On Fredholm Pfaffians and Riemann-Hilbert Problems

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Scalar-Valued Kernels

‘A Riemann-Hilbert approach to
Fredholm determinants
of Hankel composition operators:
Scalar-valued kernels’

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Hankel Composition

For an integral operator acting on $L^2(\Delta)$

$$(Kf)(x) = \int_\Delta K(x,y)f(y)dy,$$

we consider kernels of the form:

- Additive; $\Delta = \mathbb{R}_+$:

$$K(x,y) = \int_0^\infty \phi(x+t)\psi(t+y)dt$$

- Multiplicative; $\Delta = [0,1]$:

$$K(x,y) = t\int_0^1 \phi(xt)\psi(ty)dt$$

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1. We wish to create (as large as possible) a class of Fredholm determinants of integral operators induced by 2×2 matrix-valued kernels which can be reduced to scalar-valued kind.
2. We want to identify what further assumptions are needed about these kernels such that their associated determinants admit a Riemann-Hilbert characterisation, such that we can obtain asymptotic results.

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Fredholm Determinants

Consider an integral operator M acting on $L^2(\Delta) \oplus L^2(\Delta)$,
$\Delta := \bigcup_{j=1}^{m}(a_{2j-1}, a_{2j}) \subset \mathbb{R}$,$m \in \mathbb{N}$, $-\infty < a_1 < ... < a_{2m} < \infty$,

defined by the 2×2 matrix-valued kernel

$$M(x,y) = \begin{bmatrix} M_{11}(x,y) & M_{12}(x,y) \\ M_{21}(x,y) & M_{22}(x,y) \end{bmatrix}.$$

•$M_{11}, M_{22} =$ trace class integral operators on $L^2(\Delta)$

•$M_{12}, M_{21} =$ Hilbert-Schmidt class integral operators on $L^2(\Delta)$

Regularised 2-Determinant

The *regularised 2-determinant* of M is then

$$D_{M,2}(\Delta) := \det\left((I-M)e^{M}\right)e^{-\operatorname{tr}(M_{11}+M_{22})},$$

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where 'det' is the ordinary Fredholm determinant

$$\begin{aligned} D_M(\Delta) &:= \det(I-M) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{\alpha_1,\ldots,\alpha_n=1}^{2} \int_{\Delta^n} \det\bigl(M_{\alpha_i,\alpha_j}(x_i,x_j)\bigr)_{i,j=1}^{n} \, dx_1 \cdots dx_n. \end{aligned}$$

Fredholm Pfaffian

Consider a 2×2 matrix-valued skew-symmetric kernel

$$K(x,y) = \begin{bmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{bmatrix},$$

that induces an integral operator K on $L^2(\Delta) \oplus L^2(\Delta)$ and where skew-symmetry of $K(x,y)$ means

$$K_{11}(x,y) = -K_{11}(y,x), \qquad K_{22}(x,y) = -K_{22}(y,x),$$

$$K_{21}(x,y) = -K_{12}(y,x) \quad \text{on} \quad \Delta \times \Delta.$$

•K_{12}, K_{21} = trace class integral operators on $L^2(\Delta)$

•K_{11}, K_{22} = Hilbert-Schmidt class integral operators on $L^2(\Delta)$

Regularised 2-Pfaffian

We define the *regularised 2-Pfaffian* of K as

$$P_{K,2}(\Delta) := \mathrm{pf}\Big(\mathrm{e}^{-\frac{1}{2}JK}(J + JKJ)\mathrm{e}^{-\frac{1}{2}KJ}\Big)\mathrm{e}^{\frac{1}{2}\mathrm{tr}(K_{21}-K_{12})},$$

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where J is the integral operator on $L^2(\Delta) \oplus L^2(\Delta)$, with distributional kernel

$$J(x,y) = \begin{bmatrix} 0 & \delta_y(x) \\ -\delta_y(x) & 0 \end{bmatrix}, \quad \int_\Delta f(x)\delta_y(x)\mathrm{d}x = f(y), \; y \in \Delta \cup \partial\Delta,$$

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and 'pf' is the ordinary Fredholm Pfaffian

$$P_K(\Delta) \coloneqq \mathrm{pf}(J - K) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Delta^n} \mathrm{pf}\big(K(x_i, x_j)\big)_{i,j=1}^{n} dx_1 \cdots dx_n.$$

Extensions

If $M_{ij} \in$ trace class on $L^2(\Delta)$, then

$$D_{M,2}(\Delta) = \det(I - M) =: D_M(\Delta),$$

Furthermore, if $K_{ij} \in$ trace class on $L^2(\Delta)$,

$$P_{K,2}(\Delta) = \mathrm{pf}(J + JKJ) = \mathrm{pf}(J - K).$$

So $D_{M,2}$ and $P_{K,2}$ are legitimate extensions of D_M and P_K, respectively.

Pfaffian $\leftrightarrow$ Determiant

If we craft M such that it is of the form

$$M(x,y) := -(JK)(x,y) = \begin{bmatrix} -K_{21}(x,y) & -K_{22}(x,y) \\ K_{11}(x,y) & K_{12}(x,y) \end{bmatrix}, \tag{1}$$

we then have the equivalence

$$\big(P_{K,2}(\Delta)\big)^2 = D_{M,2}(\Delta).$$

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we then have the equivalence

$$\left(P_{K,2}(\Delta)\right)^2 = D_{M,2}(\Delta).$$

This is a generalisation of Cayley's theorem: 'Pfaffians are square roots of determinants'. Specifically, that this is also true for their regularised counterparts.

The Symplectic Derived Class

We say K is of *symplectic derived type* if the elements K_{ij} are of the form

$$K_{21}(x,y) = -S(x,y), \qquad K_{22}(x,y) = \frac{\partial}{\partial y} S(x,y),$$
$$\frac{\partial}{\partial x} K_{11}(x,y) = S(x,y) \quad \text{on} \quad \Delta \times \Delta,$$

in terms of a function $S : \Delta \times \Delta \to \mathbb{R}$, the *main kernel*, that is continuously differentiable on $\Delta \times \Delta$, and that obeys the dominance assumptions

$$|S(x,y)| \leq g_1(y), \quad \left| \frac{\partial}{\partial x} S(y,x) \right| \leq g_2(y) \qquad \forall \, (x,y) \in \Delta \times \Delta,$$

with $g_k \in L^2(\Delta)$.

The Symplectic Derived Class

In turn, this leads to the operator M in (1) being of the form

$$M = \begin{bmatrix} S & G \\ H & S^* \end{bmatrix} : L^2(\Delta) \oplus L^2(\Delta) \to L^2(\Delta) \oplus L^2(\Delta),$$

where $G, H : L^2(\Delta) \to L^2(\Delta)$ have skew-symmetric, trace class kernels

$$G(x,y) = -\frac{\partial}{\partial y} S(x,y), \qquad \frac{\partial}{\partial x} H(x,y) = S(x,y) \quad \text{on } \Delta \times \Delta,$$

and S^* is the real adjoint of S.

The Orthogonal Derived Class

We say K is of *orthogonal derived type* if it is of the same form as the symplectic derived type, with a slight modification to K_{11}:

$$\frac{\partial}{\partial x}K_{11}(x,y) = S(x,y) \longrightarrow K_{11}(x,y) = H(x,y) - \epsilon(x-y),$$

where

$$\epsilon(x) := \frac{1}{2}\mathrm{sgn}(x) = \begin{cases} +\frac{1}{2}, & x > 0 \\ -\frac{1}{2}, & x < 0 \end{cases}, \qquad \epsilon(0) := 0.$$

This then leads to the form of M in (1) being

$$M = \begin{bmatrix} S & G \\ H - \epsilon & S^* \end{bmatrix} : L^2(\Delta) \oplus L^2(\Delta) \to L^2(\Delta) \oplus L^2(\Delta).$$

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Symplectic Reductions

If $I - 2S$ is invertible on $L^2(\Delta)$,

$$D_M(\Delta)|_{L^2(\Delta)\oplus L^2(\Delta)} = D_{2S}(\Delta)|_{L^2(\Delta)} \det\big(\delta_{jk} - F_{jk}(\Delta)\big)_{j,k=1}^{2m},$$

with

$$F_{jk}(\Delta) := (-1)^k\big((I - 2S^*)^{-1}H\big)(a_k, a_j).$$

Orthogonal Reductions

If $I - S$ is invertible on $L^2(\Delta)$,

$$D_{M,2}(\Delta)|_{L^2(\Delta)\oplus L^2(\Delta)} = D_S(\Delta)|_{L^2(\Delta)} \det\big(\delta_{jk} - G_{jk}(\Delta)\big)_{j,k=1}^{2m},$$

with

$$\begin{aligned} G_{jk}(\Delta) :=& (-1)^k \big((I - S^*)^{-1} H\big)(a_j, a_k) \\ & - \frac{1}{2} \sum_{\ell=1}^{2m} \sigma_\ell(k) \big((I - S^*)^{-1} H\big)(a_j, a_\ell), \end{aligned}$$

where $\sigma_j(k) \in \{\pm 1\}$.

Commutation Identities

How can we prove these reductions? If D denotes the weak derivative,

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$$S^*H = HS - \sum_{k=1}^{2m} (-1)^k H(\delta_{a_k} \otimes \delta_{a_k})H,$$

$$HG = (S^*)^2 + \sum_{k=1}^{2m} (-1)^k H(\delta_{a_k} \otimes \delta_{a_k})S^*,$$

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$$HG = (S^*)^2 + \sum_{k=1}^{2m} (-1)^k H(\delta_{a_k} \otimes \delta_{a_k})S^*,$$

$$DS^* = G, \qquad SD = DS^* + \sum_{k=1}^{2m} (-1)^k DH(\delta_{a_k} \otimes \delta_{a_k}),$$

$$HD = S^* + \sum_{k=1}^{2m} (-1)^k H(\delta_{a_k} \otimes \delta_{a_k}).$$

For the Symplectic Case

With $\sigma_3 := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, we have that $\sigma(M) \setminus \{0\} \subset \sigma(N)$, where

$$N := \sigma_3 \begin{bmatrix} S^* & -S^* - \sum_{k=1}^{2m} (-1)^k H(\delta_{a_k} \otimes \delta_{a_k}) \\ S^* & -S^* - \sum_{k=1}^{2m} (-1)^k H(\delta_{a_k} \otimes \delta_{a_k}) \end{bmatrix}$$

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Proof Sketch.

$$\begin{bmatrix} H & S^* \\ -H & -S^* \end{bmatrix} M = N \begin{bmatrix} H & S^* \\ -H & -S^* \end{bmatrix}$$

$$N \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \lambda \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \qquad \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} := \begin{bmatrix} H & S^* \\ -H & -S^* \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Matching Spectra

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$$D_M(\Delta) = \prod_{j=1}^{\infty}(1 - \lambda_j(M)) = \prod_{j=1}^{\infty}(1 - \lambda_j(N))$$

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Proof Sketch.

Need to establish that

$$\begin{gathered}\text{algebraic multiplicity of } \lambda \text{ for } M = \operatorname{rank}\big(P_{\lambda,M}\big) \\ = \operatorname{rank}\big(P_{\lambda,N}\big) = \text{algebraic multiplicity of } \lambda \text{ for } N \\ \forall\, \lambda \in \sigma(M) \setminus \{0\},\end{gathered}$$

where $P_{\lambda,T}$ is the spectral projection of a bounded operator T on a Banach space at λ.

Determinantal Manipulations

It remains to perform some determinantal operations,

$$\begin{aligned} D_M(\Delta) &= \det \begin{bmatrix} I - S^* & S^* + \sum_{k=1}^{2m}(-1)^k H(\delta_{a_k} \otimes \delta_{a_k}) \\ S^* & I - S^* - \sum_{k=1}^{2m}(-1)^k H(\delta_{a_k} \otimes \delta_{a_k}) \end{bmatrix} \\ &= \det \begin{bmatrix} I - S^* & S^* + \sum_{k=1}^{2m}(-1)^k H(\delta_{a_k} \otimes \delta_{a_k}) \\ I & I \end{bmatrix} \\ &= \det \left(I - 2S^* - \sum_{k=1}^{2m}(-1)^k H(\delta_{a_k} \otimes \delta_{a_k}) \right) \\ &= D_{2S}(\Delta) \det \left(I - \sum_{k=1}^{2m}(-1)^k (I - 2S^*)^{-1} H(\delta_{a_k} \otimes \delta_{a_k}) \right), \end{aligned}$$

where ‘det’ here is an ordinary $2m \times 2m$ matrix determinant.

For the Orthogonal Case

The derivation of the orthogonal reduction results proceeds similarly with one extra complication. If one attempts to construct directly a version of N, now accounting for the added ϵ, terms appear which are not Hilbert-Schmidt and so the Fredholm determinant is undefined.

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$$N_n := \begin{bmatrix} HD & S^* \\ HD - \eta_n D & S^* \end{bmatrix}$$

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$$N_n := \begin{bmatrix} HD & S^* \\ HD - \eta_n D & S^* \end{bmatrix}$$

One then simply needs to establish that the norm on $L^2(\Delta)$ of $(\eta_n - \epsilon)Df \to 0$ as $n \to \infty$, and then perform similar determinantal manipulations as for the symplectic case.

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4. Orthogonal derived with truncated Wiener-Hopf type main kernel

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Symplectic Main Kernel

Set $\Delta = (t, \infty) \subset \mathbb{R}$ and

$$S(x,y) := \frac{1}{2}\left(\int_0^\infty \phi(x+u)\phi(u+y)du - \frac{1}{2}\phi(x)\int_y^\infty \phi(z)dz\right),$$

with $\phi : \mathbb{R} \to \mathbb{R}$ continuously differentiable, both ϕ and its derivative $D\phi$ decaying exponentially fast at $+\infty$ and $D\phi$ obeying the dominance assumption

$$|D\phi(x+y)| \leq g_3(x) \qquad \forall\,(x,y) \in (0,\infty)\times\mathbb{R} \quad \text{with} \quad g_3 \in L^2(0,\infty).$$

Further Algebraic Manipulations

Let $D_M(t) = \lim_{a_2\to\infty} D_M((t,a_2)), t \in \mathbb{R}$ denote the Fredholm determinant of M, with the aforementioned assumptions placed on ϕ. Then (matching Krajenbrink)

$$D_M(t) = D_Q(t)\left[1 + \frac{1}{2}\int_t^\infty \big((I-Q)^{-1}\phi\big)(x)\Phi(x)\mathrm{d}x\right],$$

with $\Phi(x) := \int_x^\infty \phi(z)\mathrm{d}z$ and provided $I-2S$ and $I-Q$ are invertible on $L^2(t,\infty)$. Here $Q : L^2(t,\infty) \to L^2(t,\infty)$ is the integral operator with trace class kernel

$$Q(x,y) := \int_0^\infty \phi(x+u)\phi(u+y)\mathrm{d}u, \qquad x,y \in (t,\infty),$$

and $D_Q(t)$ is the Fredholm determinant of Q acting on $L^2(t,\infty)$.

Proof Sketch.

$$\begin{tikzcd}
D_M(\Delta) = D_{2S}(\Delta)\det\left(\delta_{jk} - F_{jk}(\Delta)\right)_{j,k=1}^{2m} \arrow[d, "m = 1\ ;\ a_1 = t,\ a_2 = \infty"] \\
D_M((t,\infty)) = D_{2S}((t,\infty))(1+\tau(t)),
\end{tikzcd}$$

where $\tau(t) \coloneqq ((I-2S^*)^{-1}H)(t,t)$, and using that $((I-2S^*)^{-1}H)(x,a_2) \to 0$ pointwise in $x \in (t,\infty)$ as $a_2 \to \infty$, due to the decay properties of ϕ at $+\infty$.

Infinite-Dimensional Sherman-Morrison Identity

Proof Sketch.

If $I - K + \alpha \otimes \beta$ is invertible, then

$$(I - K + \alpha \otimes \beta)^{-1} = (I - K)^{-1} - \frac{(I-K)^{-1}(\alpha \otimes \beta)(I-K)^{-1}}{1 + \langle \beta, (I-K)^{-1}\alpha \rangle}.$$

So we can show that $\tau := \tau(t)$ satisfies

$$\tau = -\tau + \frac{2\tau^2}{1 + 2\tau},$$

and we chose the solution which satisfies the boundary condition $\tau(+\infty) = 0$, and then factor out $(I - Q)$ of D_{2S}. ☐

Orthogonal Main Kernel

$$S(x,y) := \int_0^\infty \phi(x+u)\phi(u+y)\mathrm{d}u + \frac{1}{2}\phi(x)\left(1 - \int_y^\infty \phi(z)\mathrm{d}z\right)$$

$$\downarrow$$

$$D_{M,2}(t) = D_Q(t)\left[1 - \int_t^\infty \big((I-Q)^{-1}\phi\big)(x)\big(1-\Phi(x)\big)\mathrm{d}x\right]$$

where $\Phi(x) = \int_x^\infty \phi(z)\mathrm{d}z$.

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Riemann-Hilbert Problem

Let $t \in \mathbb{R}$ and $\phi \in W^{1,1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. Find $X(z) = X(z; t, \phi) \in \mathbb{C}^{2\times 2}$ so that

(1) $z \mapsto X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) $z \mapsto X(z)$ admits continuous pointwise limits $X_{\pm}(z) := \lim_{\epsilon \downarrow 0} X(z \pm \mathrm{i}\epsilon), z \in \mathbb{R}$ that satisfy

$$X_+(z) = X_-(z) \begin{bmatrix} 1 - |r(z)|^2 & -\bar{r}(z)\mathrm{e}^{-\mathrm{i}tz} \\ r(z)\mathrm{e}^{\mathrm{i}tz} & 1 \end{bmatrix};$$

$$r(z) := -\mathrm{i}\int_{-\infty}^{\infty} \phi(y)\mathrm{e}^{-\mathrm{i}zy}\mathrm{d}y,$$

(3) As $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$, $X(z) = \mathbb{I} + \frac{1}{z}X_1 + o\big(z^{-1}\big)$

What We Gain

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$$\frac{\mathrm{d}}{\mathrm{d}t} \ln D_Q(t) = \mathrm{i} X_1^{11}(t,\phi),$$

and

$$((I-Q)^{-1}\phi)(t) = X_1^{12}(t,\phi) = X_1^{21}(t,\phi).$$

$t \to -\infty$ Result

Suppose $\phi \in W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies

$$|\phi(x)| \le \mathrm{e}^{-a|x|}, \qquad |D\phi(x)| \le \mathrm{e}^{-a|x|} \qquad \forall\, x \in \mathbb{R},$$

with $a \ge 2 + \epsilon$ for some $\epsilon > 0$.

$t \to -\infty$ Result

Suppose $\phi \in W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies

$$|\phi(x)| \leq \mathrm{e}^{-a|x|}, \qquad |D\phi(x)| \leq \mathrm{e}^{-a|x|} \qquad \forall\, x \in \mathbb{R},$$

with $a \geq 2+\epsilon$ for some $\epsilon > 0$. If $\|Q\| < 1$ in operator norm on $L^2(t,\infty)$, then $D_M(t)$ is given asymptotically as $t \to -\infty$ by

$$\begin{aligned}\ln D_M(t) = {} & ts(0) + \int_0^\infty xs(x)s(-x)\mathrm{d}x \\ & -\frac{1}{2\pi}\int_{-\infty}^{\infty} \Im\left\{\frac{r'(\lambda)}{r(\lambda)}\right\} \ln\left(1-|r(\lambda)|^2\right)\mathrm{d}\lambda \\ & +2\ln\left(\frac{1}{2}\sqrt[4]{\frac{1+\mathrm{i}r(0)}{1-\mathrm{i}r(0)}} + \frac{1}{2}\sqrt[4]{\frac{1-\mathrm{i}r(0)}{1+\mathrm{i}r(0)}}\right) + \mathcal{O}\left(t^{-\infty}\right),\end{aligned}$$

where $s(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left(1-|r(\lambda)|^2\right)\mathrm{e}^{\mathrm{i}x\lambda}\mathrm{d}\lambda$.

Bounds on $r(z)$

Consider a region

$$\Omega := \{z \in \mathbb{C} : |\Im z| \leq \epsilon\}.$$

Bounds on $r(z)$

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From our assumptions about the growth of ϕ, we can show that for $z \in \Omega$

$$|r(z)| \leq \int_{-\infty}^{\infty} |\phi(y)| e^{(\Im z) y} dy \leq \frac{2a}{a^2 - \epsilon^2} < 1.$$

Doing similarly with the bound on $D\phi$, we have that for $z \in \Omega$, both $|r(z)|$ and $|zr(z)|$ are less than one.

As such, we can move our RHP off $\mathbb{R}$, so long as we remain in Ω.

Deift-Zhou Non-linear Steepest Descent

Setting $r(z) = r_1$ and $\bar{r}(z) = r_2$, we recognise

$$\begin{bmatrix} 1 - r_1 r_2 & -r_2 e^{-itz} \\ r_1 e^{itz} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{r_1 e^{itz}}{1 - r_1 r_2} & 1 \end{bmatrix} \begin{bmatrix} 1 - r_1 r_2 & 0 \\ 0 & \frac{1}{1 - r_1 r_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{r_2 e^{-itz}}{1 - r_1 r_2} \\ 0 & 1 \end{bmatrix}$$

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Define

$$Y(z) := X(z) \begin{cases} \begin{bmatrix} 1 & \frac{r_2 e^{-itz}}{1-r_1 r_2} \\ 0 & 1 \end{bmatrix} & \Im z \in (0, \epsilon) \\ \begin{bmatrix} 1 & 0 \\ \frac{r_1 e^{itz}}{1-r_1 r_2} & 1 \end{bmatrix} & \Im z \in (-\epsilon, 0) \\ \mathbb{I} & \text{o/w} \end{cases}$$

Jumps for Y

$$Y_+(z) = Y_-(z) \begin{bmatrix} 1 & -\frac{r_2 e^{-itz}}{1-r_1 r_2} \\ 0 & 1 \end{bmatrix}, \qquad \Im z = \epsilon$$

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Model of Y

$$P_+(z) = P_-(z) \begin{bmatrix} 1 - r_1 r_2 & 0 \\ 0 & \frac{1}{1-r_1 r_2} \end{bmatrix}, \qquad z \in \mathbb{R}$$

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Plemelji formula $\implies$

$$\ln f(z) = \int_{-\infty}^{\infty} \frac{\ln a(\lambda)}{\lambda - z} \frac{d\lambda}{2\pi i},$$

$$\implies P(z) = \exp\left[\frac{\sigma_3}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - r_1(\lambda) r_2(\lambda))}{\lambda - z} d\lambda\right].$$

Jumps for R

Define a new function $R(z) \coloneqq Y(z)P^{-1}(z)$When $z \in \mathbb{R}$,

$$R_+(z) = Y_+P_+^{-1} = Y_- \begin{bmatrix} 1 - r_1r_2 & 0 \\ 0 & \frac{1}{1-r_1r_2} \end{bmatrix} \left(P_- \begin{bmatrix} 1 - r_1r_2 & 0 \\ 0 & \frac{1}{1-r_1r_2} \end{bmatrix} \right)^{-1}$$

$$= Y_-P_-^{-1} = R_-(z).$$

For the jump at $\Im z = \epsilon$ (can also choose to use the jump at $\Im z = -\epsilon$)

$$R_+(z) = Y_+(z)P^{-1}(z) = R_-(z)P(z) \begin{bmatrix} 1 & -\frac{r_1e^{-itz}}{1-r_1r_2} \\ 0 & 1 \end{bmatrix} P^{-1}(z).$$

Small Norm Problem

If $G(z;t) :=$ ‘Jump Matrix of R’, one can show

$$||G(\circ;t)-\mathbb{I}||_{L^2(\Im z=\pm\epsilon)\cap L^\infty(\Im z=\pm\epsilon)} \xrightarrow{t\to-\infty} 0,$$

exponentially fast.

$$\implies R(z)=\mathbb{I}+\frac{1}{2\pi i}\int_{\Im\lambda=\pm\epsilon} R_-(\lambda)(G(\lambda)-\mathbb{I})\frac{d\lambda}{\lambda-z},$$

and

$$||R_- - \mathbb{I}||_{L^2} \le Ce^{t\epsilon},$$

as $t\to-\infty$.

$\frac{d}{dt} \ln D_Q(t)$

Reversing the transformations, one finds

$$X_1 = R_1 + P_1 \implies X_1^{11} = P_1^{11} + R_1^{11},$$

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$$\frac{d}{dt}\ln D_Q(t) \xrightarrow{t\to-\infty} -\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln(1-|r(\lambda)|^2)d\lambda + \mathcal{O}(t^{-\infty})$$

Constant Factor Computation

After indefinite integration in t

$$\ln D_Q(t) = ts(0) + \varpi + \mathcal{O}(t^{-\infty}), \quad t \to -\infty.$$

(Recall $s(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left(1 - |r(\lambda)|^2\right) \mathrm{e}^{\mathrm{i}x\lambda} \mathrm{d}\lambda$)

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The RHP is then solved again for $t \to -\infty$, this time to find $\partial_\gamma \ln\det(I - \gamma Q)$, which can be definitely γ-integrated over $[0,1]$ to find terms constant in t.

The Orthogonal Case

The orthogonal derived class gives an almost identical asymptotic result for $t \to -\infty$, with some variation in the constant term which comes from the matrix determinant.
One finds this from the same Riemann-Hilbert problem as the symplectic asymptotic result.

$$\begin{aligned}\ln D_{M,2}(t) = {} & ts(0) + \int_0^\infty xs(x)s(-x)\mathrm{d}x \\ & -\frac{1}{2\pi}\int_{-\infty}^{\infty} \Im\left\{\frac{r'(\lambda)}{r(\lambda)}\right\}\ln\left(1-|r(\lambda)|^2\right)\mathrm{d}\lambda \\ & +\frac{1}{2}\ln\left(\frac{1-\mathrm{i}r(0)}{1+\mathrm{i}r(0)}\right) + \mathcal{O}\left(t^{-\infty}\right)\end{aligned}$$

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Symplectic Algebraic Result

$$S(x,y) := \frac{1}{4\pi}\int_{-\infty}^{\infty}\phi(u)\mathrm{e}^{\mathrm{i}u(x-y)}\mathrm{d}u, \qquad x,y\in\Delta=(-t,t),$$

where $\phi:\mathbb{R}\to\mathbb{R}$ is an even function such that $\phi\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $\mathbb{R}\ni x\mapsto x\phi(x)\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$.

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$$D_M(t) = D_Q(t)\left[1+\frac{1}{2}\int_{-t}^{t}\big((I-Q)^{-1}Q\big)(x,t)\mathrm{d}x\right]$$
$$\times\left[1-\frac{1}{2}\int_{-t}^{t}Q(x,t)\mathrm{d}x-\frac{1}{2}\int_{-t}^{t}\left(\int_{-x}^{x}Q(z,t)\mathrm{d}z\right)\big((I-Q)^{-1}Q\big)(x,t)\mathrm{d}x\right],$$

provided $I-2S$ is invertible on $L^2(-t,t)$.

Here, Q is the integral operator with kernel $Q(x,y):=2S(x,y)$.

$t \to \infty$ Result

Suppose $\phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is an even function with $x\phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $|\phi(x)| < 1$ for $x \in \mathbb{R}$, such that $z \mapsto \phi(z)$ is analytic in some neighbourhood U of $\mathbb{R}$ with $\lim_{\substack{|z| \to \infty \\ z \in U}} \phi(z) = 0$.

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If $\|Q\| < 1$ in operator norm on $L^2(-t,t)$, then $D_M(t)$ is given asymptotically as $t \to +\infty$ by

$$\begin{aligned} \ln D_M(t) = {} & -ts(0) + \frac{1}{4}\int_0^\infty xs(x)s(-x)\mathrm{d}x \\ & + 2\ln\left(\frac{1}{2}\sqrt[4]{1-\phi(0)} + \frac{1}{2\sqrt[4]{1-\phi(0)}}\right) + \mathcal{O}\left(t^{-\infty}\right), \end{aligned}$$

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- We form two classes of integral operators induced by 2×2 matrix-valued kernels: symplectic derived and orthogonal derived.
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- Once we have scalar-valued kernels, we identity two further classes of ‘main kernels’, the associated Fredholm determinants of which admit a Riemann-Hilbert characterisation. We consider these as Fredholm determinants now acting on one specified interval.
- We obtain asymptotics from the associated RHP via a transformation into a small-norm problem which can be solved via general theory.

Thank You for Your Attention!