Elliptic Orthogonal Polynomials and OPRL

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Victor Alves

Universidade de São Paulo
victorjulio@alumni.usp.br

Summary

Orthogonal Polynomials

Elliptic curves

Elliptic Orthogonality

Decomposition of EOP in terms of OPRL

Based on "Elliptic orthogonal polynomials and OPRL", joint work with Martinez-Finkelshtein, to appear in Journal of Mathematical Analysis and Applications.

Elliptic Orthogonal Polynomials

Orthogonal Polynomials

What are Orthogonal Polynomials?

$$p_m(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$$

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

What are Orthogonal Polynomials?

Let $d\mu(x)$ be a borel finite positive measure on $\mathbb{C}$ with finite moments.

A polynomial $p_n(z) = z^n +$ lower order terms is ***orthogonal*** when

$$\int_{\mathbb{C}} p_n(x) z^k \, d\mu(x) = 0, \quad 0 \leqslant k < n.$$

Equivalently, p_n is orthogonal to every polynomial of degree $< n$.

Above is called *non-hermitian orthogonality* (the integral is not a well defined inner product - unless $\operatorname{supp} \mu \subset \mathbb{R}$).

Existence of orthogonal polynomials

The existence of p_n is equivalent to the non-vanishing of the determinant

$$D_n := \det(m_{i+j})_{i,j=0}^{n-1}, \quad m_k := \int z^k \, d\mu(z).$$

It can be immediately verified from the determinantal identity

$$p_n(z) = \frac{1}{D_n} \det \begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & \cdots & m_n & m_{n+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} & m_{2n-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}.$$

The Three Term Recurrence Relation

There exists two sequences a_n and b_n such that

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where $p_{-1} \equiv 0$. Moreover, for $h_n := \int p_n(x)^2 \, d\mu(x)$, we have

$$a_n = \frac{h_n}{h_{n-1}}, \quad b_n = \frac{1}{h_n} \int x p_n(x)^2 \, d\mu(x).$$

Other properties.

- Zeroes of p_n lie in the convex hull of supp μ.
- The reproducing Kernel of OPs

$$K_n(z,w) = \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{h_j}$$

satisfies the Christoffel Darboux Identity

$$K_n(x,y) = \frac{1}{h_n} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}.$$

Orthogonal polynomial in the real line (OPRL)

When $\operatorname{supp} \mu \subset \mathbb{R}$ we say that p_n is an *orthogonal polynomial in the real line* (OPRL). In this case:

- p_n has real coefficients. In particular $p_n(\bar{z}) = \overline{p_n(z)}$;
- The zeros of p_n are simple and interlace with those of p_{n+1}

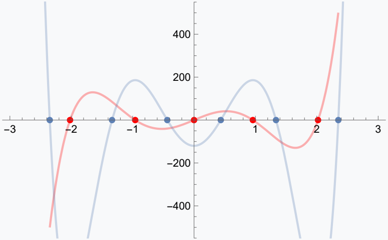


Figure: Plot of the 5-th and the 6-th Hermite polynomials.

Elliptic Orthogonal Polynomials

Elliptic

Genus 1 Compact Riemann Surfaces

Every compact Riemann surface of genus 1 $\mathcal{T}$ is a torus.

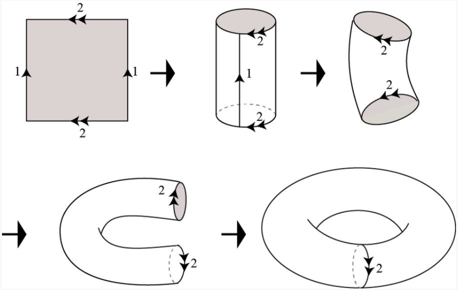


Figure: “Survey of Graph Embeddings into Compact Surfaces”

Genus 1 Compact Riemann Surfaces

For each such $\mathcal{T}$, there exists a representation of the form

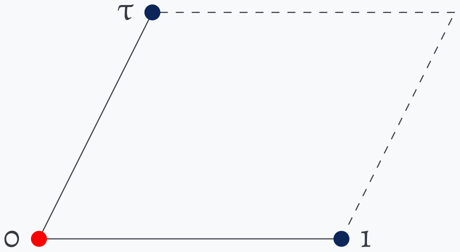
$$\mathcal{T} = \frac{\mathbb{C}}{\Lambda_\tau}, \quad \Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}, \quad \operatorname{Im}\tau > 0.$$

Meromorphic functions in $\mathcal{T}$ are identified with *elliptic functions* - meromorphic functions in $\mathbb{C}$ with two linearly independent periods, in this case, 1 and τ.

Fundamental paralellogram

Elliptic functions of $\mathcal{T}$ are completely determined by their values in the *fundamental parallelogram*

$$\Delta := \{t + s\tau; s, t \in [0, 1)\}.$$



Meromorphic functions on $\mathcal{T}$:

- Weierstrass $\wp$ function

$$\wp(z) := \frac{1}{z^2} + \sum_{w \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

$\wp$ is an even elliptic function with periods 1 and τ and a unique pole of order two at 0.

$\wp$ satisfies the ODE

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \quad e_j := \wp(\omega_j)$$

where ω_j is the *j-th half period* of the lattice Λ_τ, i.e.,

$$\omega_1 := \frac{1}{2}, \quad \omega_2 := \frac{1}{2} + \frac{\tau}{2}, \quad \omega_3 := \frac{\tau}{2}.$$

Meromorphic functions on $\mathcal{T}$:

The Weierstrass $\wp$ function can be understood as a uniformizing parameter: the map

$$z \mapsto (\wp(z), \wp'(z))$$

is a biholomorphism between $\mathcal{T}$ and the elliptic curve $\mathcal{C}$ of equation

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

Meromorphic functions on $\mathcal{T}$

- Weiestrass ζ (zeta) function

$$\zeta(z) := \frac{1}{z} - \int_0^z \left[\wp(w) - \frac{1}{w^2}\right] \, \mathrm{d}w$$

  where the path of integration does not intersect any element of Λ_τ. ζ satisfies $\zeta'(z) = -\wp(z)$.

ζ is NOT an elliptic function, but it is an odd meromorphic quasi-periodic function satisfying

$$\zeta(u + 2\omega_i) = \zeta(u) + \eta_i, \quad \eta_i := \zeta(\omega_i).$$

"Polynomial-like" functions on $\mathcal{T}$

A polynomial p on $\mathbb{C}$ is a meromorphic function with a unique single pole at ∞, whose order of the pole is exactly its degree.

We use above definition as a motivator to define *polynomial-like* functions on $\mathcal{T}$. We fix a point to act as ∞ in this setup, i.e., ***the degree of our functions shall correspond to the order of the pole at this point***.

$0 \in \mathcal{T}$ is a natural candidate since it corresponds to the unique point at infinity of $\mathcal{C}$.

Definition. The ***polynomial degree*** of a meromorphic function f is the order of its pole at $0 \in \mathcal{T}$.

Example: $\wp$ has polynomial degree 2, $\zeta(z) - \zeta(z-a)$ has polynomial degree 1.

Degree 1 polynomials in $\mathcal{T}$

There is no meromorphic function f of polynomial degree 1 that is analytic on $\mathcal{T} \setminus \{0\}$.

$$0 = \frac{1}{2\pi i} \int_{\partial \Delta} f(s)\, ds = \sum \text{residues inside } \Delta.$$

To allow the possibility of degree 1 polynomial functions, we shall admit the possibility of an extra simple pole $a \neq 0$.

This can be further justified by the theory of Padé approximation. This was the initial framework utilized by Bertola to consider these orthogonal meromorphic functions in higher genus.

Elliptic a-polynomials

Definition. An ***elliptic a-polynomial*** f is a meromorphic function on $\mathcal{T}$, that is analytic on $\mathcal{T} \setminus \{0, a\}$, whose possible pole at a is of maximum order 1.

There are unique $c \in \mathbb{C}$ and $n \geqslant 0$ such that

$$f(z) = \frac{1}{z^n}\left(c + \mathcal{O}(z)\right), \quad z \to 0.$$

we call n and c the *degree* and the *leading coefficient* of f, respectively.

When $c = 1$ we say that f is *monic*.

Space of elliptic a-polynomials

The space of elliptic a-polynomials is the Riemann-Roch space $L(n \cdot 0 + a)$.

A monic basis for this base is given by

$$\begin{aligned} b_0(z) &:= 1, \\ b_1(z) &:= \zeta(z) - \zeta(z-a) - \zeta(a), \\ b_{2k}(z) &:= \wp(z)^k, \\ b_{2k+3}(z) &:= -\frac{1}{2}\wp'(z)\wp^k(z), \quad k \geqslant 0. \end{aligned}$$

Orthogonality

Let Γ be a contour on $\mathcal{T}$, W a function defined on Γ and

$$d\mu(s) = W(s)ds$$

a positive probability measure.

We say that an elliptic a-polynomial f_n of polynomial degree n is orthogonal (shortly, an EOP) with respect to μ when

$$\int_\Gamma f_n(z) b_j(z)\, d\mu(z) = 0, \quad 0 \leqslant j < n.$$

We fix the notation F_n for the n-th monic EOP.

Analogous properties to usual orthogonality

The n-th monic orthogonal elliptic a-polynomial F_n exists if, and only if,

$$D_n := \det(\mu_{i,j})_{i,j=0}^{n-1} \neq 0, \quad \mu_{i,j} := \int b_i(z) b_j(z) \, d\mu(z).$$

Similarly, f_n admits a determinamtal formula

$$F_n(z) = \frac{1}{D_n} \begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n-1} & \mu_{0.n} \\ \mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n-1} & \mu_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n-1} & \mu_{n-1,n} \\ b_0(z) & b_1(z) & \cdots & b_{n-1}(z) & b_n(z) \end{pmatrix}.$$

Five Term Recurrence Relation

Assume $D_n \neq 0$ for all $n \geqslant 0$ and let f_n be the n-th orthonormal EOP. There exists constants $A_k, B_k, C_k \in \mathbb{C}$ such that

$$\wp(z)f_k(z) = A_k f_{k+2}(z) + B_k f_{k+1}(z) + C_k f_k(z) + B_{k-1} f_{k-1}(z) + A_{k-2} f_{k-2}(z),$$

where $f_{-1} \equiv f_{-2} \equiv 0$. Moreover,

$$A_k = \int_\Gamma \wp(z) f_k(z) f_{k+2}(z) W(z)\, dz,$$
$$B_k = \int_\Gamma \wp(z) f_k(z) f_{k+1}(z) W(z)\, dz,$$
$$C_k = \int_\Gamma \wp(z) f_k(z)^2 W(z)\, dz.$$

Why we cannot find a three term recurrence

In general we cannot multiply f_k by b_1, the pole at a can become of order 2 and, in that case, $f_k b_1$ is not an EOP anymore.

The five term recurrence reduces to three term when we heavily restrict W, Γ and a. An analogous phenomena to OPRL with even weights.

Christoffel-Darboux Formula

The reproducing Kernel of EOP

$$K_n(z,w) = \sum_{j=0}^{n-1} f_j(z) f_j(w)$$

satisfies the CD-like identity

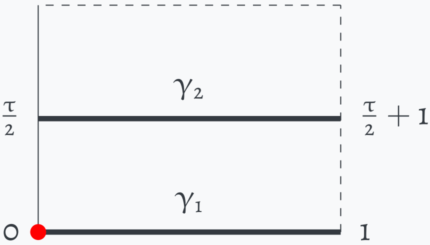
$$K_n(z,w) = A_{n-1}\frac{f_{n+1}(z)f_{n-1}(w) - f_{n+1}(w)f_{n-1}(z)}{\wp(z)-\wp(w)}$$
$$+A_{n-2}\frac{f_n(z)f_{n-2}(2) - f_n(2)f_{n-2}(z)}{\wp(z)-\wp(w)} + B_{n-1}\frac{f_n(z)f_{n-1}(w) - f_n(w)f_{n-1}(z)}{\wp(z)-\wp(w)}.$$

EOP and Real Orthogonality

The real locus of $\mathcal{T}$

$$\{z \in \mathcal{T}; z = \bar{z}\}$$

has at maximum 2 connected components (Harnack's Theorem). It is exactly 2 if, and only if, $\tau \in i\mathbb{R}^+ = 0$, and in this case they are given by



Definitions:

- $\gamma_1 = [0, 1)$;
- $\gamma_2 = \left[\frac{\tau}{2}, \frac{\tau}{2} + 1\right)$;

EOP and Real Orthogonality

The analogous case of OPRL in $\mathcal{T}$ comes by taking the following assumptions:

- The orthogonality curve Γ is one of the contours γ_1, γ_2;
- The point a lies in the set $(\gamma_1 \cup \gamma_2) \setminus \Gamma$.

Consequences:

- b_j is real valued in $\gamma_1 \cup \gamma_2$;
- F_n can be obtained by Gram-Schmidt;
- *Schwarz identity* $F_n(\bar{s}) = \overline{F_n(s)}$

EOP and Real Orthogonality

Under this assumptions one can prove an Andriéief-type identity:

$$D_n = \frac{1}{n!} \int_{\operatorname{supp} \mu^n} \left(\det \left[b_{j-1}(x_i) \right]_{i,j=1}^{n} \right)^2 \prod_{i=1}^{n} d\mu(x_i).$$

Therefore, since the basis b_j is real-valued over $\gamma_1 \cup \gamma_2$, the fact that

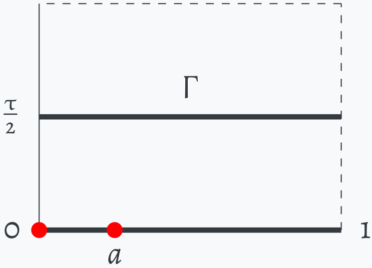
$$\mu_{i,j} = \int_{\Gamma} b_i(z) b_j(z) \, d\mu(z) \in \mathbb{R}$$

implies that $D_n \neq 0$ and thus the n-th EOP exists.

Setup of interest

Assumptions:

- $a \in \gamma_1$;
- $\Gamma = \gamma_2$;



Zeroes

Theorem

- *When n is odd, then f_n has a pole at a, and has exactly $n+1$ simple zeroes at γ_2, that interlace with those of f_{n+1}.;*
- *When n is even, f_n has exactly n simple zeroes at γ_2, and a zero at γ_1 iff it has a pole at a.*

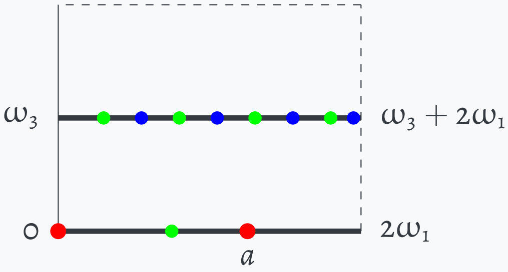


Figure: $n = 3$. Blue points marks zeros of F_n, the green of F_{n+1}.

Decomposition of EOP in terms of OPRL

Behavior of $\wp$ over γ_2

The Weierstrass $\wp$ function maps γ_2 twice over the interval (e_3, e_2).

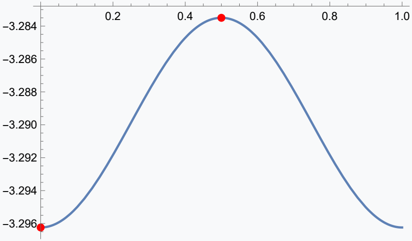


Figure: Plot of $\wp\left(\frac{\tau}{2} + t\right)$ for $\tau = \frac{3i}{2}$.

Let $x = \wp(z)$ for z in the first-half of γ_2. Then $-z$ lies in the second half.

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Basis functions in the x variable

$$b_1(z) = -\frac{\sqrt{\prod_{i=1}^{3}(x-e_i)} + \frac{\wp'(a)}{2}}{x-\wp(a)},$$

$$b_1(-z) = \frac{\sqrt{\prod_{i=1}^{3}(x-e_i)} - \frac{\wp'(a)}{2}}{x-\wp(a)},$$

$$b_{2l}(z) = x^l,$$

$$b_{2l+3}(z) = -x^l\sqrt{\prod_{i=1}^{3}(x-e_i)}.$$

General Decomposition Theorem

There exists polynomials $p_{1,n}$ and $p_{2,n}$ such that

$$F_n(z) = p_{1,n}(x) + b_1(z)p_{2,n}(x) + \frac{\wp'(a)}{2}p_{3,n}(x).$$

where $p_{3,n}(x) := \frac{p_{2,n}(x) - p_{2,n}(\wp(a))}{x - \wp(a)}$. Let

$$w_k^{\pm}(x) := \left(\sqrt{\prod_{l=1}^{3}(x - e_l)}\right)^k \frac{1}{\wp(a) - x}\frac{W(z) \pm W(-z)}{2}$$

$$q_n(x) := p_{1,n}(x)(\wp(a) - x) + \frac{\wp'(a)}{2}p_{2,n}(\wp(a)).$$

Decomposition of EOP in terms of OPRL

Theorem (A.-Martinez-Finkelshtein, 2026)

Let $n = 2m + k$, $k \in \{0, 1\}$. The pair of polynomials $p_{n,2}$ and q_n satisfies the vector orthogonality relations on the interval $[e_3, e_2]$:

$$\int_{e_3}^{e_2} \left[q_n(x) w_{-1}^{+}(x) + p_{n,2}(x) w_0^{-}(x) \right] x^l \,\mathrm{d}x = 0, \quad 0 \leqslant l < m + \frac{k}{2};$$

$$\int_{e_3}^{e_2} \left[q_n(x) w_0^{-}(x) + p_{n,2}(x) w_1^{+}(x) \right] x^l \,\mathrm{d}x = 0, \quad 0 \leqslant l < m - 2;$$

$$\int_{e_3}^{e_2} q_n(x) \left(w_0^{-}(x) + \frac{\wp'(a)}{2} w_{-1}^{+}(x) \right) \frac{\mathrm{d}x}{\wp(a) - x} + \int_{e_3}^{e_2} p_{n,2}(x) \left(w_1^{+}(x) + \frac{\wp'(a)}{2} w_0^{-}(x) \right) \frac{\mathrm{d}x}{\wp(a) - x} = 0.$$

Assuming that W is even

When W is even

$$w_k^-(x) = \left(\sqrt{\prod_{l=1}^{3}(x-e_l)}\right)^k \frac{1}{\wp(a)-x}\frac{W(z)-W(-z)}{2} = 0$$

In this case the orthogonality conditions specializes.

Usual orthogonality when W is even

Corollary

Let W be an even weight on $\Gamma = \gamma_2$. For $n = 2m + k$, $k \in \{0, 1\}$,

$$\int_{e_3}^{e_2} q_n(x) x^l w_{-1}^{+}(x) = 0, \quad 0 \leqslant l \leqslant \deg q_n - 2 + k$$

$$\int_{e_3}^{e_2} p_{n,2}(x) x^l w_1^{+}(x) = 0, \quad 0 \leqslant l \leqslant \deg p_{n,2} - 1 - k$$

Note that for k even, $p_{n,2}$ is orthogonal and q_n is quasi orthogonal. For k odd, $p_{n,2}$ is quasi orthogonal and q_n is orthogonal.

Orthogonality when W is even and $a = \frac{1}{2}$

In the case we assume full symmetry of W and a. We have

$$F_{2j}(z) = p_j(x), \quad F_{2j+1}(z) = b_1(z)\widehat{p_j}(x)$$

where p_j and $\widehat{p_j}$ is a polynomial of degree j orthogonal with respect to the weight

$$w(x) = \frac{W(z)}{\sqrt{(e_1 - x)(e_2 - x)(x - e_3)}},$$

$$\tilde{w}(x) = \sqrt{\frac{(x - e_3)(e_2 - x)}{(e_1 - x)^3}} W(z).$$

in $[e_3, e_2]$, respectivelly.

Example: EOP coming from OPRL

Let $\wp$ be the Weierstrass $\wp$ function of the torus defined by the square lattice

$$\Lambda = 2\omega_1\mathbb{Z}+2i\omega_1\mathbb{Z}, \quad \omega_1 = \frac{32\pi}{\Gamma\left(\frac{1}{4}\right)^4}, \quad e_3 = -1 = -e_1, \quad e_2 = 0.$$

Denote by $P_n^{(\alpha,\beta)}$ the n-th monic OP with respect to

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1.$$

Example: EOP coming from OPRL

For $n = 2j + k$, define:

$$F_n(s) = \frac{1}{2^j} P_j^{(\alpha,\beta)}(2\wp(s)+1), \quad k = 0,$$

$$F_n(s) = \frac{1}{2^{j+1}} \frac{\wp'(s)}{1-\wp(s)}$$
$$\left[P_j^{(\alpha+1,\beta+1)}(2\wp(s)+1) - \frac{\lambda_j}{\lambda_{j-1}} P_{j-1}^{(\alpha+1,\beta+1)}(2\wp(s)+1) \right], \quad k = 1.$$

Where

$$\lambda_n = \lambda_n(\alpha,\beta) = \int_{-1}^{1} \frac{P_n^{(\alpha+1,\beta+1)}(x)}{3-x} (1-x)^{\alpha+1}(1+x)^{\beta+1}\, dx.$$

EOP coming from OPRL

Theorem (A.-Martinez-Finkelshtein, 2026)

F_n is the n-th EOP with $a = \omega_1$, with respect to the weight

$$W(s) = |\wp(s)|^{\alpha+\frac{1}{2}} (\wp(s)+1)^{\beta+\frac{1}{2}} (1-\wp(s))^{\frac{1}{2}} > 0, \quad s \in \Gamma.$$

- This is the first explicit family of a-EOP in literature.

Open questions

- For general orthogonality in $\mathcal{T}$, can you still find some information regarding localization of zeros without the real orthogonality assumption?

In the complex plane, the position of the zeros in the convex hull comes from the fact that the orthogonal polynomials minimize the L^2 norm in the space of polynomials. This later property is satisfied for the EOPs by definition (for the appropriate space)

A more general question is what is the convex hull in the torus?

Open questions

- Can this be utilized to construct an EOP ensemble in the torus?

A very important application of orthogonal polynomials is in the theory of random matrices, as the OP ensemble describes the eigenvalues of unitary ensembles over Hermitian matrices.

Can a similar thing be done in the elliptic setup? A Riemann-Hilbert formulation for these polynomials was presented by Marco Bertola (and there is a similar construction for elliptic functions without the pole at a, given by Desiraju et all).

Asymptotic analysis have already been done by Bertola. Can this be utilized with the CD-like formula to find anything?

Open questions

- What about even higher genus?

Most of the arguments of the work can be generalized for hyperelliptic curves.

- How this relates with other forms of higher genus orthogonality that appears in recent literature?

In particular in the recent developments concerning the decomposition of Matrix Valued Orthogonal Polynomials in terms of scalar orthogonality in higher genus (works of Bertola, Charlier, Kuijlaars and Duits, for example).

Even in genus 0 you can find different orthogonality types – Hermite orthogonality becomes Laurent orthogonality, for example.

Thank you!

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