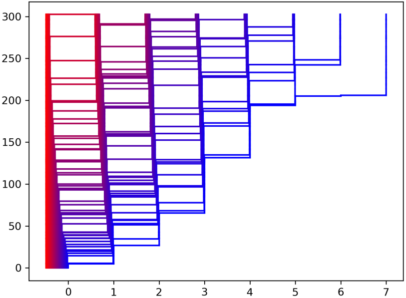

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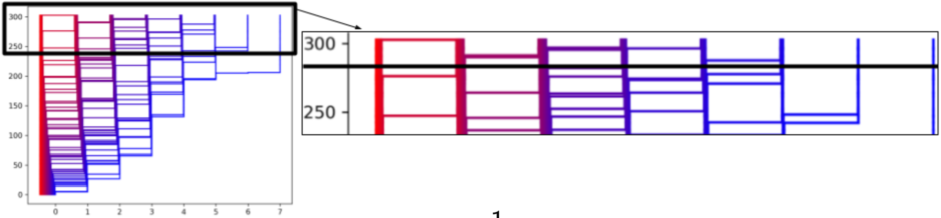

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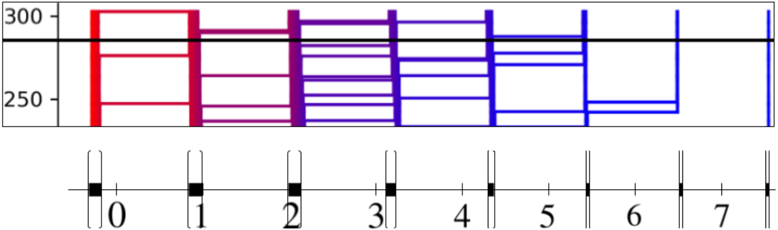


The support of $\mu^{(\gamma;T)}$

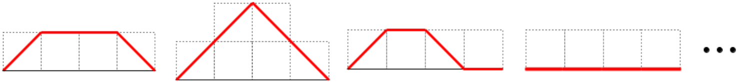
Moments of the limiting measure

Theorem (C.–Dolega '25; part #1)

The probability measure $\mu^{(\gamma;T)}$ is uniquely determined by its moments, e.g. if $\gamma = T$, then:

$$\int_{\mathbb{R}} x^k \mu^{(\gamma;\gamma)}(dx) = \sum_{P \in Motzkin(k)} \frac{weight(P)}{1 + \#horizontal\ steps\ at\ height\ 0}$$

$P =$



$$\mathrm{wt}(P) = \gamma(1+\gamma)^3,\ \gamma^2(2+\gamma)(1+\gamma),\ \frac{1}{2}\gamma^2(1+\gamma)^2,\quad \frac{1}{5}\gamma^4,\quad \cdots$$

What's new?

$$\widetilde{\xi}_i^{(\theta)} := \theta(1-i)+x_i\frac{\partial}{\partial x_i}+\theta\sum_{j=1}^{i-1}\frac{x_i}{x_i-x_j}(1-s_{i,j})+\theta\sum_{j=i+1}^{N}\frac{x_j}{x_i-x_j}(1-s_{i,j}).$$

Unlike Dunkl operators: Even if $F(\vec{x})$ symmetric, we have

$$\left(\widetilde{\xi}_i^{(\theta)}\right)^k F(\vec{x})\Big|_{\vec{x}=(1^N)} \neq \left(\widetilde{\xi}_j^{(\theta)}\right)^k F(\vec{x})\Big|_{\vec{x}=(1^N)}, \text{ if } i \neq j.$$

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The Hecke relations

$$\boxed{\widetilde{\xi}_j^{(\theta)}\, s_{j-1,j} = s_{j-1,j}\, \widetilde{\xi}_{j-1}^{(\theta)} - \theta\cdot \mathrm{id}, \ \ \forall\, 2\leqslant j\leqslant N,}$$

reduce the prelimit k-th moment to a polynomial function of

$$\widetilde{\xi}_1^{(\theta)} F(\vec{x})\Big|_{\vec{x}=(1^N)},\ \left(\widetilde{\xi}_1^{(\theta)}\right)^2 F(\vec{x})\Big|_{\vec{x}=(1^N)},\ \cdots,\ \left(\widetilde{\xi}_1^{(\theta)}\right)^k F(\vec{x})\Big|_{\vec{x}=(1^N)},$$

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$$\widetilde{\xi}_i^{(\theta)} := \theta(1-i) + x_i \frac{\partial}{\partial x_i} + \theta \sum_{j=1}^{i-1} \frac{x_i}{x_i - x_j}(1 - s_{i,j}) + \theta \sum_{j=i+1}^{N} \frac{x_j}{x_i - x_j}(1 - s_{i,j}).$$

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and $\widetilde{\xi}_1^{(\theta)} = x_1 \frac{\partial}{\partial x_1} + \theta \sum_{j=2}^{N} \frac{x_j}{x_1 - x_j}(1 - s_{1,j}) \ \sim$ Dunkl operator!

Moment generating function of the limiting measure $\mu^{(\gamma;T)}$

The moments $m_k^{(\gamma;T)} := \int_{-\infty}^{\infty} x^k \mu^{(\gamma;T)}(dx)$ can be assembled into:

Theorem (C.–Dolega '25; part #2)

As formal power series in z^{-1},

$$ {}_1F_1\big(\gamma; z; -T\big) = \exp\left(\gamma \cdot \widetilde{\mathcal{L}}\left\{ \sum_{k=1}^{\infty} \frac{m_k^{(\gamma;T)}}{k!} (-\mathbf{x})^k - \frac{e^{\gamma\mathbf{x}} - \gamma\mathbf{x} - 1}{\gamma\mathbf{x}} \right\}(z)\right), $$

where:

- $ {}_1F_1\big(a; b; x\big) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \quad (t)_n := t(t+1)\cdots(t+n-1),$

  is the confluent hypergeometric function;
- $\widetilde{\mathcal{L}}$ *is the formal Laplace transform:*

$$ \widetilde{\mathcal{L}}\left\{ \sum_{n\geqslant 0} \frac{f_n}{n!} \mathbf{x}^n \right\}(z) := \sum_{n\geqslant 0} f_n z^{-n-1}. $$

Density of $\mu^{(\gamma;T)}$ via the zeroes of an entire function

By the **Fourier inversion theorem**:

Theorem (C.–Dolega '25; part II)

For any $\gamma, T > 0$, *the density of the limiting measure is*

$$\frac{d\mu^{(\gamma;T)}}{dx}(x) = \frac{1}{\gamma}\left\{ \mathbf{1}_{\left[-\gamma, -\ell_1^{(\gamma;T)}\right]}(x) + \sum_{k=1}^{\infty} \mathbf{1}_{\left[1-\ell_k^{(\gamma;T)}, -\ell_{k+1}^{(\gamma;T)}\right]}(x) \right\},$$

where $\ell_1^{(\gamma;T)}, \ell_2^{(\gamma;T)}, \dots \in \mathbb{R}$ *satisfy:*

$$-\gamma < -\ell_1^{(\gamma;T)}, \qquad 1 - \ell_k^{(\gamma;T)} < -\ell_{k+1}^{(\gamma;T)}, \ \forall\, k \geqslant 1.$$

Moreover, $\ell_1^{(\gamma;T)} > \ell_2^{(\gamma;T)} > \cdots$ *are the zeroes of the entire function*

$$\boxed{G^{(\gamma;T)}(z) := \frac{1}{\Gamma(z)} \cdot {}_1F_1\big(\gamma; z; -T\big), \ z \in \mathbb{C}.}$$

Density of $\mu^{(\gamma;T)}$ via the spectrum of a Jacobi operator

The entire function $G^{(\gamma;T)}(z)$ is the **characteristic function** of a Jacobi matrix:

Theorem (C.–Dolega '25; part II)

For any $\gamma, T > 0$,

$$\frac{d\mu^{(\gamma;T)}}{dx}(x) = \frac{1}{\gamma}\left\{\mathbf{1}_{\left[-\gamma,-\ell_1^{(\gamma;T)}\right]}(x) + \sum_{k=1}^{\infty}\mathbf{1}_{\left[1-\ell_k^{(\gamma;T)},-\ell_{k+1}^{(\gamma;T)}\right]}(x)\right\},$$

and $\left(-\ell_1^{(\gamma;T)} < -\ell_2^{(\gamma;T)} < \cdots\right) =$ *spectrum of the Jacobi matrix*

$$J^{(\gamma;T)} = \begin{bmatrix} T & \sqrt{T(\gamma+1)} & & & \\ \sqrt{T(\gamma+1)} & T+1 & \sqrt{T(\gamma+2)} & & \vdots \\ & \sqrt{T(\gamma+2)} & T+2 & \sqrt{T(\gamma+3)} & \vdots \\ & & \sqrt{T(\gamma+3)} & T+3 & \\ & \cdots & \cdots & & \ddots \end{bmatrix}$$

Outline of the talk

The Gaussian β-ensemble and semicircle distribution

LLN for random β-partitions at high temperature

The limiting measure: moment problem and Jacobi operators

A remark on a deformation of free convolution

Discrete β-DBM with arbitrary initial condition

Assume the discrete β-DBM has initial conditions

$$\ell^{(N)}(0) := \left(\ell_1^{(N)}(0) > \cdots > \ell_N^{(N)}(0)\right), \text{ s.t. } \frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i^{(N)}(0)} \xrightarrow{N\to\infty} \boldsymbol{\nu}.$$

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Theorem (C.–Dolega '25; part #2)

Let $\ell^{(N)}(T) = \big(\ell_1^{(N)}(T) > \ldots > \ell_N^{(N)}(T)\big)$ *be the Markov chain at time* $T > 0$. *Then*

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i^{(N)}(T)} \xrightarrow[N\theta\to\gamma]{N\to\infty} \mu^{(\gamma;T;\boldsymbol{\nu})},$$

for a probability measure $\mu^{(\gamma;T;\boldsymbol{\nu})}$ *uniquely determined by moments, or by the following equalities:*

$$\boxed{\kappa_n^{(\gamma)}\left[\mu^{(\gamma;T;\boldsymbol{\nu})}\right] = \kappa_n^{(\gamma)}\left[\mu^{(\gamma,T)}\right] + \kappa_n^{(\gamma)}[\boldsymbol{\nu}], \text{ for all } n \geqslant 1.}$$

A deformation of (quantized) free convolution

Question

Given two probability measures μ, ν *with finite* $\kappa_n^{(\gamma)}[\mu]$, $\kappa_n^{(\gamma)}[\nu]$, *does there exist a probability measure* $\mu \boxplus^{(\gamma)} \nu$ *such that*

$$\kappa_n^{(\gamma)}\big[\mu \boxplus^{(\gamma)} \nu\big] = \kappa_n^{(\gamma)}[\mu] + \kappa_n^{(\gamma)}[\nu], \text{ for all } n \geqslant 1\,?$$

Our theorem answers YES, if $\mu = \mu^{(\gamma, T)}$ and ν is of the form $\nu = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\ell_i^{(N)}(0)}$.

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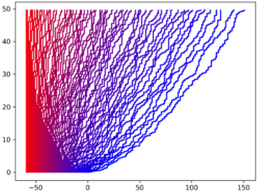
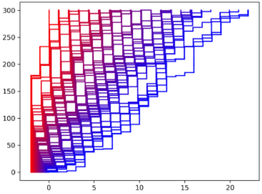
Conjecture

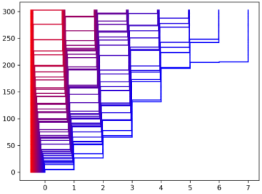
The answer is always YES.

The question is a limiting case of the conjecture of [Stanley '89] on integrality/positivity of Littlewood-Richardson coeffs. of Jack polys.

$\mu \boxplus^{(\gamma)} \nu$ is a 1-parameter γ-deformed (quantized) free convolution ([Voiculescu '92], [Speicher '94], [Bufetov–Gorin '15]).







Thank you for your attention!