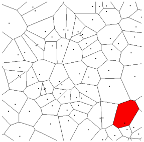
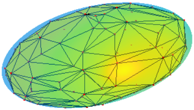
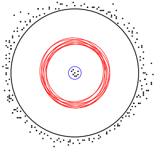
Exceptional events in convex random geometry







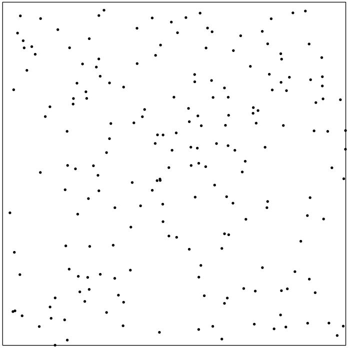
Pierre Calka

Journées Occimath, Montpellier
27 May 2026

UNIVERSITÉ DE ROUEN NORMANDIE

LMRS

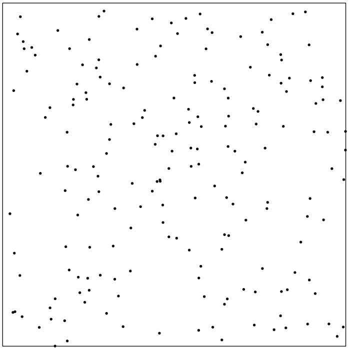
Generate random points in a compact window W



Binomial point process

Fix $n \geq 1$, generate n independent points uniformly distributed in W

Generate random points in a compact window W

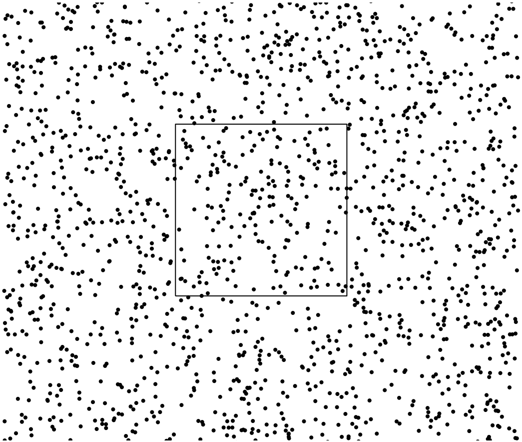


Poisson point process in W

Fix $\lambda \geq 1$, generate Pois(λ) independent points uniformly distributed in W

When W has unit volume, $\lambda :=$ intensity

Generate random points in $\mathbb{R}^d$



Homogeneous Poisson point process in $\mathbb{R}^d$

Same intersection with W and independence between disjoint windows

Characterize the homogeneous Poisson point process

$\mathrm{Vol}_d :=$ Lebesgue measure in $\mathbb{R}^d$

Intensity

$$\lambda := \frac{1}{\mathrm{Vol}_d(A)}\mathbb{E}(\#(\mathcal{P}_\lambda \cap A)), \quad A \in \mathcal{B}(\mathbb{R}^d),\ 0 < \mathrm{Vol}_d(A) < \infty$$

Capacity

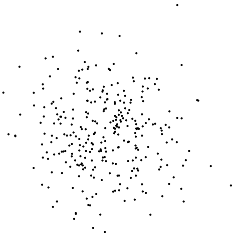
$$\mathbb{P}(\mathcal{P}_\lambda \cap K \neq \emptyset) = 1 - \exp(-\lambda \mathrm{Vol}_d(K)), \quad \text{K compact set of } \mathbb{R}^d$$

Mecke's formula

$$\mathbb{E}\Big(\sum_{x_{1:\ell} \subset \mathcal{P}_\lambda} f(x_{1:\ell}, \mathcal{P}_\lambda)\Big) = \frac{\lambda^\ell}{\ell!}\int \mathbb{E}(f(x_{1:\ell}, \mathcal{P}_\lambda \cup x_{1:\ell}))\mathrm{d}x_{1:\ell}$$

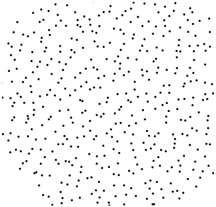
where $x_{1:\ell} := \{x_1, \ldots, x_\ell\}$ and f is a non-negative measurable function.

Generate random points in $\mathbb{R}^d$



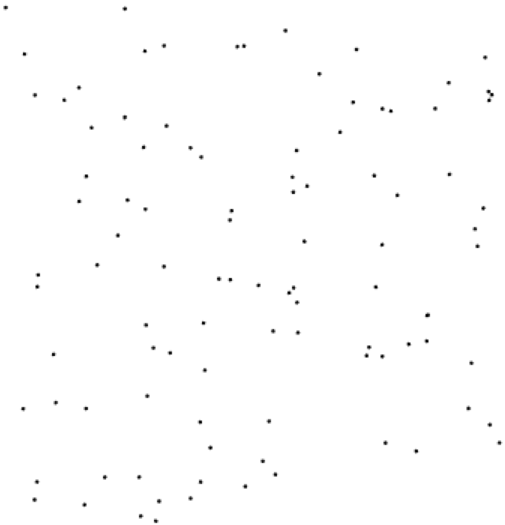
Gaussian Poisson point process

Matérn cluster point process

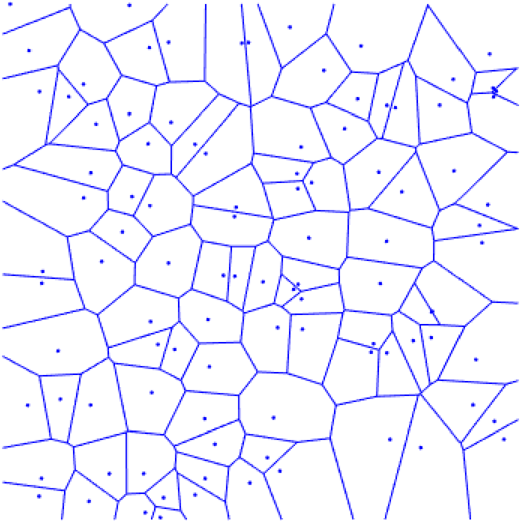


Ginibre determinantal point process

Make a deterministic geometric construction



Poisson-Voronoi tessellation



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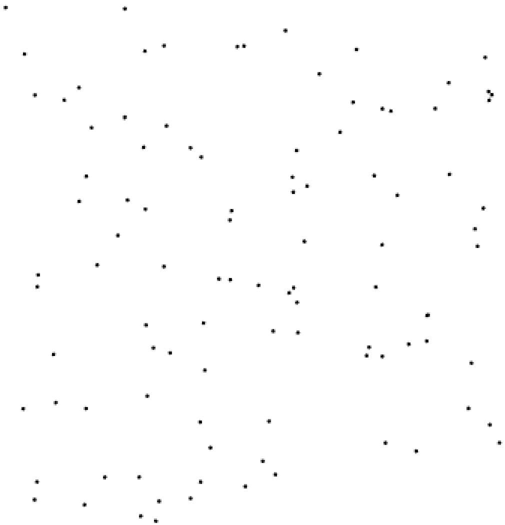
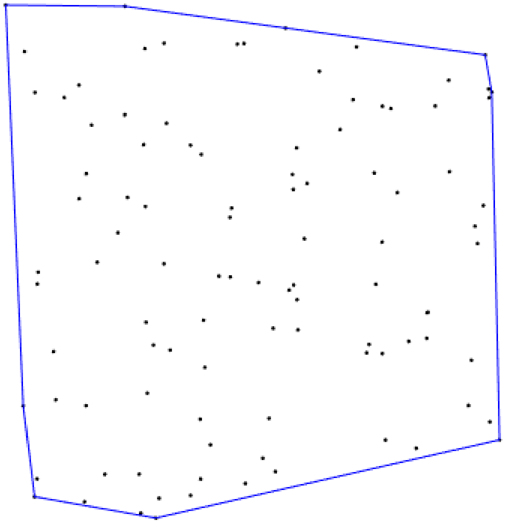


Random convex hull



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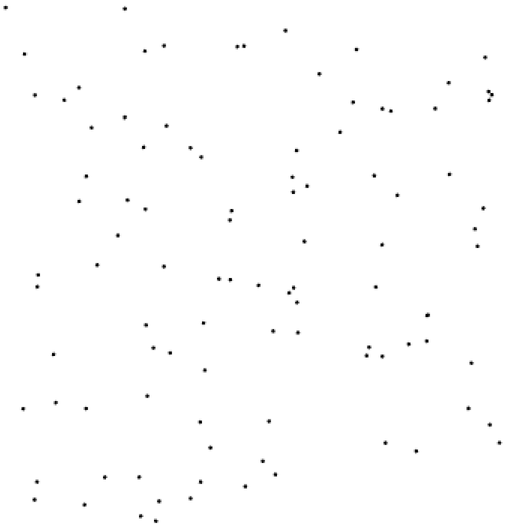
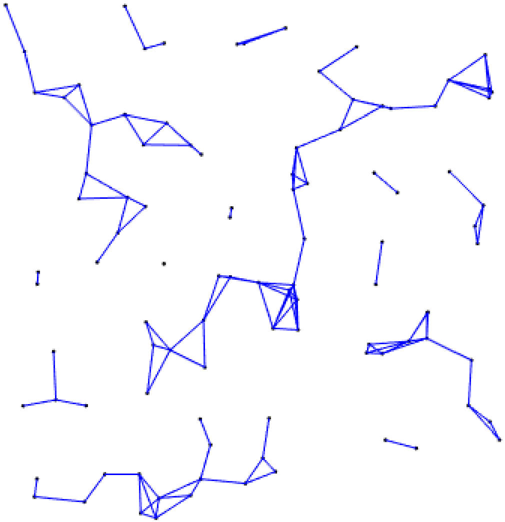


Random geometric graph



Plan

Maximal distance to the nucleus in a Poisson-Voronoi tessellation

Maximal facet volume of a random convex hull

Connected component of a random covering model

Joint works with A. Chaudron, N. Chenavier, C. D'Errico, N. Enriquez & J. E. Yukich

Poisson-Voronoi tessellation

$\mathcal{P}_\lambda$ homogeneous Poisson point process of intensity λ
$C(x, \mathcal{P}_\lambda) :=$ set of points with nearest neighbor x in $\mathcal{P}_\lambda$

References. R. Cowan (1978), D. Hug, R. Schneider (2003), A.Gusakova, Z. Kabluchko, C. Thäle (2024)

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Means over the cells, exceptional cells ?

References. R. Cowan (1978), D. Hug, R. Schneider (2003), A.Gusakova, Z. Kabluchko, C. Thäle (2024)

Poisson-Voronoi typical cell

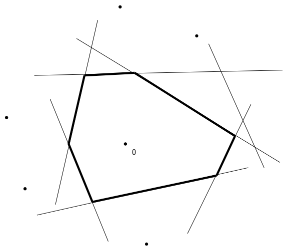
- Typical cell $\mathcal{C}_\lambda$ *uniformly chosen among all cells*

$$\mathbb{E}(f(\mathcal{C}_\lambda)) := \lim_{r\to\infty} \frac{1}{N_r} \sum_{x\in\mathcal{P}_\lambda\cap\mathcal{B}^{(\mathbb{R}^n)}(o,r)} f(C(x,\mathcal{P}_\lambda)) \text{ a.s.}$$

$$\mathbb{E}(f(\mathcal{C}_\lambda)) := \frac{1}{\lambda \mathrm{vol}^{(\mathbb{R}^n)}(B)} \mathbb{E}\left(\sum_{x\in\mathcal{P}_\lambda\cap B} f(C(x,\mathcal{P}_\lambda))\right), \; B\in\mathcal{B}(\mathbb{R}^n)$$

for any translation-invariant measurable function f

- Theorem (Slivnyak) $\mathcal{C} := \mathcal{C}_1 \overset{\text{law}}{=} C(o,\mathcal{P}_1\cup\{o\})$



Two mean calculations in $\mathbb{R}^2$

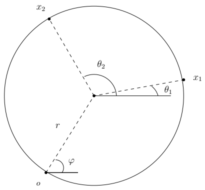
Mean area

$$\mathbb{E}(\mathrm{Vol}_2(\mathcal{C})) = \int_{\mathbb{R}^2} \mathbb{P}(x \in C(o, \mathcal{P}_\lambda \cup \{o\}))\mathrm{d}x = 2\pi \int_0^\infty e^{-\lambda\pi r^2} r\mathrm{d}r = \frac{1}{\lambda}$$

Mean number of vertices

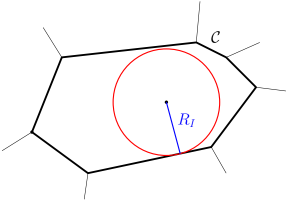
$\mathcal{N}(\cdot) :=$ number of vertices, $B(x,y,z) :=$ circumscribed disk of the triangle xyz

$$\begin{aligned}
\mathbb{E}(\mathcal{N}(\mathcal{C})) &= \mathbb{E}\left(\sum_{\{x_1,x_2\}\subset\mathcal{P}_\lambda} \mathbf{1}_{\{B(o,x_1,x_2)\cap\mathcal{P}_\lambda=\emptyset\}}\right) \\
&= \frac{\lambda^2}{2}\int \mathbb{P}(B(o,x_1,x_2)\cap\mathcal{P}_\lambda=\emptyset)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \frac{\lambda^2}{2}\int e^{-\lambda \mathrm{vol}_2(B(o,x_1,x_2))}\mathrm{d}x_1\mathrm{d}x_2 \\
&= 4\pi\lambda^2\int_0^\infty e^{-\lambda\pi r^2} r^3\mathrm{d}r \iint_{(0,2\pi)^2} \sin(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})|\sin(\frac{\theta_2-\theta_1}{2})|\mathrm{d}\theta_1\mathrm{d}\theta_2 \\
&= 6
\end{aligned}$$



Inradius

$\mathcal{C}$:= typical cell (i.e. chosen *at random*) for $\lambda = 1$
R_I := minimal distance from a point in $\partial\mathcal{C}$ to the nucleus



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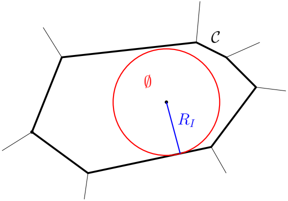

$$\mathbb{P}(R_I \geq t) = \mathbb{P}(\mathcal{P} \cap B(0,t) = \emptyset) = \exp(-\mathrm{Vol}_d(B(0,t))) = \exp(-\kappa_d t^d)$$

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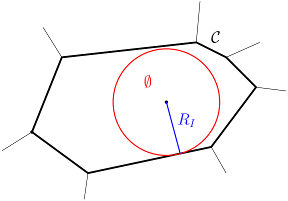

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Distribution of the elongated cells ?

References. R. Cowan (1978), D. Hug, R. Schneider (2003), A.Gusakova, Z. Kabluchko, C. Thäle (2024)

Maximal distance to the nucleus

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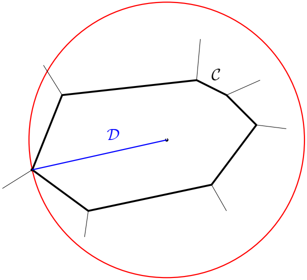

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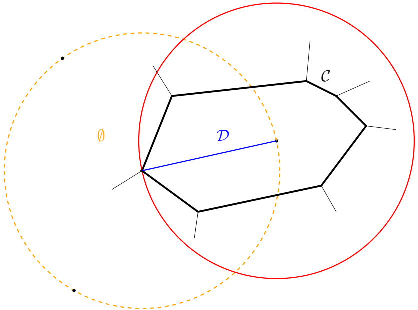

(with C. D'Errico & N. Enriquez) (κ_d := volume of the unit ball)

$$\mathbb{P}(\mathcal{D} \geq t) \underset{t\to\infty}{=} C_d t^{d(d-1)} e^{-\kappa_d t^d} + O(t^{d(d-2)} e^{-\kappa_d t^d})$$

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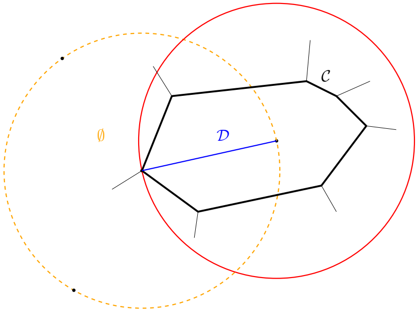

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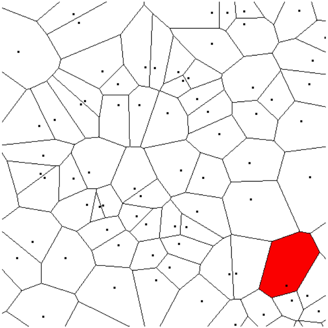
Maximal value of the maximal distance $\mathcal{D}$ in a window



$K :=$ convex body of volume 1
$\mathcal{D}_{\max,\lambda} :=$ maximal distance to the nucleus for all cells in λK

(with N. Chenavier)

$$\mathbb{P}(\kappa_d \mathcal{D}^d_{\max,\lambda} - \log(c_d \lambda^d (\log \lambda)^{d-1}) \le t) \underset{\lambda\to\infty}{\to} e^{-e^{-t}}, \quad t \in \mathbb{R}$$

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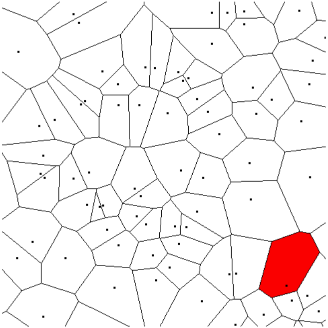

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Extreme values and extremal index

$(X_i, i \geq 1) :=$ sequence of i.i.d. random variables

$t_n :=$ threshold such that $n\mathbb{P}(X_1 \geq t_n) \underset{n\to\infty}{\to} \tau$ for fixed $\tau \in (0, \infty)$

$$\mathbb{P}(\max_{1\leq i\leq n} X_i \leq t_n) = (1 - \mathbb{P}(X_1 \geq t_n))^n \underset{t\to\infty}{\to} e^{-\tau}$$

When the variables are dependent, the exceedances come in clusters and there is $\theta \in (0, 1]$ such that

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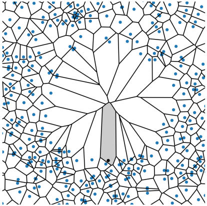
Maximum de la distance maximale $\mathcal{D}$ dans une fenêtre



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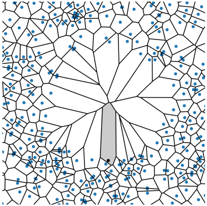

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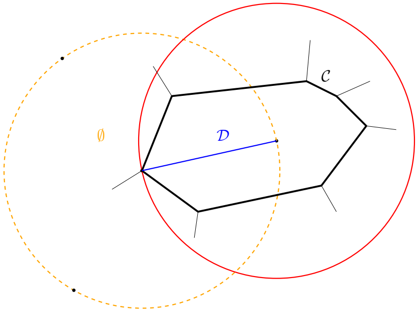

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Proof strategy

- $\mathcal{D} \geq t$ if there is a vertex which is a local maximum for the distance to o.
- Such vertices are rare car very *expensive* so
$$\mathbb{P}(\mathcal{D} \geq t) \approx \text{ mean of the number of such vertices.}$$
- Mecke's formula then change of variables to calculate that mean
- The multiplicative constant is the mean volume of a simplex with random vertices on the unit sphere.
- The question of the limit law for the maximum can interpreted as a covering problem in the plane.

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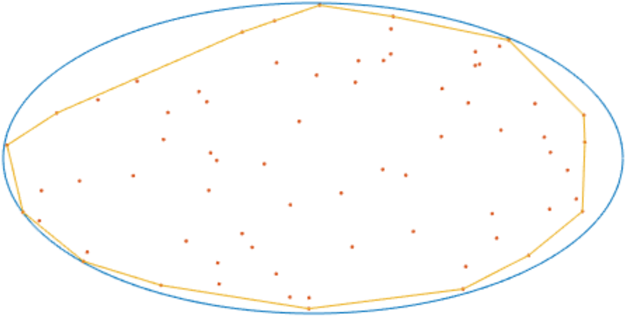
Random convex hull in a smooth window K

K smooth convex compact set in $\mathbb{R}^d$ with non empty interior

N_λ Poisson variable of mean $\lambda \mathrm{Vol}_d(K)$, $\lambda > 0$

K_λ convex hull of N_λ i.i.d. uniform points in K



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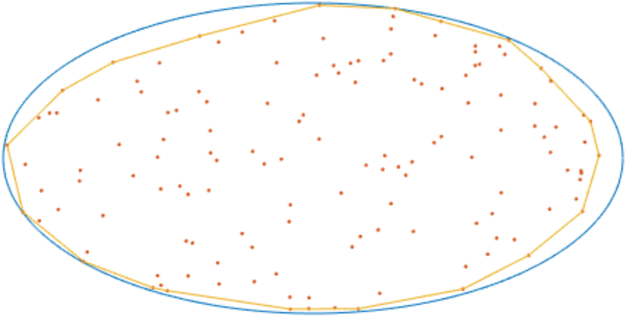

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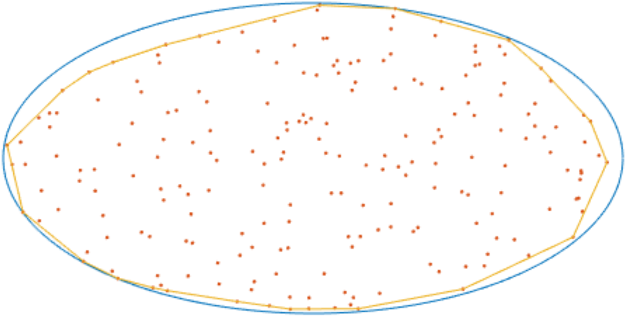

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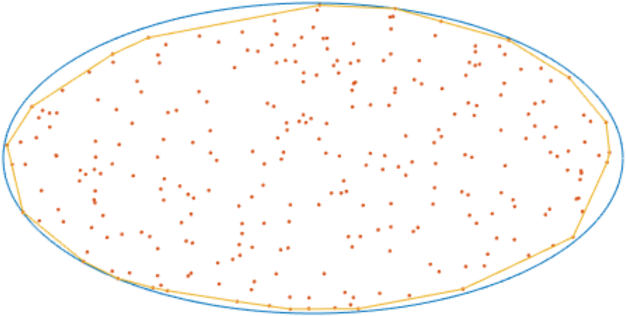
K_λ convex hull of N_λ i.i.d. uniform points in K



Random convex hull in a smooth window K

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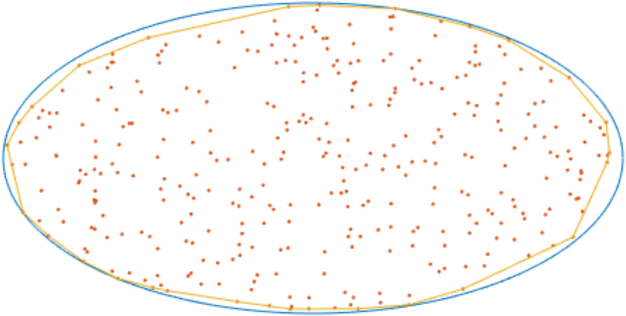

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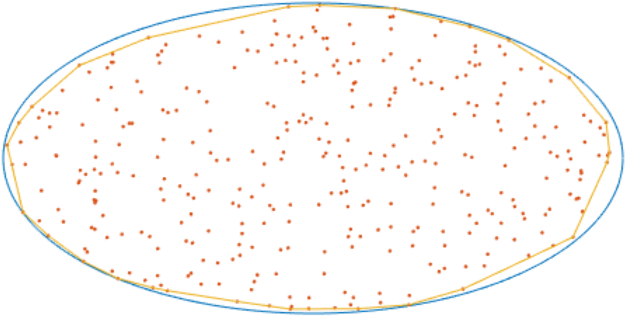
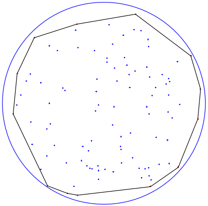
Behavior when $\lambda \to \infty$?

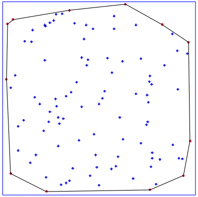


Dichotomy in the study of random convex hulls

$\mathrm{Vol}_d :=$ Lebesgue measure of $\mathbb{R}^d$, $f_k(\cdot) :=$ number of k-dimensional faces

		K smooth	K polytope
$\mathbb{E}(\mathrm{Vol}_d(K \setminus K_\lambda))$	$\underset{\lambda\to\infty}{\sim}$	$c_1(d,K)\lambda^{-\frac{2}{d+1}}$	$c_1'(d,K)\lambda^{-1}\log^{d-1}(\lambda)$
$\mathbb{E}(f_k(K_\lambda))$	$\underset{\lambda\to\infty}{\sim}$	$c_2(d,k,K)\lambda^{\frac{d-1}{d+1}}$	$c_2'(d,k,K)\log^{d-1}(\lambda)$





For any K, $\quad c(d,K)\log^{d-1}(\lambda) \leq \mathbb{E}(f_k(K_\lambda)) \leq c'(d,K)\lambda^{\frac{d-1}{d+1}}$

References. Rényi & Sulanke (1963), Schneider & Wieacker (1980), Bárány (1989), Reitzner (2003)

Random convex hull in a smooth window K

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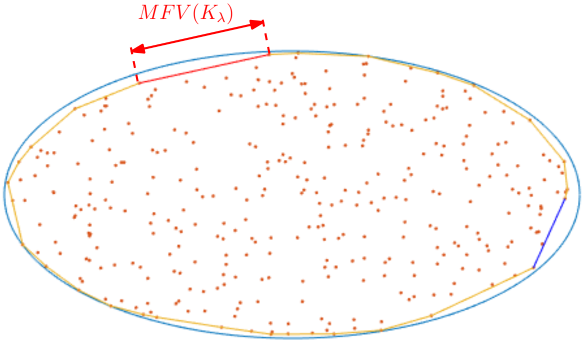
Law of the maximal facet volume when $\lambda \to \infty$?



Longitudinal extreme : maximal facet volume

$$\color{red}{MFV(K_\lambda)} := \max_{F \text{ facet of } K_\lambda} \mathrm{Vol}_{d-1}(F),$$ $\mathrm{Vol}_{d-1} := (d-1)$-dimensional volume

(with J. Yukich)

✓ $\left(a_o a_1 \frac{\log \lambda}{\lambda}\right)^{\frac{d-1}{d+1}} \color{red}{MFV(K_\lambda)} \overset{\mathbb{P}}{\to} 1$

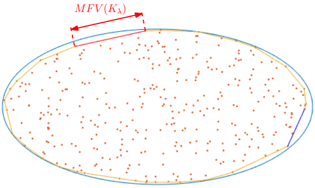
✓ $\color{red}{MFV(K_\lambda)} = \lambda^{-\frac{d-1}{d+1}}(a_0(a_1 \log \lambda + a_2 \log(\log \lambda) + a_3 + \xi_\lambda))^{\frac{d-1}{d+1}}$

où $\mathbb{P}(\xi_\lambda \le t) \underset{\lambda\to\infty}{\longrightarrow} e^{-e^{-t}}$ (Gumbel law)

$$a_0 := \frac{2\Gamma(\frac{d+3}{2}) v_{d-1}^{\frac{d+1}{d-1}}}{\pi^{\frac{d-1}{2}} \min_{\partial K} \kappa^{\frac{1}{d-1}}}$$

$$a_1 := \frac{d-1}{d+1}$$

$$v_{d-1} := \mathrm{Vol}_{d-1}(\text{regular simplex in } \mathbb{B}^{d-1})$$

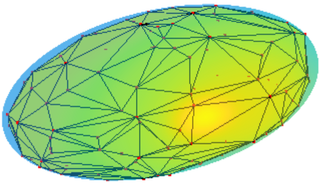


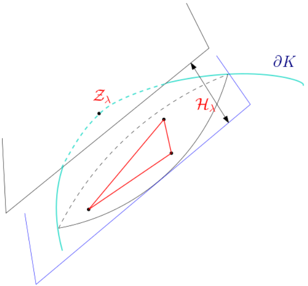
Facet with maximal volume

$\mathcal{F}_{\lambda,\mathrm{max}}$:= facet with maximal volume

Location and shape of $\mathcal{F}_{\lambda,max}$ when $\lambda \to \infty$?

The facet $\mathcal{F}_{\lambda,\mathrm{max}}$ is a random simplex included in a section of K and whose associated cap is empty.





Convergence of the location of $\mathcal{F}_{\lambda,\max}$

(with J. Yukich)

$$\kappa(\mathcal{Z}_\lambda) \xrightarrow[\lambda\to\infty]{law} \min_{z\in\partial K} \kappa(z)$$

where $\kappa(\cdot) :=$ Gauss curvature of ∂K.

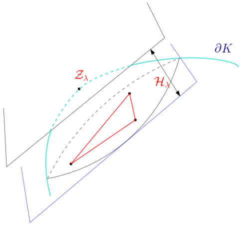
If $\mathrm{Vol}_{d-1}(\mathrm{argmin}(\kappa)) > 0$,

$$\mathcal{Z}_\lambda \xrightarrow[\lambda\to\infty]{loi} \text{Unif}\,(\mathrm{argmin}(\kappa)).$$

If $\mathrm{argmin}(\kappa) = \{z_1, \cdots, z_k\}$,

$$\mathcal{Z}_\lambda \xrightarrow[\lambda\to\infty]{loi} \sum_{i=1}^k w_i \delta_{z_i}$$

where $w_i \propto \left(\det(D^2\kappa\big|_{z_i})\right)^{-\frac{1}{2}}$.

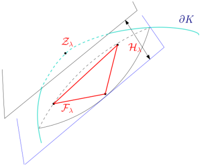


Convergence of the shape of $\mathcal{F}_{\lambda,\max}$

Affine transformation T_λ

$\mathcal{F}_{\lambda,\max} \subset$ section of K with *osculating ellipsoid*

T_λ sends this ellipsoid to $\mathbb{B}^{d-1}$.



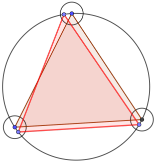
ρ-distance to the regular simplex

$\mathcal{S}_i$ simplex with vertices $s_i^{(1)}, \cdots, s_i^{(d)}$,

$\eta(\mathcal{S}_1, \mathcal{S}_2) = \max_k \min_l \|s_1^{(k)} - s_2^{(l)}\|$

$B(c_1, r_1)$ circumscribed ball of S_1

$\rho(\mathcal{S}_1) = \min\{\eta(r_1^{-1}(\mathcal{S}_1 - c_1), \mathcal{S}_{\text{reg}}) : \mathcal{S}_{\text{reg}} \text{ regular} \subset \mathbb{B}^{d-1}\}$



(with J. Yukich)

$$\forall\, \beta \in (0, \tfrac{1}{2}), \quad \mathbb{P}(\rho(T_\lambda(\mathcal{F}_{\lambda,\max})) \geq (\log \lambda)^{-\beta}) \underset{\lambda\to\infty}{\longrightarrow} 0$$

Poisson approximation

$f_n(F) = a_0^{-1} n \mathrm{Vol}_{d-1}(F)^{\frac{d+1}{d-1}} - (a_1 \log n + a_2 \log\log n + a_3)$

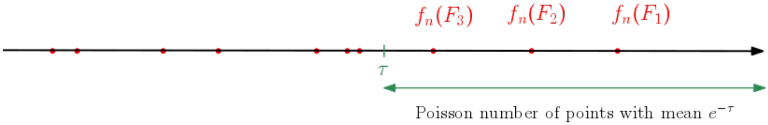
Kantorovich-Rubinstein distance between two random measures μ and ν

$d_{KR}(\mu,\nu) := \sup_{h\in \mathrm{Lip}_1} |\mathbb{E}(h(\mu) - h(\nu))|$, $\mathrm{Lip}_1 = \{$1-Lipschitz functions for $d_{TV}\}$

(with J. Yukich)

✓ Convergence of the point process $\{f_n(F), F$ facet of $K_\lambda\}$ to a Poisson process

✓ Similar result for the process of couples $(f_n(F), z(F))$ when $\dim((\partial K)^{\min}) = d - 1$



References. O. Bobrowski, M. Schulte, D. Yogeshwaran (2021)

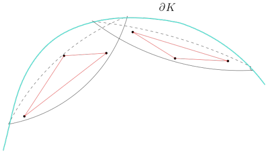
Proof strategy

▶ Estimation of the distribution tail of $\mathrm{Vol}_{d-1}(\mathcal{F}_\lambda)$ at the threshold for a *typical* facet $\mathcal{F}_\lambda$, i.e. chosen *at random*

- Mecke's formula
- Change of variables in the integral
- Geometric approximations for ∂K

▶ Mixing property of the considered variables

The exceeding facets are asymptotically independent.



▶ Theorems about the extremes of functions of point processes

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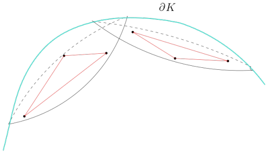

- Theorems about the extremes of functions of point processes

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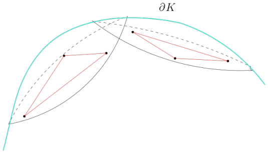


- Theorems about the extremes of functions of point processes

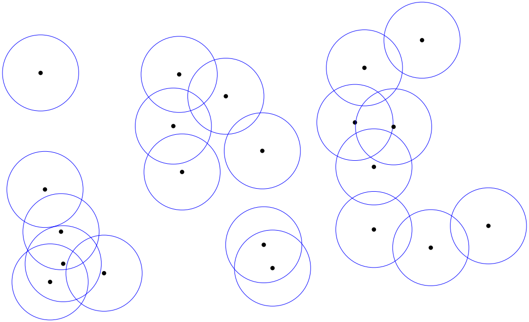
Plan

Maximal distance to the nucleus in a Poisson-Voronoi tessellation

Maximal facet volume of a random convex hull

Connected component of a random covering model

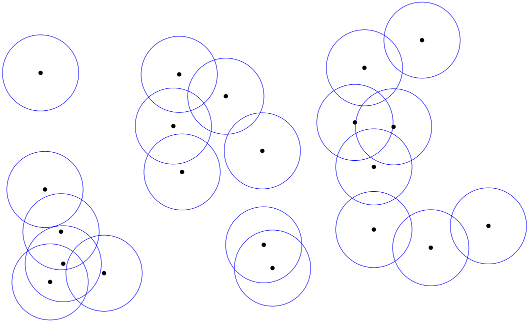
Boolean model or continuum percolation model



$$\text{Occupied region } \mathcal{O}_{\lambda,r} := \bigcup_{x \in \mathcal{P}_\lambda} B(x,r) \qquad \lambda, r > 0$$

References. E. N. Gilbert (1961), R. Meester & R. Roy (1996), J.-B. Gouéré (2008)

Boolean model or continuum percolation model

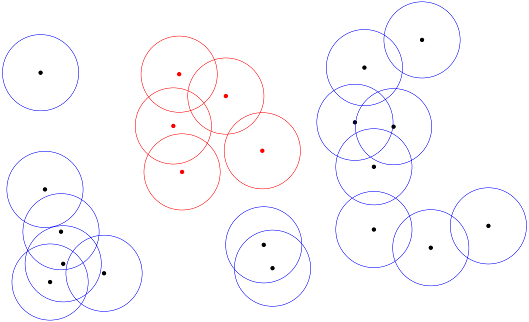


Occupied region $\mathcal{O}_{\lambda,r} := \bigcup_{x \in \mathcal{P}_\lambda} B(x, r) \quad \lambda, r > 0$

Size of the components, threshold in λ?

References. E. N. Gilbert (1961), R. Meester & R. Roy (1996), J.-B. Gouéré (2008)

The connected component seen from a typical point



By scaling invariance, one can fix $r = 1$.

$C(x, \mathcal{P}_\lambda) :=$ connected component of $\mathcal{O}_{\lambda,1}$ containing x
$\mathcal{C}_\lambda :=$ component of a typical point

Theorem (Slivnyak) $\mathcal{C}_\lambda \overset{\text{law}}{=} C(o, \mathcal{P}_\lambda \cup \{o\})$

Law of the cardinality of the typical component $\mathcal{C}$

$\mathcal{N}(\mathcal{C}_\lambda) :=$ number of points of $\mathcal{P}_\lambda$ in $\mathcal{C}_\lambda$

Mecke's formula

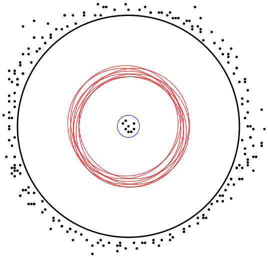
$$\mathbb{P}(\mathcal{N}(\mathcal{C}_\lambda) = k+1) = \frac{\lambda^k}{k!} \int \mathbf{1}_{CC}(x_{0:k}) \exp(-\lambda \mathrm{Vol}_d(\cup_{i=0}^k B(x_i, 1)))\mathrm{d}x_{1:k}$$

where $x_0 := o$ and

$\mathbf{1}_{CC}(x_{0:k}) :=$ indicator of the event that the balls $B(x_i, 1)$, $0 \leq i \leq k$, are connected.

Behavior when $\lambda \to \infty$?

Asymptotics for large intensity



For fixed $k \geq 1$, there is an explicit constant $\alpha_k > 0$ such that

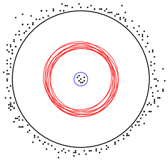
$$\mathbb{P}(\mathcal{N}(\mathcal{C}_\lambda) = k+1) \underset{\lambda\to\infty}{\sim} \frac{\alpha_k}{k!} \lambda^{-k(d-1)} e^{-\lambda\kappa_d}$$

Compression of points in $\mathcal{C}_\lambda$ at distance $\frac{1}{\lambda}$ from o

Sphericality if $\mathcal{C}_\lambda$ when k grows with λ

References. K. Alexander (1993), M. Penrose & X. Yang (2023)

Two-term expansion



(with A. Chaudron)

✓ For fixed $k \geq 1$,

$$k!\lambda^{k(d-1)}e^{\lambda\kappa_d}\mathbb{P}(\mathcal{N}(\mathcal{C}_\lambda) = k+1) \underset{\lambda\to\infty}{=} \alpha_k - \frac{1}{\lambda}\beta_{k+1} + O\left(\frac{1}{\lambda^2}\right)$$

where

$$\alpha_k := \int_{(\mathbb{R}^d)^k} e^{-\kappa_{d-1}V_1(z_{0:k})}\,\mathrm{d}z_{1:k},\ \beta_k := \int_{(\mathbb{R}^d)^k} e^{-\kappa_{d-1}V_1(z_{0:k})}V_2\left(z_{0:k}\right)\,\mathrm{d}z_{1:k}$$

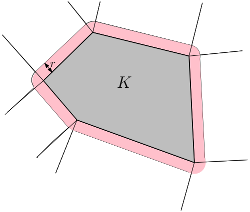
$V_1\left(z_{0:k}\right), V_2\left(z_{0:k}\right) :=$ intrinsic volumes of $\mathrm{Conv}(z_{0:k})$, $z_0 = o$

✓ Convergence rate to the limit law of the set of points in $\mathcal{C}_\lambda$

Intrinsic volumes

$K :=$ convex compact set in $\mathbb{R}^d$

Convex parallel set $K + B(0,r) = \cup_{x \in K} B(x,r), \quad r > 0$



Steiner's formula

$$\mathrm{Vol}_d(K + B(0,r)) = \sum_{k=0}^{d} \kappa_{d-k} r^{d-k} V_k(K)$$

$V_k(K) :=$ intrinsic volume of order k in K

($\kappa_d :=$ volume of the unit ball in $\mathbb{R}^d$)

Proof strategy

- Mecke's formula for the integral rewriting of the probability
- Prediction of the size of $\mathcal{C}_\lambda$ and rescaling accordingly
- Approximation of the union of balls in $\mathcal{C}_\lambda$ by its convex hull
- Use of Steiner's formula

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Merci pour votre attention !